ON ANALYTIC CONTINUATION OF LAURENT SERIES

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1. INTRODUCTION

There are various theorems, dating back to 1900, concerning analytic continuation of power series $\Sigma_0^\infty a_n z^n$ in which $a_n = g(n)$ for a holomorphic function g of some sort. Here we shall discuss certain extensions and analogs of the theorems of Wigert, Hardy, and Kronecker, which can be stated as follows.

THEOREM OF WIGERT [8, p. 288]. *$\Sigma_0^\infty a_n z^n$ defines a holomorphic function which has 1 as its only possible singularity and which vanishes at ∞ if and only if there is an entire function g of exponential type 0 such that* $g(n) = a_n$ $(n = 0, 1, 2, \cdots)$.

THEOREM OF HARDY [5, p. 338]. *Let $0 < \rho < \pi$, and let S be the circle $|z| = \rho$. In order that a series $\Sigma_0^\infty a_n z^n$ define a function holomorphic in the exterior of the curve e^{-S}, having a singularity on e^{-S}, and vanishing at ∞, it is necessary and sufficient that there exists an entire function g of exponential type ρ such that* $g(n) = a_n$ $(n = 0, 1, 2, \cdots)$.

THEOREM OF KRONECKER [4, p. 321]. *$\Sigma_0^\infty a_n z^n$ defines a rational function if and only if the infinite matrix (a_{i+j}) has finite rank.*

We begin by obtaining—Theorem 1 below—a complete generalization of the theorem of Hardy, in the sense that we are able to consider functions f that are holomorphic in the complement of an arbitrary compact set. The "coefficient functions" g that then occur are entire functions of arbitrary exponential type. However, the Laplace transform $\hat{g}$ of g can be continued holomorphically to a region whose complement B is bounded and has the property that $\bigcup_{-\infty}^{\infty}(B + 2n\pi i)$ does not separate the plane. The function f is then holomorphic in the complement of e^{-B}. It is clear from Pólya's theory of the indicator diagram (see [7] or [3, pp. 66-77]) that the above property of $\hat{g}$ is weaker than Hardy's type-restriction on g. Another feature of Theorem 1, important for the subsequent discussion of Laurent series, is that the expansions of f about 0 and ∞ are treated symmetrically.

Our results on Laurent series—Theorems 2 and 3—are concerned with a pair of series $\Sigma_{-\infty}^\infty a_n z^n$ and $\Sigma_{-\infty}^\infty b_n z^n$ that converge in two disjoint annuli. Theorem 2 gives a necessary and sufficient condition on the sequence $\{a_n - b_n\}$ in order that the given series be Laurent expansions of the same holomorphic function. Theorem 3 gives the condition that this be the case and that there be only poles and essential singularities between the two annuli. It is also mentioned here how the location of the singularities and the construction of the principal parts depend upon the sequence $\{a_n - b_n\}$. Finally, to illustrate Theorem 3, we derive a formula from the theory of elliptic functions.

Notation and terminology. $\mathfrak{C}$ will denote the complex plane. ∂E will denote the topological boundary of E $(E \subset \mathfrak{C})$. A region is a nonempty, open, connected subset of $\mathfrak{C}$. Holomorphic means analytic and single-valued in a region that may or may not be specified. If g is an entire function of exponential type, $\hat{g}$ denotes its Laplace

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transform or any holomorphic extension thereof. N will denote the set of integers, N^+ the set of nonnegative integers, and N^- the set of negative integers. For Cauchy's theorem, cycle, winding number, and so forth, we refer to [1]. We shall frequently use sets in numerical expressions with the obvious meanings. For example, if γ is a cycle, $n(\gamma, E) = 1$ means $n(\gamma, z) = 1$ for all $z \in E$. This notation, incidentally, implies $\gamma \subset \mathfrak{C} \sim E$. (We shall use the same symbol for a cycle and the associated point set.) Finally, ext γ denotes the (unique, open) unbounded component of $\mathfrak{C} \sim \gamma$.

2. PRELIMINARY LEMMAS

In establishing Theorem 1, we shall make repeated use of three topological lemmas. Lemmas 1 and 2 may be of general interest.

LEMMA 1. *Let* h *be holomorphic in a simply connected region* D, *let* γ *be a cycle in* D, *and let* $c \in D$. *Then*

$$h(c) \in \operatorname{ext} h(\gamma) \Rightarrow c \in \operatorname{ext} \gamma .$$

Proof. Let us suppose to the contrary that c belongs to one of the bounded components U of $\mathfrak{C} \sim \gamma$. Then, by the maximality of U, $\partial U \subset \gamma$, and by the simple connectivity of D, $U \subset D$. Thus $\overline{U} \subset D$, and therefore $h(\overline{U})$ is compact. Let E be any connected set in ext $h(\gamma)$ containing both h(c) and a point of $\mathfrak{C} \sim h(\overline{U})$. Then $E \cap h(\overline{U})$ is closed in E. But since $\partial U \subset \gamma$, $E \cap h(\overline{U}) = E \cap h(U)$, which is open in E. Thus we have contradicted the connectedness of E.

LEMMA 2. *Let* h *be holomorphic in a simply connected region* D, *and let* E *be a compact subset of* D. *Then*

$$\bigcup_{\substack{\gamma \subset D \\ n(\gamma, E)=1}} \operatorname{ext} h(\gamma) = \mathfrak{C} \sim h(E)$$

if and only if $\mathfrak{C} \sim h(E)$ *is connected.*

Proof. First we observe that the "only if" statement is trivially correct. Next we assert that

$$\bigcup_{\gamma} \operatorname{ext} h(\gamma) \subset \mathfrak{C} \sim h(E)$$

whether $\mathfrak{C} \sim h(E)$ is connected or not. Indeed, by Lemma 1,

$$c \in E \Rightarrow c \notin \operatorname{ext} \gamma \Rightarrow h(c) \notin \operatorname{ext} h(\gamma)$$

for any γ as described. Finally, let us assume $\mathfrak{C} \sim h(E)$ is connected and $c \in \mathfrak{C} \sim h(E)$. Let K be any closed, connected, unbounded set such that

$$c \in K \subset \mathfrak{C} \sim h(E).$$

Let $V = h^{-1}(\mathfrak{C} \sim K)$. Then V is open and $E \subset V \subset D$. Let γ be a cycle in V with $n(\gamma, E) = 1$. (To see how such a cycle can be constructed we refer to [1, p. 113].) Then $h(\gamma) \subset \mathfrak{C} \sim K$. Therefore $c \in \operatorname{ext} h(\gamma)$.

LEMMA 3. *Let* B *be compact. Then* $\mathfrak{C} \sim e^{-B}$ *is connected if and only if* $\mathfrak{C} \sim (B + 2N\pi i)$ *is connected.*

Proof. Suppose $\mathfrak{C} \sim e^{-B}$ is connected, and let $s_0 \in \mathfrak{C} \sim (B + 2N\pi i)$. Let σ be a real number such that $\sigma < \Re B$. We shall show that s_0 can be joined to the line $\{s: \Re s = \sigma\}$ by an arc contained in $\mathfrak{C} \sim (B + 2N\pi i)$. Let K be a Jordan arc joining e^{-s_0} to the circle $\{z: |z| = e^{-\sigma}\}$ such that $K \subset \mathfrak{C} \sim (e^{-B} \cup \{0\})$. Let log be continuous on K, with $\log e^{-s_0} = -s_0$. Then $-\log K$ is the required arc. The proof of the converse can be omitted. Indeed, for any entire function h and any closed set A, connectedness of $\mathfrak{C} \sim h^{-1}(A)$ implies connectedness of $\mathfrak{C} \sim A$.

3. A GENERALIZATION OF HARDY'S THEOREM

THEOREM 1. *Let* g *be an entire function of exponential type. Suppose* $\hat{g}$ *is holomorphic in a region* $\mathfrak{C} \sim B$, *where* B *is a compact set such that* $\mathfrak{C} \sim (B + 2N\pi i)$ *is connected. Let*

$$A = e^{-B}. \tag{1}$$

Then there exists a function f, *holomorphic in* $\mathfrak{C} \sim A$, *such that* $f(\infty) = 0$,

$$f(z) = \sum_{n \in N^+} g(n) z^n \qquad (|z| < |A|), \tag{2}$$

$$f(z) = \sum_{n \in N^-} -g(n) z^n \qquad (|z| > |A|). \tag{3}$$

Conversely, suppose f *is holomorphic in a region* $\mathfrak{C} \sim A$, *where* A *is compact,* $0 \in \mathfrak{C} \sim A$, *and* $f(\infty) = 0$. *Then there exist an entire function* g *of exponential type and a compact set* B *such that* $\hat{g}$ *is holomorphic in* $\mathfrak{C} \sim B$, $\mathfrak{C} \sim (B + 2N\pi i)$ *is connected, and* (1), (2), (3) *hold.*

Proof. Let g, $\hat{g}$, and B be as described. Let

$$f_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{\hat{g}(s)}{1 - ze^s} \, ds \qquad (n(\gamma, B) = 1, \ z \in \text{ext } e^{-\gamma}).$$

Then f_γ is holomorphic in ext $e^{-\gamma}$. We assert that if $f_{\gamma'}$ is another such function element, then

$$f_\gamma(z) = f_{\gamma'}(z) \qquad (z \in \text{ext } e^{-\gamma} \cap \text{ext } e^{-\gamma'}).$$

By Cauchy's theorem this assertion is implied by the equation

$$n(\gamma, s_0) = n(\gamma', s_0) \qquad (s_0 \in B \cup \{s: e^{-s} = z\}).$$

For $s_0 \in B$, this equation holds by construction. For $e^{-s_0} = z$, Lemma 1 gives

$$e^{-s_0} \in \text{ext } e^{-\gamma} \Rightarrow s_0 \in \text{ext } \gamma \Rightarrow n(\gamma, s_0) = 0,$$

and the same for γ'. Thus all the f_γ have a common extension f, holomorphic in

$$\bigcup_{n(\gamma,B)=1} \operatorname{ext} e^{-\gamma}$$

and with $f(\infty) = 0$. By Lemma 3, $\mathfrak{C} \sim e^{-B}$ is connected. Therefore Lemma 2 gives

$$\bigcup_{n(\gamma,B)=1} \operatorname{ext} e^{-\gamma} = \mathfrak{C} \sim e^{-B}.$$

Thus f is holomorphic in $\mathfrak{C} \sim A$, A being defined by (1). To prove (2), let us choose a cycle γ such that $n(\gamma, B) = 1$ and $0 \in \operatorname{ext} e^{-\gamma}$. Then, for $|z| < |e^{-\gamma}|$, we have the relations

$$f(z) = f_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \hat{g}(s)\,ds \sum_{n \in N^+} z^n e^{ns}$$

$$= \sum_{n \in N^+} \left\{ \frac{1}{2\pi i}\int_\gamma e^{ns}\hat{g}(s)\,ds \right\} z^n = \sum_{n \in N^+} g(n) z^n.$$

(Here we have used Pólya's inversion formula [3, p. 84, Theorem 5.3.5], slightly modified according to Cauchy's theorem.) Since f is holomorphic in $\mathfrak{C} \sim A$, (2) follows, and a similar computation establishes (3).

Conversely, let f and A be given as described. Let K be a curve in $\mathfrak{C} \sim A$ joining 0 and ∞ and such that $D = \mathfrak{C} \sim K$ is a simply connected region and $D \sim A$ is a region. Let $\log$ be holomorphic in D, and let

$$g(t) = -\frac{1}{2\pi i}\int_\delta f(z)\, e^{-(t+1)\log z}\,dz \qquad (\delta \subset D,\ n(\delta, A) = 1).$$

Then g is an entire function of exponential type which, by an application of Cauchy's theorem, is independent of the choice of δ. To prove (2) and (3), let us write

$$f(z) = \sum_{n \in N} c_n z^n \qquad (|z| < |A|),$$

$$f(z) = \sum_{n \in N} d_n z^n \qquad (|z| > |A|),$$

where $c_n = 0$ for $n \in N^-$ and $d_n = 0$ for $n \in N^+$. By the integral formula for Laurent coefficients and another application of Cauchy's theorem, we obtain the formula

$$d_n - c_n = \frac{1}{2\pi i}\int_\delta f(z)\, z^{-n-1}\,dz = -g(n) \qquad (n \in N).$$

Choosing $n \in N^+$, we obtain (2), and choosing $n \in N^-$, we obtain (3). Now

$$\hat{g}(s) = \int_0^\infty e^{-st} g(t)\,dt = -\frac{1}{2\pi i}\int_\delta \frac{f(z)}{z(s + \log z)}\,dz \qquad (\Re\, s > -\Re \log \delta).$$

The last integral provides a holomorphic extension of $\hat{g}$ to ext(-log δ). By an argument similar to that used before, based on Cauchy's theorem and Lemma 1, we may show that all choices of δ subject to the conditions stated provide mutually consistent extensions. Therefore $\hat{g}$ is holomorphic in $\bigcup_\delta$ ext(-log δ). Since $D \sim A$ is a region, the same is true of

$$-\log(D \sim A) = (-\log D) \sim (-\log A).$$

It follows easily that $\mathfrak{C} \sim (-\log A)$ is connected. Therefore, by Lemma 2,

$$\bigcup_\delta \operatorname{ext}(-\log \delta) = \mathfrak{C} \sim (-\log A).$$

Let $B = -\log A$. Then (1) holds and $\hat{g}$ is holomorphic in $\mathfrak{C} \sim B$. Finally, $\mathfrak{C} \sim (B + 2N\pi i)$ is connected, by Lemma 3.

Remark. It can happen that $\mathfrak{C} \sim B$ is a maximal domain of holomorphy for $\hat{g}$, but $\mathfrak{C} \sim e^{-B}$ is not maximal for the corresponding function f. (For example, let $g(t) = \sin \pi t$, $B = \{\pi i, -\pi i\}$.) It appears, however, that this cannot happen if the map from B onto e^{-B} is one-to-one, as is the case for $B = -\log A$ above.

4. A COROLLARY CONCERNING LAURENT SERIES

THEOREM 2. *Let the series* $\Sigma_{n \in N} a_n z^n$ *and* $\Sigma_{n \in N} b_n z^n$ *converge for* $r_1 < |z| < r_2$ *and* $r_3 < |z| < r_4$, *respectively, where* $0 \le r_1 < r_2 \le r_3 < r_4 \le \infty$. *Then these series are Laurent expansions of the same holomorphic function if and only if there exist an entire function* g *of exponential type and a compact set* B *such that* $\hat{g}$ *is holomorphic in* $\mathfrak{C} \sim B$, $\mathfrak{C} \sim (B + 2N\pi i)$ *is connected,*

(4) $$r_2 \le |e^{-B}| \le r_3,$$

and

(5) $$g(n) = a_n - b_n \qquad (n \in N).$$

Proof. Suppose h is holomorphic in a region $\{z: r_1 < |z| < r_4\} \sim A$ with $r_2 \le |A| \le r_3$ and

(6) $$h(z) = \sum_{n \in N} a_n z^n \qquad (r_1 < |z| < r_2),$$

(7) $$h(z) = \sum_{n \in N} b_n z^n \qquad (r_3 < |z| < r_4).$$

Let

(8) $$k(z) = \sum_{n \in N^-} a_n z^n + \sum_{n \in N^+} b_n z^n \qquad (r_1 < |z| < r_4)$$

and

(9) $$f(z) = h(z) - k(z) \qquad (r_1 < |z| < r_4,\ z \notin A).$$

Then

$$f(z) = \sum_{n \in N^+} (a_n - b_n) z^n \qquad (r_1 < |z| < r_2),$$

$$f(z) = \sum_{n \in N^-} -(a_n - b_n) z^n \qquad (r_3 < |z| < r_4).$$

But these series converge for $|z| < r_2$ and $|z| > r_3$, respectively, and therefore f has a holomorphic extension to the region $\mathfrak{C} \sim A$ with $f(\infty) = 0$. Hence, by Theorem 1, there exist a function g and a set B as described above such that (4) and (5) are satisfied.

Conversely let g and B be as described, and let (4) and (5) hold. Let f and A correspond to g and B, as in Theorem 1. Since $\mathfrak{C} \sim A$ is connected, so is

$$\{z: r_1 < |z| < r_4\} \sim A,$$

and we can define the holomorphic function h by equations (8) and (9). Finally, (6) and (7) follow directly from (1), (2), (3), (4), (5), (8), and (9).

Remark. In the second half of Theorem 2, the convergence assumptions regarding $\Sigma_{n \in N} a_n z^n$ and $\Sigma_{n \in N} b_n z^n$ were not entirely necessary. Only that part occurring in (8) was needed, the rest following from (4) and the other assumptions.

5. A SPECIAL CASE OF THEOREM 2

THEOREM 3. *Let the series* $\Sigma_{n \in N} a_n z^n$ *and* $\Sigma_{n \in N} b_n z^n$ *converge for* $r_1 < |z| < r_2$ *and* $r_3 < |z| < r_4$, *respectively, where* $0 \leq r_1 < r_2 \leq r_3 < r_4 \leq \infty$. *Suppose there are entire functions* $p_1, \cdots, p_K$ *of zero type, none identically zero, and distinct, nonzero complex numbers* $z_1, \cdots, z_K$ *such that*

$$a_n - b_n = \sum_{k=1}^{K} p_k(n) z_k^{-n} \qquad (n \in N). \tag{10}$$

Then

$$r_2 \leq |z_k| \leq r_3 \qquad (k = 1, \cdots, K), \tag{11}$$

and there exists a function h, *holomorphic in* $\{z: r_1 < |z| < r_4\}$ *except for singularities at the* z_k, *having the given series as Laurent expansions. Moreover, if*

$$h(z) = \sum_{q \in N} c_{qk} (z - z_k)^q \qquad (z \text{ near } z_k), \tag{12}$$

then

$$c_{qk} = -z_k^{-q} \sum_{m=0}^{-q-1} \binom{-q-1}{m} (-1)^m p_k(q+m) \qquad (q \in N^-). \tag{13}$$

In particular, if p_k *is a polynomial, then* z_k *is a pole of* h *of order* $1 +$ degree p_k.

Conversely, suppose the given series are Laurent expansions of a function h *that is holomorphic in* $\{z: r_1 < |z| < r_4\}$ *except for isolated singularities* $z_1, \cdots, z_K$ *satisfying* (11). *Then there exist entire functions* $p_1, \cdots, p_K$ *of zero type such that* (10) *holds.*

Proof. For each k, let

$$e^{-s_k} = z_k, \qquad g_k(t) = p_k(t)\, e^{s_k t}.$$

Then $\hat{g}_k$ is holomorphic in $\mathbb{C} \sim \{s_k\}$. By Theorem 1 there exists a function f_k holomorphic in $\mathbb{C} \sim \{z_k\}$ such that

$$f_k(z) = \sum_{n \in N^+} p_k(n)\, z_k^{-n} z^n \qquad (|z| < |z_k|)$$

and

$$f_k(z) = \sum_{n \in N^-} -p_k(n)\, z_k^{-n} z^n \qquad (|z| > |z_k|).$$

Let

$$f_k(z) = \sum_{q \in N^-} c_{qk} (z - z_k)^q \qquad (z \neq z_k).$$

Then if $n(\gamma_k, z_k) = 1$ and $q \in N^-$, we have the formulas

$$\begin{aligned} c_{qk} &= \frac{1}{2\pi i} \int_{\gamma_k} f_k(z)(z - z_k)^{-q-1}\, dz \\ &= \sum_{m=0}^{-q-1} \binom{-q-1}{m} (-z_k)^m \frac{1}{2\pi i} \int_{\gamma_k} f_k(z)\, z^{-q-1-m}\, dz \\ &= -\sum_{m=0}^{-q-1} \binom{-q-1}{m} (-z_k)^m p_k(q+m)\, z_k^{-q-m} \\ &= -z_k^{-q} \sum_{m=0}^{-q-1} \binom{-q-1}{m} (-1)^m p_k(q+m). \end{aligned}$$

Now, for $|z| < \min |z_k|$,

$$\sum_{k=1}^{K} f_k(z) = \sum_{n \in N^+} z^n \sum_{k=1}^{K} p_k(n)\, z_k^{-n} = \sum_{n \in N^+} (a_n - b_n) z^n.$$

By assumption, the last series converges if $|z| < r_2$. But the z_k are distinct, non-removable singularities of $\Sigma_{k=1}^K f_k(z)$. We conclude that $r_2 \leq \min |z_k|$, and a similar argument yields the other half of (11). Then the required function h is given by

$$h(z) = \sum_{k=1}^{K} f_k(z) + \sum_{n \in N^-} a_n z^n + \sum_{n \in N^+} b_n z^n.$$

The converse can be proved more directly without Theorem 1. Indeed, suppose there exists a function h as described. Then, with the help of (12), a calculation shows that

$$b_n - a_n = \sum_{k=1}^{K} \operatorname{Res}[h(z) z^{-n-1}, z_k] = \sum_{k=1}^{K} z_k^{-n} \sum_{q \in N^-} c_{qk} \binom{-n-1}{-q-1} z_k^q.$$

Finally, since $\lim_{q \to -\infty} |c_{qk}|^{1/|q|} = 0$, the function p_k defined by

$$p_k(n) = - \sum_{q \in N^-} c_{qk} \binom{-n-1}{-q-1} z_k^q$$

is entire of zero type. (We refer to [4, p. 339] for the required estimates.)

Remarks. $a_n - b_n$ can be written as in (10), with the p_k all polynomials, if and only if the doubly infinite matrix $(a_{i+j} - b_{i+j})$ has finite rank. Thus we have a criterion, analogous to that in Kronecker's theorem, that there be only poles between the two annuli. It would be interesting to have a condition on the sequence $\{a_n - b_n\}$ in order that (10) hold in general, and then to have a means of recovering the z_k and $p_k(n)$. On the other hand, there are situations (see for example Section 6) in which we have formulas for a_n and b_n, so that (10) is explicitly known. Finally, we observe that the first few lines of the above proof essentially form a proof of Wigert's theorem.

6. AN APPLICATION OF THEOREM 3

We shall now illustrate the use of Theorem 3 in a case where there is only one pole between the two annuli of holomorphy. Let $R > 1$, and let

$$h_0(z) = \sum_{n \in N \sim \{0\}} \frac{n z^n}{R^n - 1} \qquad (1 < |z| < R).$$

This series (familiar in the theory of elliptic functions) appears, for example, in the orthogonal expansion of the Bergman kernel of an annulus [2, pp. 2, 9, 10]. We shall show that

$$h_0(z) = p(\log z) + c_0 \qquad (1 < |z| < R),$$

where p is Weierstrass's elliptic function with primitive periods $\log R$ and $2\pi i$, and where

$$c_0 = -\frac{1}{2\pi i} \int_a^{a+2\pi i} p(w)\, dw$$

for any path of integration avoiding the poles of p. More generally, let

$$h_k(z) = \sum_{n \in N \sim \{0\}} \frac{n z^n}{R^{(k+1)n} - R^{kn}} \qquad (R^k < |z| < R^{k+1},\ k \in N).$$

Then it follows immediately from Theorem 3 that all the h_k are function elements of the same holomorphic function, for

$$\frac{n}{R^{kn} - R^{(k-1)n}} - \frac{n}{R^{(k+1)n} - R^{kn}} = nR^{-kn} \qquad (n \in N \sim \{0\}),$$

as required for (10). Hence there exists a function h, holomorphic in the complement of $\{R^k\colon k \in N\} \cup \{0\}$, such that

$$h(z) = h_k(z) \qquad (R^k < |z| < R^{k+1},\ k \in N).$$

Also, by (13),

$$h(z) = R^{2k}(z - R^k)^{-2} + R^k(z - R^k)^{-1} + \cdots \qquad (z \text{ near } R^k).$$

To complete the analysis, let us define $g(w) = h(e^w)$. Then g is holomorphic in the complement of $\{k \log R + 2q\pi i\colon k, q \in N\}$, the indicated lattice points being second-order poles. Further, g has periods $\log R$ and $2\pi i$, for if $R^{k-1} < |e^w| < R^k$, then

$$g(w + \log R) = h_k(Re^w) = h_{k-1}(e^w) = g(w).$$

Thus g is an elliptic function, and from the location of its poles we know that the periods found are primitive. Also g is even:

$$g(-w) = h_k(e^{-w}) = h_{-k-1}(e^w) = g(w).$$

Hence we can write

$$g(w) = c_{-2} w^{-2} + c_0 + c_2 w^2 + \cdots \qquad (w \text{ near } 0).$$

Comparing this with the Laurent expansion of h about 1, we obtain $c_{-2} = 1$. Therefore [6, p. 81] $g(w) = \wp(w) + c_0$. Hence

$$\wp(w) + c_0 = \sum_{n \in N \sim \{0\}} \frac{ne^{nw}}{R^{(k+1)n} - R^{kn}} \qquad (k \log R < \Re w < (k+1)\log R).$$

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