A REMARK ON MAXIMAL SUBRINGS

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A well-known theorem in group theory asserts that a finite group is solvable if it contains a maximal subgroup which is nilpotent and the Sylow 2-subgroup of which is sufficiently restricted (see [2], [5], [1], [3]). A similar "commutativity theorem" (without any finiteness conditions) holds for rings. It is the purpose of this note to prove the following proposition.

THEOREM. *If the maximal subring* M *of the ring* R *is solvable, then* M *is an ideal (containing all the additive commutators* ab - ba *of* R). *The set of all nilpotent elements of* R *is a solvable ideal; it is weakly nilpotent if* M *is weakly nilpotent.*

As in [4], we call an ideal I of the ring R *solvably (nilpotently) embedded* in R if for every homomorphism σ of R such that $I^\sigma \neq 0$ there is an ideal $J \neq 0$ of R^σ contained in I^σ such that $J^2 = 0$ ($R^\sigma J = JR^\sigma = 0$). The ring R is called *solvable (weakly nilpotent)* if it is a solvably (nilpotently) embedded ideal of itself.

Before proving the theorem we shall present our tools in a slightly more general form than is actually necessary. We shall make free use of propositions (S) and (N) of [4].

LEMMA 1. *Each solvable ideal* S *of the ring* R *is solvably embedded in* R.

Proof. By the general properties of the sum S(R) of all solvably embedded ideals of R (see Proposition (S) of [4]), we may assume that S(R) = 0. We shall now assume that the statement of the lemma is false, in other words, that $S \neq 0$, and then exhibit an ideal I of S with $I \neq I^2 = 0$. This contradiction yields the desired result. So let $A \neq 0$ be an ideal of S with $A^2 = 0$. Then clearly $(SA)^2 = 0$, and SA is a left ideal of R. Thus also the two-sided ideal SAR of R satisfies the equation $(SAR)^2 = 0$. Hence, if $SAR \neq 0$ we have arrived at the desired contradiction. If $SA \neq 0$ but SAR = 0, then SA is an ideal of R, and again we have a contradiction. But if SA = 0, then A is a left ideal of R, and hence $(AR)^2 = 0$. Thus either $AR \neq 0$ or A is an ideal of R; both cases yield the desired contradiction.

LEMMA 2. *If* N *is a weakly nilpotent ideal of the ring* R *such that* $R^2 \subseteq N$, *then the ring* R *is weakly nilpotent.*

Proof. By the general properties of the sum N(R) of all nilpotently embedded ideals of R (see Proposition (N) of [4]), we may assume that N(R) = 0, in other words, that no nonzero ideal in R annihilates R from both sides. We shall now assume that the statement of the lemma is false (that is, $R \neq 0$) and then exhibit an ideal of R that annihilates R from both sides. This contradiction yields the result. Since N(R) = 0, we see that $R^2 \neq 0$, hence $N \neq 0$. Let Z be the ideal of N consisting of all the elements of N that annihilate N from both sides; clearly Z is an ideal of R. If Z does not annihilate R from both sides, then $RZ \neq 0$, say, and

$$R(RZ) = (R)^2 Z \subseteq NZ = 0.$$

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The ideal RZR of R annihilates R from either side. Thus if $RZR \neq 0$, we get the desired contradiction. But if $RZR = 0$, then RZ is already a nonzero ideal of R that annihilates R from either side, and we still get the contradiction.

Remark. By iteration of the above lemma, it is obvious that R is weakly nilpotent if only R/N is nilpotent and N is weakly nilpotent. We have not been able to decide whether it suffices to assume that R/N is weakly nilpotent, in order to infer the weak nilpotency of R.

The property $\mathfrak{P}$ of rings is called *ideal* if the fact that the ring R has property $\mathfrak{P}$ implies that every ideal of R has property $\mathfrak{P}$. The property $\mathfrak{P}$ of rings is called *conservative* if the fact that a left (right) ideal I of a ring R has property $\mathfrak{P}$ implies that the ideal IR (RI) also has property $\mathfrak{P}$. The applicability of ideal properties rests on the following trivial fact.

LEMMA 3. *If in the left ideal* L *of the ring* R *there exists a bilateral ideal* I *of* L *having the ideal property* $\mathfrak{P}$, *then* LI *is a left ideal of* R *having property* $\mathfrak{P}$.

We mention some simple examples of properties that are ideal and conservative: nilpotency (of finite class), local nilpotency, the property of satisfying the equation $nr = 0$ for some fixed integer n. It is an open question (known as the *Koethe* problem) whether the property of being a nil ring is conservative.

LEMMA 4. *Both the ring properties of being solvable and of being weakly nilpotent are ideal and conservative.*

Proof. That both properties are ideal is obvious. —Let L be a solvable (weakly nilpotent) left ideal of R, and suppose LR is not solvable (weakly nilpotent). By the properties of solvably (nilpotently) embedded ideals, we may assume that no nonzero ideal I of R is solvably (nilpotently) embedded in LR, in particular, that LR contains no ideal $I \neq 0$ of R such that $I^2 = 0$ ($ILR = LRI = 0$). If we exhibit such an ideal I of R contained in LR, then we have a contradiction to the assumption that LR is not solvable (weakly nilpotent). Let V be the ideal of L consisting of all elements v of L with $vR = 0$.

First we consider the case where L is solvable: Let A be an ideal of L maximal with respect to the condition $A \supset V \supseteq A^2$. If $LA \not\subseteq V$, then the ideal LAR of R is contained in LR and satisfies the relation $(LAR)^2 \subseteq LA^2R \subseteq VR = 0$; this yields the desired contradiction. If $LA \subseteq V$, choose an ideal B of L that is maximal with respect to the condition $B \supset A \supseteq B^2 \not\subseteq V$. Consider now the ideal LBR of R. This ideal is contained in LR, and $(LBR)^2 \subseteq LB^2R \subseteq LAR = 0$. Again we have the desired contradiction, and the lemma is established for solvability.

Now let L be weakly nilpotent. Let $\overline{N}_\nu$ be the general term of the upper annihilator chain of the weakly nilpotent ring L/V, and let N_ν be the preimage of $\overline{N}_\nu$ in L. If N_α is the first member of this chain satisfying the condition $LN_\alpha R \neq 0$, then by definition of the upper annihilator chain, the ordinal α is not a limit ordinal. The ideal LN_α of R is contained in LR and satisfies the conditions

$$(LN_\alpha R)LR \subseteq LN_{\alpha-1}R = 0 \quad \text{and} \quad LR(LN_\alpha R) \subseteq LN_{\alpha-1}R = 0.$$

This contradiction completes the proof of the lemma.

If S is a subring of the ring R, then the *left transporter* of R into S is the set

$$T_s(R; S) = \{x \in R; Rx \subseteq S\}$$

[we denote it by T_s, where no confusion is to be feared]. The intersection

$$S_s(R; S) = T_s \cap S \; [= A_s]$$

is called the *left attraction* of S in R. The *right transporter* T_d and the *right attraction* are defined in the same way. The intersection

$$T(R; S) = T_s \cap T_d \; [= T]$$

is the *transporter* of R into S; the intersection

$$A(R; S) = A_s \cap A_d \; [= A]$$

is the *attraction* of S in R.

Evidently, T_s and A_s are left ideals of R. In the subring $\{T_s, S\}$ of R the left ideal T_s is also a right ideal: for $t \in T_s$ and $s \in S$, $Rts = (Rt)s \subseteq S$, and $ts \in T_s$. Hence, in particular, A_s is an ideal of S. Furthermore,

$$T_d T_s \subseteq S \cap T = A; \qquad T_s T_d \text{ is an ideal;} \qquad T_s^2,\, T_s^2 \subseteq S.$$

Since S is a right ideal in $\{T_s, S\}$ and a left ideal in $\{T_d, S\}$, both T and S are ideals in $\{T, S\}$. In particular, we have established the following proposition.

LEMMA 5. *If a proper subring* S *of the ring* $R \neq 0$ *is not a proper ideal in any larger subring of* R, *then* $T = A \subset S$. *The relation* $S = A$ *holds exactly if* S *is an ideal of* R.

LEMMA 6. *If* $\mathfrak{P}$ *is an ideal and conservative property of rings, and if the ring* R *has no nonzero ideals with the property* $\mathfrak{P}$, *then for every subring* S *of* R *with the property* $\mathfrak{P}$, *the left attraction* $A_s(R; S)$ *is* 0.

Proof. Let the subring S have property $\mathfrak{P}$. The left attraction A_s of S in R, being an ideal of S, also has property $\mathfrak{P}$. Because $\mathfrak{P}$ is conservative, the ideal $A_s R$ of R has property $\mathfrak{P}$; in other words, $A_s R = 0$ by assumption. But then the left ideal A_s is even an ideal in R, and since it has property $\mathfrak{P}$, we conclude that $A_s = 0$.

The *left idealizer* of the subring S of the ring R is the set

$$I_s(S \subseteq R) = \{x \in R;\, xS \subseteq S\};$$

this is the largest subring of R that contains S as a left ideal.

LEMMA 7. *If the subring* S *is a left ideal of every proper subring* T *of* R *containing it, then every ideal* J *of* S *satisfying* $JS \subseteq A_s(R; S)$ *lies in* $A_s(R; I_s(S \subseteq R))$.

Proof. If S is a left ideal of R, then the much stronger relation

$$J \subseteq S = A_s(R; S)$$

holds. If S is not a left ideal, then $I_s(S \subseteq R)$ is a maximal subring of R. If the statement of the lemma were false, then there would exist an element of the form rj ($r \in R$, $j \in J$) that together with $I_s(S \subseteq R)$ generates all of R. Thus every element of R may be expressed as a sum of elements of the form

$$\prod_{\nu=1}^{m} t_\nu (rj)^{\alpha_\nu}, \qquad t_\nu \in I\,(S \subseteq R),\ \alpha_\nu \text{ nonnegative integers.}$$

If such a summand is multiplied by J, then the product is contained in S if rj is the final factor of the summand. If the final factor lies in $I_s(S \subseteq R)$, then either the summand is already contained in $I_s(S \subseteq R)$ or else the final factor is preceded by rj and hence the summand is of the form $\cdots rjt$. But

$$\cdots rjtJ = \cdots rj(tJ) \subseteq \cdots rjS \subseteq \cdots rA_s(R; S) \subseteq S.$$

Thus, no element of the form rj may lie outside $I_s(S \subseteq R)$, and the lemma is proved.

If in this lemma we assume that S is already a maximal subring, and observe that in the proof tJ is contained not only in M but also in J, then the argument that proved Lemma 7 yields (together with a trivial induction step) the following result.

COROLLARY. *If* M *is a maximal subring of the ring* R, *then every nilpotent ideal of* M *is contained in* $A_s(R; M)$.

Evidently, this corollary entails several criteria for the nonsimplicity of rings.

Proof of the theorem. To show that M is an ideal, we shall construct a transfinitely ascending sequence of ideals J_ν of R, contained in M and satisfying the conditions

$$J_{\nu+1} \supseteq J_\nu, \qquad (J_{\nu+1})^2 \subseteq J_\nu; \qquad \text{if } M \neq J_\nu, \text{ then } J_{\nu+1} \neq J_\nu.$$

Since this sequence ascends transfinitely (if necessary), there exists some ordinal μ such that $J_\mu = M$.

Choose $J_0 = 0$. Assume that for all ordinals $\alpha < \beta$ an ideal J_α of R has been chosen in M subject to the above conditions. We shall now choose a suitable J_β.

If β is a limit ordinal, choose $J_\beta = \bigcup_{\alpha<\beta} J_\alpha$.

If β is of the form $\alpha + 1$, consider the factor ring R/J_α. The attraction $A_s(R/J_\alpha; M/J_\alpha)$ is locally nilpotent and contains all the nilpotent ideals of M/J_α, by the corollary. Hence, if $M/J_\alpha \neq 0$, then either $A_s(R/J_\alpha; M/J_\alpha)R/J_\alpha = 0$, in which case $A_s(R/J_\alpha; M/J_\alpha)$ is an ideal—or $A_s(R/J_\alpha; M/J_\alpha)R/J_\alpha$ is an ideal in R/J_α; in any case, $A_s(R/J_\alpha; M/J_\alpha)R/J_\alpha$ is by Lemma 1 solvably embedded in R/J_α, and thus there exists an ideal I in R/J_α with $I \neq I^2 = 0$. If now $I \cap M/J_\alpha \neq 0$, we choose the preimage of this intersection to be $J_{\alpha+1}$. If $I \cap M/J_\alpha = 0$, then every ideal K of M/J_α with $K \neq K^2 = 0$ attracts I from either side into M/J_α by the above corollary, and hence annihilates I from either side. Thus K is an ideal of R/J_α; its preimage may be chosen as $J_{\alpha+1}$.

Thus, for every ordinal ν an ideal J_ν of R in M is defined, and M is an ideal of R.

Evidently, the factor ring R/M has prime order (hence it is commutative, and all the commutators ab - ba of R lie in M). If R/M is a zero-ring, then R is solvable. If R/M is a field, then none of the nonzero elements of R/M can be the image of a nilpotent element of R; hence in this case all the nilpotent elements of R are contained in M.

If M is weakly nilpotent, then there is nothing to prove, provided R/M is a field. But if $R^2 \subseteq M$, then Lemma 2 tells us that R is weakly nilpotent.

Remark. It would be of interest to obtain an analogous theorem for the case where M is locally nilpotent, or even nil. Our method of proof does not seem to work, because we have made essential use of the existence of nilpotent ideals in M.

Note that our theorem and its proof hold also for an arbitrary associative algebra over a commutative field with M a maximal subalgebra.

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