THE SHAPE OF LEVEL SURFACES OF HARMONIC FUNCTIONS IN THREE DIMENSIONS

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1. INTRODUCTION

Consider the Green's function g(P) of a region D in E_3, with pole at the origin O. If D is star-shaped relative to O, then the regions $D_k = \{P: g(P) > k\}$ are star-shaped relative to O (Gergen, [4]); and if D is convex, then the regions D_k are also convex (Gabriel, [2]).

We now obtain corresponding results for harmonic functions where the pole at the origin is replaced by a continuum (star-shaped relative to the origin or convex, in the respective cases) on which the functions are constant.

HYPOTHESIS H. *Let* C_1 *and* C_0 *be two closed subsets of* E_3 (C_1 *not empty*), *and let* $\phi(P)$ *denote a real-valued function on* E_3, *subject to the following conditions.*

(i) $\phi(P)$ *is continuous on* E_3,

(ii) $\phi(P) = 1$ *on* C_1,

(iii) $\phi(P) = 0$ *on* C_0,

(iv) $\phi(P) \to 0$ *as* $P \to \infty$,

(v) $\phi(P)$ *is harmonic on* $D = (C_0 \cup C_1)' = E_3 - (C_0 \cup C_1)$.

Since the set C_0 may be empty, the situation just described includes the case where $\phi(P) = 1$ on a closed, nonempty set C_1, $\phi(P) \to 0$ as $P \to \infty$, and $\phi(P)$ is harmonic on $C_1' = E_3 - C_1$ (see [2, pp. 397, 401]). We assume the existence of a function satisfying the stated conditions; some conditions on C_1 and C_0 sufficient for the existence are given, for example, in [1, pp. 290-312].

Note that C_1 and C_0 are disjoint because of conditions (ii) and (iii), and that C_1 is bounded because of conditions (ii) and (iv). In addition, by an application of the principle of the maximum in the strong form, we can deduce from our conditions that $0 \leq \phi(P) \leq 1$ on E_3.

We shall denote the Euclidean distance of a point P from the origin by $|P|$, the Euclidean distance between points P and Q by $|P - Q|$, and the distance of a point P from a set C by d(P, C).

2. STAR-SHAPED REGIONS

By definition, a set C is star-shaped relative to the origin O if λP is in C whenever P is in C and $0 \leq \lambda \leq 1$.

THEOREM 1. *Let* C_1, C_0, *and* ϕ *satisfy Hypothesis* H, *and let* C_1 *and* $C_0' = E_3 - C_0$ *be star-shaped relative to* O; *then the regions* $D_k = \{P: \phi(P) > k\}$ *are star-shaped relative to* O.

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LEMMA 1. *Under the hypotheses of the theorem,* D *is connected.*

Proof. Let δ be the distance between C_0 and C_1 (for C_0 empty, let δ be any positive number). Since C_0 is closed and C_1 is compact, δ is positive. Take a point R in C_1 at maximum distance from O, and any real number Δ greater than $|R|$. On each plane through OR, start from OR to divide the disk $\{P: |P| \leq \Delta\}$ into closed acute sectors A_i determined by circular arcs of length less than δ. Let R_i be a point on the compact set $A_i \cap C_1$ at maximum distance from O. Since no point of C_0 lies at distance less than δ from R_i, there exists an arc L_i across A_i not meeting $C_0 \cup C_1$. Since C_1 and C_0' are star-shaped, the arcs L_i can be joined by radial segments to form a curve K not meeting $C_0 \cup C_1$. For the same reason, every point of D can be joined by a radial segment to some K, and the curves K can be joined by a segment on the extended segment OR. Hence D is arc-wise connected.

LEMMA 2. *Under the hypotheses of Theorem* 1, $0 < \phi(P) < 1$ *on* D.

Proof. Since D is connected, the strong form of the principle of the maximum gives both inequalities.

LEMMA 3. *Under the hypotheses of Theorem* 1, ϕ *is nonincreasing on each radius.*

Proof. Suppose Lemma 3 is false. Then there exist two points P_0 and $\lambda_0 P_0$ $(0 < \lambda_0 < 1)$ in D with $\phi(\lambda_0 P_0) < \phi(P_0)$, and the function $\psi(P) = \phi(P) - \phi(\lambda_0 P)$ has a positive least upper bound m on E_3. By condition (iv) in Hypothesis H, $|\phi(P)| < m/2$ when $|P|$ is greater than some positive δ. Hence $\psi(P) < m/2$ also for $|P| > \delta$. Hence m is the least upper bound of ψ on the compact set $\{P: |P| \leq \delta\}$, and is attained there. But m is not attained when P is in C_0, since $\psi \leq 0$ in C_0. Nor is m attained if P is in C_1, since C_1 is star-shaped so that $\psi = 0$ in C_1. Also, if $\lambda_0 P$ is in C_0, then $\psi(P) = 0$ since C_0' is star-shaped; thus m is not attained in that case. Finally, m cannot be attained at P if $\lambda_0 P$ is in C_1, since then $\psi(P) \leq 0$. Hence m is attained at some point P_1 such that both P_1 and $\lambda_0 P_1$ are in D.

Let d be the lesser of $d(P_1, C_0)$ and $d(\lambda_0 P_1, C_1)/\lambda_0$; the second is certainly finite. Then the set $N = \{P: |P - P_1| < d\}$ is contained in C_0'. Also

$$\lambda_0 N = \{\lambda_0 P: P \text{ in } N\}$$

is contained in C_0' since C_0' is star-shaped. But $\lambda_0 N = \{Q: |Q - \lambda_0 P_1|/\lambda_0 < d\}$, and hence is contained in C_1'. Therefore N is contained in C_1', since C_1 is star-shaped. Thus N and $\lambda_0 N$ are contained in D, and therefore $\psi(P)$ is harmonic in N. By the principle of the maximum, $\psi(P) = m$ on N. Now either (a) $|P_1 - R| = d$ for some R in C_0, or (b) $|P_1 - R| = d$ for some $\lambda_0 R$ in C_1. In case (a), $\psi(R) = 0 - \phi(\lambda_0 R) \leq 0$, while in case (b), $\psi(R) = \phi(R) - 1 \leq 0$. However, $\psi(P) = m$ for some points in any neighbourhood of R. This contradicts continuity.

Theorem 1 follows immediately from Lemma 3.

COROLLARY. *Under the hypotheses of Theorem* 1, *the radial derivative* $\partial\phi/\partial r$ *is strictly negative in* D. *Thus* grad $\phi \neq 0$ *throughout* D.

Proof. The function $r\,\partial\phi/\partial r$ is harmonic and nonpositive in D. Thus if $r\,\partial\phi/\partial r$ were zero at some point of D, $r\,\partial\phi/\partial r$ would be zero throughout D, so that ϕ would be radially constant in D. Since each radius meets the set C_1, it would follow that $\phi(P) = 1$ throughout D, contrary to Lemma 2.

3. CONVEX REGIONS

THEOREM 2. *Let* C_1, C_0, *and* ϕ *satisfy Hypothesis* H, *and let* C_1 *and* C_0' *be convex. Then the sets* $D_k = \{P: \phi(P) > k\}$ *are convex.*

LEMMA 4. *If the hypotheses of Theorem* 2 *are satisfied, and if* P *and* Q *are two points in* D *such that* $\phi(P) = \phi(Q)$, *then* $\phi(R) > \phi(P)$ *for every point* R *on the open segment* PQ.

Proof. For all point pairs P, Q with $\phi(P) = \phi(Q)$ and for all points R on the corresponding closed segment PQ, define

$$\theta(P, Q, R) = \phi(P) + \phi(Q) - 2\phi(R).$$

The function $\theta(P, Q, R)$ is continuous and bounded on its domain of definition, and its least upper bound m is nonnegative.

If $m = 0$, then $\phi(R) \geq \phi(P) = \phi(Q)$ for all P, Q, R in the domain of θ. If we assume that the lemma is false, then there exist some P_0, Q_0 in D, and an R_0 in the open segment $P_0 Q_0$, with $\phi(R_0) \leq \phi(P_0) = \phi(Q_0)$. Thus if $m = 0$ and the lemma is false, then $\phi(R_0) = \phi(P_0) = \phi(Q_0)$. Hence $\theta = 0$ at P_0, Q_0, R_0, and $m = 0$ is attained. Since P_0 and Q_0 are in D, it follows from Lemma 2 that

$$0 < \phi(P_0) = \phi(Q_0) < 1,$$

hence $0 < \phi(R_0) < 1$. Thus R_0 is in D, and P_0, Q_0, R_0 are in D.

If $m > 0$, condition (iv) in Hypothesis H implies the existence of a $\delta > 0$ such that $\theta(P, Q, R) < m/2$ whenever $|P| > \delta$ or $|Q| > \delta$, and therefore m is the maximum value of θ on the compact set

$$\{(P, Q, R): |P| \leq \delta, \ |Q| \leq \delta, \ \phi(P) = \phi(Q), \ R \in PQ\}.$$

Now $\theta(P, Q, R) = 0$ whenever two of the points P, Q, and R coincide. Also, $\theta(P, Q, R) \leq 0$ whenever P or Q lies in C_0; and $\theta(P, Q, R) = 0$ when P or Q lies in C_1, since C_1 is convex. If R lies in C_0, then (by the convexity of C_0') either P or Q lies in C_0, hence $\phi(P) = \phi(Q) = \phi(R) = 0$, and again $\theta(P, Q, R) = 0$. If R lies in C_1, then $\theta(P, Q, R) \leq 0$ because $\phi(R) = 1$. Thus, for $m > 0$, θ takes its maximum at some P, Q, R distinct and in D.

It follows that in both cases (either $m = 0$ and the lemma assumed to be false, or $m > 0$) we could conclude that θ takes its maximum at some P, Q, R with P, Q, and R distinct and in D, R in PQ, and $\phi(P) = \phi(Q)$. On the other hand, by the Corollary in Section 2, grad $\phi \neq 0$ everywhere in D. By a theorem of R. M. Gabriel [2, p. 389], ϕ is radially constant in D with respect to some center O*.

For any point S in D, consider a ray J from S on which ϕ is constant on each segment SP from S on J lying in D. If J is completely contained in D, then ϕ is constant on J, and, by condition (iv), $\phi = 0$ on J and $\phi(S) = 0$. If J is not contained in D, then the minimum of $|S - P|$ for P in $J \cap (C_0 \cup C_1)$ is attained either in C_0, in which case $\phi(S) = 0$, or in C_1, in which case $\phi(S) = 1$. Hence, in all cases, the result contradicts Lemma 2. This proves that $m = 0$ and the lemma is true.

Proof of Theorem 2. If $\phi(P) \geq \phi(Q) > k$ and $\phi(R) \leq k$ for some R in PQ, then there exists a point P' in PR with $\phi(P') = \phi(Q)$. This situation is impossible, by Lemma 4.

4. A COUNTEREXAMPLE

In relation to the results in Section 2, it is appropriate to consider an example suggested by W. J. Wong, which shows that if C_1 and C_0' are merely assumed to be simply connected, then the regions D_k need not be simply connected, and grad ϕ can be zero in D. We shall require bounds for the change in ϕ with change in C_1. Our technique is an adaption of a method used by Gergen in [3].

Suppose C_1^- is C_1 with a piece removed, with corresponding ϕ^-. Then $\phi(P) - \phi^-(P)$ is harmonic in D, continuous in E_3, 0 on C_0, and nonnegative on C_1. Hence $\phi(P) - \phi^-(P)$ is nonnegative on D. Let A be the piece of the boundary D^* of D removed in forming C_1^-, and let $g(Q; P, D)$ be the Green's function of D with pole P. If D^* is sufficiently smooth (see [5, p. 237]), then, for P in D,

$$\phi(P) - \phi^-(P) = \int_{D^*} (\phi(Q) - \phi^-(Q)) \frac{\partial g(Q; P, D)}{-4\pi\, \partial n}\, d\sigma \leq \int_A \frac{\partial g(Q; P, D)}{-4\pi\, \partial n}\, d\sigma.$$

Let K be any compact set in D. Again provided that D^* is sufficiently smooth (see [6, p. 259]), $\frac{\partial g(Q; P, D)}{-4\pi\, \partial n}$ has a finite upper bound M_K for P in K and Q in C_1^*. Hence $\phi(P) - \phi^-(P) \leq M_K a(A)$, where $a(A)$ is the area of A.

Now apply this result to the following system. Let the set C_0' be an open sphere with center X, and the set C_1 a solid torus inside C_0', with the same center of symmetry X. We form C_1^- from C_1 by removing a section bounded by two half-planes having the major axis of C_1 as common edge. Then C_1^- is a simply connected continuum. It has only one axis of symmetry, which cuts the inner surface of C_1 at Y and Z, say, the latter being removed in the forming of C_1^-. Since $\phi(X) < 1$, we can take k between $\phi(X)$ and 1.

First, take $K = \{P, P'\}$, where P is in XY and P' is in XZ with $k < \phi(P) = \phi(P') < 1$. By forming C_1^- appropriately, make $M_K a(A) < \phi(P) - k$. This gives $\phi^-(P) > k$ and $\phi^-(P') > k$, while $\phi^-(X) < k$. Hence the component of grad ϕ^- along YZ is zero somewhere in PP'. With symmetry, this shows that grad $\phi^- = 0$ there.

Second, take $K = \{P: \phi(P) = k\}$. For suitably formed C_1^-, $M_K a(A) < k - \phi(X)$. Hence $\phi^-(P) > \phi(X)$ on K. On the major axis of C_1, $\phi^-(P) \leq \phi(P) \leq \phi(X)$. This shows that $\{P: \phi^-(P) > \phi(X)\}$ is not simply connected.

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