NONLINEAR PERTURBATION OF A LINEAR SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

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Given two systems of ordinary differential equations,

$$\dot{x} = A(t)x + f(t,\ x), \tag{1}$$

$$\left(\cdot = \frac{d}{dt}\right)$$

$$\dot{x} = A(t)y \tag{2}$$

and a fundamental matrix $Y(t)$ for (2), we pose the following problems:

(i) If $x(t)$ is a solution of (1), does there exist a constant $n \times 1$ matrix b such that

$$x(t) = Y(t)[b + o(1)] \text{ as } t \to \infty\,? \tag{3}$$

(ii) If b is a constant $n \times 1$ matrix, does there exist a solution $x(t)$ of (1) such that (3) holds?

In Theorem 1 we generalize the results of Z. Szmydt [11, Theorems 1 and 2], and we give a positive answer to Problem (i). Theorem 1 is also a generalization of a result of R. Bellman [2], who studied the case in which $f(t,\ x)$ is linear.

A positive answer to Problem (ii) is given in Theorem 2. This theorem depends on the Lemma stated below, which is a very special case of one of the author's earlier results [8, Theorem 1].

A special case of Theorems 1 and 2 is considered in the Corollary following Theorem 2.

In Theorem 3 we give a generalization of a result of W. Trench [12]. See also [1, Theorem 2], [5] and [9]. Our Theorem 3 is a positive answer to Problem (ii) for the case in which second-order systems are considered. Trench deals with second-order scalar equations under linear perturbations. We deal with second-order systems under perturbations not necessarily linear. The proof of Theorem 3 depends on the Corollary mentioned above.

Results related to problems (i) and (ii) may be found in [3], [6], [7], and [10]. Other references can be found in the book by L. Cesari [4].

We denote by $\|z\| = \Sigma_j\,|z_j|$ the norm of any $n \times 1$ matrix $z = \mathrm{col}(z_1,\ \cdots,\ z_n)$ and by $\|Z\| = \Sigma_{i,j}\,|Z^i_j|$ the norm of any $n \times n$ matrix $Z = (Z^i_j)$. Our results are dependent upon the following hypothesis.

HYPOTHESIS H. *For every positive constant* M *there exists a nonnegative function* $h_M(t)$ *such that if* Y *is a fundamental matrix for* (2), *then*

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$$\int_0^\infty h_M(t)\,dt < \infty \quad \textit{and} \quad \|Y^{-1}(t)\,f(t,\ Y(t)x)\| \le h_M(t)$$

for all (t, x) *with* $t \ge 0$, $\|x\| \le M$.

In the sequel it is always supposed that $f(t, x)$ is an $n \times 1$ continuous matrix for $t \ge 0$, that $\|x\| < \infty$, and that $A(t)$ is an $n \times n$ continuous matrix for $t \ge 0$.

LEMMA. *If Hypothesis* H *holds, with* $A(t) = 0$ *in* (2), *and* b *is a constant* $n \times 1$ *matrix, then there exists a solution* $x(t)$ *of* $\dot{x} = f(t, x)$ *for which*

$$x(t) = b + o(1) \quad \textit{as } t \to \infty .$$

THEOREM 1. *Suppose that Hypothesis* H *is satisfied. If* $\phi(t)$ *is a solution of* (1) *such that* $Y^{-1}(t)\phi(t)$ *is bounded as* $t \to \infty$, *then there exists a constant* $n \times 1$ *matrix* b *for which*

$$\phi(t) = Y(t)[b + o(1)] \quad \textit{as } t \to \infty .$$

Proof. If $z(t) = Y^{-1}(t)\phi(t)$, then $z(t)$ satisfies the equation

$$\dot{z}(t) = Y^{-1}(t)\,f(t,\ Y(t)z(t)),$$

and there exists a positive constant M such that $\|z(t)\| \le M$ for all $t \ge 0$. Thus

$$z(t) = z(t_0) + \int_{t_0}^{t} Y^{-1}(s)\,f(s,\ Y(s)z(s))\,ds,$$

and $\|Y^{-1}(t)\,f(t,\ Y(t)z(t))\| \le h_M(t)$ for all $t \ge t_0$. It follows that

$$\int_{t_0}^{\infty} Y^{-1}(s)\,f(s,\ Y(s)z(s))\,ds$$

is finite. Consequently,

$$\begin{aligned} z(t) &= z(t_0) + \int_{t_0}^{\infty} Y^{-1}(s)\,f(s,\ Y(s)z(s))\,ds + \int_{\infty}^{t} Y^{-1}(s)\,f(s,\ Y(s)z(s))\,ds \\ &= b + \int_{\infty}^{t} Y^{-1}(s)\,f(s,\ Y(s)z(s))\,ds, \end{aligned}$$

and therefore $z(t) = b + o(1)$ as $t \to \infty$. This implies that

$$\phi(t) = Y(t)\,z(t) = Y(t)[b + o(1)] \quad \text{as } t \to \infty .$$

The proof of the theorem is complete.

THEOREM 2. *If Hypothesis* H *is satisfied and* b *is a constant* $n \times 1$ *matrix, then there exists a solution* $\phi(t)$ *of* (1) *such that*

$$\phi(t) = Y(t)[b + o(1)] \quad \textit{as} \quad t \to \infty.$$

Proof. If we make the transformation $x = Y(t)z$ in the system (1), then

$$\dot{z} = Y^{-1}(t) f(t, Y(t)z). \tag{4}$$

It follows from the Lemma, applied to the system (4), that there exists a solution $\psi(t)$ of (4) such that $\psi(t) = b + o(1)$ as $t \to \infty$. Therefore $\phi(t) = Y(t)\psi(t)$ is a solution of (1), and $\phi(t) = Y(t)[b + o(1)]$ as $t \to \infty$.

The proof of the theorem is complete.

COROLLARY. *Suppose that there exists a nonnegative function* $h(t)$ *such that* $\int_0^\infty h(t)\,dt < \infty$ *and*

$$\|Y^{-1}(t) f(t, Y(t)x)\| \le h(t)\|x\|$$

for all (t, x) *with* $t \ge 0$, $\|x\| < \infty$.

If $\phi(t)$ *is a solution of* (1), *then there exists a constant* $n \times 1$ *matrix* b *for which* $\phi(t) = Y(t)[b + o(1)]$ *as* $t \to \infty$. *Conversely, if a constant* $n \times 1$ *matrix* b *is given, then there exists a solution* $\phi(t)$ *of* (1) *such that* $\phi(t) = Y(t)[b + o(1)]$ *as* $t \to \infty$.

Proof. By using Gronwall's Lemma [4; 3.2.i, p. 35] we can easily show that, for every solution $\phi(t)$ of (1), $Y^{-1}(t)\phi(t)$ is bounded as $t \to \infty$. Thus the corollary is an immediate consequence of Theorems 1 and 2.

Finally, we consider the second-order systems

$$\ddot{x} = A(t)x + f(t, x), \tag{5}$$

$$\ddot{y} = A(t)y, \tag{6}$$

where $A(t)$ is a diagonal $n \times n$ matrix, that is, $A(t) = \operatorname{diag}(a_1(t), \cdots, a_n(t))$.

Let ϕ_j^1, ϕ_j^2 be linearly independent solutions of the second-order scalar equation

$$\ddot{y}_j = a_j(t)\, y_j \qquad (j = 1, \cdots, n).$$

Let

$$\lambda(t) = \max_{j=1,\cdots,n} \{|\phi_j^1(t)|^2, |\phi_j^2(t)|^2\},$$

and suppose $\|f(t, x)\| \le h(t)\|x\|$ for all (t, x) with $t \ge 0$, $\|x\| < \infty$. Concerning the systems (5) and (6), the following theorem holds.

THEOREM 3. *Suppose that* $\int_0^\infty h(t)\lambda(t)\,dt < \infty$. *If* $x(t)$ *is a solution of* (5), *then there exist constants* b_j^1, b_j^2 $(j = 1, \cdots, n)$ *such that*

$$\begin{aligned} x_j(t) &= \phi_j^1(t)\,[b_j^1 + o(1)] + \phi_j^2(t)\,[b_j^2 + o(1)], \\ \dot{x}_j(t) &= \dot{\phi}_j^1(t)\,[b^1 + o(1)] + \dot{\phi}_j^2(t)\,[b_j^2 + o(1)] \end{aligned} \tag{7}$$

as $t \to \infty$ $(j = 1, \cdots, n)$.

Conversely, if the constants b_j^1, b_j^2 $(j = 1, \cdots, n)$ *are given, there exists a solution* $x(t)$ *of* (5) *satisfying* (7).

Proof. Without loss of generality we may assume that

$$\phi_j^1(0) = \dot{\phi}_j^2(0) = 1, \quad \dot{\phi}_j^1(0) = \phi_j^2(0) = 0 \qquad (j = 1, \cdots, n).$$

For $j = 1, \cdots, n$, let us define the $2n \times 1$ matrices $z = \operatorname{col}(z_1, \cdots, z_{2n})$ and $u = \operatorname{col}(u_1, \cdots, u_{2n})$ by the relations

$$z_{2j-1} = x_j, \qquad z_{2j} = \dot{x}_j,$$

$$u_{2j-1} = y_j, \qquad u_{2j} = \dot{y}_j.$$

Associated with the systems (5) and (6) are the systems

$$\dot{z} = B(t)z + g(t, z), \tag{5'}$$

$$\dot{u} = B(t)u, \tag{6'}$$

where

$$B(t) = \operatorname{diag}(B_1(t), \cdots, B_n(t)), \qquad B_j(t) = \begin{pmatrix} 0 & 1 \\ a_j(t) & 0 \end{pmatrix},$$

$$g(t, z) = \operatorname{col}(0, f_1(t, x), \cdots, 0, f_n(t, x)).$$

We shall show that we can apply the above corollary to the systems (5') and (6'). Then our theorem will be an immediate consequence of this fact. The matrix $U(t) = \operatorname{diag}(U_1(t), \cdots, U_n(t))$, where

$$U_j(t) = \begin{pmatrix} \phi_j^1(t) & \phi_j^2(t) \\ \dot{\phi}_j^1(t) & \dot{\phi}_j^2(t) \end{pmatrix},$$

is a fundamental matrix of (6'). Clearly,

$$U_j^{-1}(t) \begin{pmatrix} g_{2j-1}(t, U(t)z) \\ g_{2j}(t, U(t)z) \end{pmatrix} = \begin{pmatrix} \dot{\phi}_j^2(t) & -\phi_j^2(t) \\ -\dot{\phi}_j^1(t) & \phi_j^1(t) \end{pmatrix} \begin{pmatrix} 0 \\ g_{2j}(t, U(t)z) \end{pmatrix}$$

and

$$g_{2j}(t, U(t)z) = f_j(t, \phi_1^1(t) z_1 + \phi_1^2(t) z_2, \cdots, \phi_n^1(t) z_{2n-1} + \phi_n^2(t) z_{2n}).$$

An easy computation shows that

$$\|U^{-1}(t)\,g(t,\ U(t)z)\|$$
$$\leq \sum_{i,j} [\,|\phi_i^1(t)| + |\phi_i^2(t)|\,]\,[\,|\phi_j^1(t)| + |\phi_j^2(t)|\,]\,h(t)\,\|z\|$$
$$\leq \sum_j 2n[\,|\phi_j^1(t)|^2 + |\phi_j^2(t)|^2\,]\,h(t)\,\|z\| \leq 4n^2 h(t)\lambda(t)\,\|z\|.$$

We can now apply the Corollary to the systems (5') and (6'), and our theorem is proved.

The proof of Theorem 3 shows that there are some obvious extensions of the theorem. For example, the conclusions of Theorem 3 hold if we deal with the system

$$\ddot{x} = A(t)x + f(t,\ x,\ \dot{x})$$

with $\|f(t,\ x,\ \dot{x})\| \leq h(t)(\|x\| + \|\dot{x}\|)$ and $\int_0^\infty h(t)\lambda(t)\,dt < \infty$, where

$$\lambda(t) = \max_{j=1,\cdots,n} \{\,|\phi_j^1|^2,\ |\phi_j^2|^2,\ |\dot{\phi}_j^1|^2,\ |\dot{\phi}_j^2|^2\}\ .$$

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