ON AN ANALOGUE OF LITTLEWOOD'S DIOPHANTINE APPROXIMATION PROBLEM

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1. Littlewood asked whether for each pair of real numbers θ and ϕ and each $\varepsilon > 0$ there exists a positive integer n such that

$$n \|n\theta\| \|n\phi\| < \varepsilon,$$

where $\|\alpha\|$ denotes the difference between α and the nearest integer. Davenport and Lewis [1] recently obtained a negative answer for an analogous question concerning formal power series. They proved that there exist pairs of formal power series

$$\Theta(t) = \theta_1 t^{-1} + \theta_2 t^{-2} + \cdots, \qquad \Phi(t) = \phi_1 t^{-1} + \phi_2 t^{-2} + \cdots$$

such that, for all triples of polynomials u(t), v(t), w(t) with $u(t) \neq 0$, the product

$$|u(t)|_K \, |v(t) - \Theta(t)u(t)|_K \, |w(t) - \Phi(t)u(t)|_K$$

is bounded below by a positive constant. Here the valuation of a formal power series relative to the real number field K is defined by

$$|a_m t^m + a_{m-1} t^{m-1} + \cdots|_K = e^m \quad \text{if } a_m \neq 0,$$

where m may be positive, negative or zero.

The construction of Davenport and Lewis was by an inductive process, and it is the purpose of this note to give some explicit examples of series Θ, Φ with the above property. We show that the conclusion of Davenport and Lewis holds if $\Theta = e^{1/t}$ and $\Phi = e^{2/t}$; in fact we prove that *if* u(t), v(t), w(t) *are any polynomials with real coefficients* ($u(t) \neq 0$), *then*

$$(1) \qquad |u(t)|_K \, |v(t) - e^{1/t}u(t)|_K \, |w(t) - e^{2/t}u(t)|_K \geq e^{-5}.$$

We note that in this example the two power series have rational coefficients and converge for all $t \neq 0$. From the nature of the proof it will be apparent that similar results hold for other pairs of series, such as $e^{\lambda/t}$ and $e^{\mu/t}$ or e^{λ/t^2} and e^{μ/t^2}, if λ, μ are distinct nonzero real numbers. In Section 5 we mention a generalization that can be proved by the same method.

2. Let m, n be positive integers. Then there exist real polynomials $P_0(x)$, $Q_0(x)$, $R_0(x)$ of degree at most $h = m + n - 2$, not all identically zero, such that

$$(2) \qquad \begin{cases} Q_0(x) - e^x P_0(x) = b_{m+h} x^{m+h} + b_{m+h+1} x^{m+h+1} + \cdots, \\ R_0(x) - e^{2x} P_0(x) = c_{n+h} x^{n+h} + c_{n+h+1} x^{n+h+1} + \cdots. \end{cases}$$

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For this merely requires that the $3(h+1)$ coefficients of the polynomials should satisfy a certain system of $(m+h)+(n+h) = 3h+2$ homogeneous linear equations, and the system indeed has a nontrivial solution. Clearly none of $P_0(x)$, $Q_0(x)$, $R_0(x)$ can vanish identically, for then (2) would imply the vanishing of the others.

We define further polynomials $P_i(x)$, $Q_i(x)$, $R_i(x)$, for $i = 1, 2$, by

$$(3) \qquad P_{i+1}(x) = 2P_i(x) + P_i'(x), \qquad Q_{i+1}(x) = Q_i(x) + Q_i'(x), \qquad R_{i+1}(x) = R_i'(x),$$

where the accent denotes the derivative with respect to x. Define also

$$\xi_i(x) = Q_i(x) - e^x P_i(x), \qquad \eta_i(x) = R_i(x) - e^{2x} P_i(x) \qquad (i = 0, 1, 2).$$

Then it is easily verified that

$$\xi_{i+1}(x) = \xi_i(x) + \xi_i'(x), \qquad \eta_{i+1}(x) = \eta_i'(x) \qquad (i = 1, 2).$$

From (2) it follows that for $i = 0, 1, 2$ the lowest possible powers of x in $\xi_i(x)$ and in $\eta_i(x)$ are x^{m+h-i} and x^{n+h-i}, respectively. Thus

$$(4) \qquad |\xi_i(t^{-1})|_K \le e^{-m-h+i}, \qquad |\eta_i(t^{-1})|_K \le e^{-n-h+i}.$$

Finally we observe that the determinant

$$\Delta(x) = \begin{vmatrix} P_0(x) & Q_0(x) & R_0(x) \\ P_1(x) & Q_1(x) & R_1(x) \\ P_2(x) & Q_2(x) & R_2(x) \end{vmatrix}$$

is not identically zero. For (3) implies that it is a polynomial with highest coefficient $-2pqr$, where p, q, r are the highest nonzero coefficients in $P_0(x)$, $Q_0(x)$, $R_0(x)$, respectively.

3. Now let $u(t)$ be a polynomial with real coefficients, of degree $k \ge 0$, and let $v(t)$, $w(t)$ be any polynomials with real coefficients. Let

$$|v(t) - e^{1/t}u(t)|_K = e^{-a}, \qquad |w(t) - e^{2/t}u(t)|_K = e^{-b}.$$

To prove (1) we have to show that

$$(5) \qquad k - a - b \ge -5.$$

Suppose first that

$$(6) \qquad a \ge 2, \quad b \ge 2.$$

We use the construction of Section 2 with $m = a - 1$, $n = b - 1$. Since the determinant $\Delta(x)$ is not identically zero, it has two rows, say the rows i_1 and i_2, such that the determinant

$$E(t) = \begin{vmatrix} P_{i_1}(t^{-1}) & Q_{i_1}(t^{-1}) & R_{i_1}(t^{-1}) \\ P_{i_2}(t^{-1}) & Q_{i_2}(t^{-1}) & R_{i_2}(t^{-1}) \\ u(t) & v(t) & w(t) \end{vmatrix}$$

does not vanish identically. We shall compare two inequalities for $|E(t)|_K$.

First, since $t^{2h}E(t)$ is a polynomial in t, not identically zero, we see that

$$|E(t)|_K \geq e^{-2h}. \tag{7}$$

On the other hand, by a linear combination of columns,

$$E(t) = \begin{vmatrix} P_{i_1}(t^{-1}) & \xi_{i_1}(t^{-1}) & \eta_{i_1}(t^{-1}) \\ P_{i_2}(t^{-1}) & \xi_{i_2}(t^{-1}) & \eta_{i_2}(t^{-1}) \\ u(t) & v(t) - e^{1/t}u(t) & w(t) - e^{2/t}u(t) \end{vmatrix}.$$

Using (4) and the fact that $|P_i(t^{-1})|_K \leq 1$, we obtain the inequality

$$|E(t)|_K \leq e^M,$$

where

$$M = \max(-m - n - 2h + 3 + k, \ -n - h + 2 - a, \ -m - h + 2 - b)$$

$$= \max(-2h + k - a - b + 5, \ -2h - 1),$$

by virtue of the relations $m = a - 1$, $n = b - 1$, $h = m + n - 2$. Comparison with (7) gives the desired result (5).

4. There remains the possibility that (6) is not satisfied. It will suffice to consider the case $b \leq 1$, since the case $a \leq 1$ is similar. Then we need merely prove that $k - a \geq -4$, in other words, that

$$|u(t)|_K \, |v(t) - e^{1/t}u(t)|_K \geq e^{-4}.$$

For this a simpler form of the preceding argument is used. We require only the polynomials $P_0(x)$, $Q_0(x)$, $P_1(x)$, $Q_1(x)$, and instead of $E(t)$ we consider the determinant

$$E_1(t) = \begin{vmatrix} P_i(t^{-1}) & Q_i(t^{-1}) \\ u(t) & v(t) \end{vmatrix},$$

where $i = 0$ or 1.

5. The extension to a product of more than three factors presents no difficulty, and we obtain the following result.

THEOREM. *If* $\lambda_1, \cdots, \lambda_r$ *are distinct real numbers, none of them* 0, *and* $u(t)$, $u_1(t)$, $\cdots$, $u_r(t)$ *are real polynomials with* $u(t) \neq 0$, *then*

$$|u(t)|_K \prod_{j=1}^{r} |u_j(t) - e^{\lambda_j/t} u(t)|_K \geq e^{-R},$$

where $R = \frac{1}{2}(r^3 + r)$.

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REFERENCE

1. H. Davenport and D. J. Lewis, *An analogue of a problem of Littlewood,* Michigan Math. J. 10 (1963), 157-160.

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