ON REPRESENTABLE RELATION ALGEBRAS

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In this note we show that the class of representable relation algebras is not finitely axiomatizable; thus we answer a question raised by Tarski [10] in 1954. The proposition is an easy consequence of the following three important known results: (1) the correspondence between projective geometries and certain relation algebras established by Lyndon [7]; (2) the theorem of Bruck and Ryser [1] on the nonexistence of projective planes of certain finite orders; and (3) the fundamental theorem about ultraproducts (Theorem 5.1 of [4]). The author is grateful to the referee for making some helpful suggestions while this note was being prepared for publication.

A *relation algebra* is a universal algebra of the type $\mathfrak{A} = \langle A, +, \cdot, -, ;, {}^{\smile}, 1' \rangle$ that satisfies certain axioms, due to Tarski (see Chin and Tarski [2]); $\langle A, +, \cdot, - \rangle$ is characterized as a Boolean algebra satisfying some seven additional equational postulates concerned with $;$, ${}^{\smile}$, and $1'$. A relation algebra is *representable* if it is isomorphic to a relation algebra of the form $\langle A, \cup, \cap, \sim, |, {}^{-1}, I \rangle$ where, for some set D, $\langle A, \cup, \cap, \sim \rangle$ is a Boolean algebra of subsets of $D \times D$ (with unit set not necessarily equal to $D \times D$), where I is the identity relation on D, and where, for any R, S contained in $D \times D$, $R | S$ is the relative product of R and S and R^{-1} is the converse of R. It was hoped that every relation algebra is representable. However, Lyndon in [5] showed that this is not the case. In [10] Tarski showed that, at any rate, the class of representable relation algebras can be characterized by a set of equations.

A relation algebra $\mathfrak{A}$ is *integral* if $x = 0$ or $y = 0$ for all $x, y \in A$ for which $x ; y = 0$. An integral relation algebra is always simple (in the sense of universal algebra). A relation algebra is *representable over a group* if it is isomorphic to a relation algebra $\langle A, \cup, \cap, \sim, \cdot, {}^{-1}, \{e\} \rangle$ where, for some group G with neutral element e, $\langle A, \cup, \cap, \sim \rangle$ is a Boolean algebra of subsets of G, with $\{e\} \in A$, and for any R, S contained in G, $R \cdot S$ is the complex product of R and S, and $R^{-1} = \{x^{-1} : x \in R\}$. A relation algebra representable over a group is an integral representable relation algebra (the question whether the converse holds is one of the main outstanding problems in the theory of relation algebras).

Lyndon considered a special class of relation algebras, characterized by the following conditions:

(i) $\langle A, +, \cdot, - \rangle$ *is a Boolean algebra,*

(ii) $x ; (y ; z) = (x ; y) ; z$,

(iii) $x ; y = y ; x$,

(iv) $(x + y) ; z = x ; z + y ; z$,

(v) $x ; 1' = x$,

(vi) $x^{\smile} = x$,

(vii) $1' \leq x ; y$ *if and only if* $x \cdot y \neq 0$,

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for all $x, y, z \in A$. We shall call algebras $\langle A, +, \cdot, -, ;, ^\smile, 1' \rangle$ satisfying these conditions *Lyndon algebras*. It is easily seen that every Lyndon algebra is an integral relation algebra in which $1'$ is an atom.

The main result of this note is the following theorem.

THEOREM 1. *There exists an ultraproduct* $\mathfrak{A}$ *of Lyndon algebras* $\mathfrak{A}_m$ *such that* $\mathfrak{A}$ *is representable over a group, while no* $\mathfrak{A}_m$ *is representable.*

Using the fundamental result that all elementary sentences are preserved under the formation of ultraproducts, we conclude from Theorem 1 the following result.

THEOREM 2. *If* $\mathcal{K}$ *is an elementary class containing all Lyndon algebras, then neither the class of all representable relation algebras in* $\mathcal{K}$ *nor the class of all algebras in* $\mathcal{K}$ *representable over a group is finitely axiomatizable. In particular, the class of all representable relation algebras and likewise the class of all relation algebras representable over a group is not finitely axiomatizable.*

Now, to begin the proof of Theorem 1, we first make some remarks about Boolean algebras. For each nonempty set G, let $\mathfrak{A}(G)$ be a complete atomic Boolean algebra with G as its set of atoms.

LEMMA 1. *If* G_t *is a nonempty set for each* $t \in T$ *and* D *is an ultrafilter over* T, *then* $\Pi_{t \in T}\, \mathfrak{A}(G_t)/D$ *can be isomorphically embedded in* $\mathfrak{A}(\Pi_{t \in T}\, G_t/D)$.

Proof. That a Boolean algebra is atomic can be expressed by an elementary sentence. Hence by Theorem 5.1 of [4], $\Pi_{t \in T}\, \mathfrak{A}(G_t)/D$ is atomic. If x is an atom of $\Pi_{t \in T}\, \mathfrak{A}(G_t)/D$, then, again by Theorem 5.1 of [4], we can write $x = f/D$, where $f \in \Pi_{t \in T}\, A(G_t)$ and $\{t \in T\colon f_t \in G_t\} \in D$; thus for some $g \in \Pi_{t \in T}\, G_t$, we can write $x = g/D$. Hence there is a one-to-one correspondence between the set of atoms of $\Pi_{t \in T}\, \mathfrak{A}(G_t)/D$ and the set $\Pi_{t \in T}\, G_t/D$. The lemma is now a consequence of a well-known theorem of Boolean algebra.

It may be of interest that the isomorphism mentioned in this lemma is not in general onto, and in fact is not onto for the application of the lemma made below. For example, if $T = \omega$ and $\overline{\overline{G}}_s < \overline{\overline{G}}_t < \omega$ whenever $s < t < \omega$ and if D is nonprincipal, then by Theorem 6.5 of [4] the cardinality of $\Pi_{t \in T}\, \mathfrak{A}(G_t)/D$ is $2^{\aleph_0}$ while the cardinality of $\mathfrak{A}(\Pi_{t \in T}\, G_t/D)$ is $2^{2^{\aleph_0}}$.

The following construction was given by Lyndon [7]. Suppose G is any set, and $I \in G$. We define an operation ; on $\mathfrak{A}(G)$ as follows: $p ; I = I ; p = p$ and $p ; p = p + I$ for all $p \in G$, and

$$p ; q = \sum_{r \in G,\, r \neq p, q, I} r \quad \text{for all } p, q \in G \text{ with } p \neq q \neq I \neq p;$$

further, for any $a, b \in \mathfrak{A}(G)$.

$$a ; b = \sum_{p < a,\, q < b,\, p, q \in G} p ; q .$$

Also, let $^\smile$ be the identity on $\mathfrak{A}(G)$. Then set $\mathfrak{A}(G, I) = \langle A, +, \cdot, -, ;, ^\smile, I \rangle$, where $\mathfrak{A}(G) = \langle A, +, \cdot, - \rangle$. By [7], the following proposition holds.

THEOREM A. *If* $\overline{\overline{G}} \geq 5$ *and* $I \in G$, *then* $\mathfrak{A}(G, I)$ *is a Lyndon algebra.*

(The proof given in [7] has one very minor error. Indeed, when $n = 3$, the case in which p, q, r are distinct and collinear must be treated in a different way; for $(pq)r = p \cup q \cup r \cup I$ in this case—and hence by symmetry $(pq)r = p(qr)$.)

We use one of the standard definitions of a projective plane, and we assume in particular that a line is simply the collection of all points lying on it. A plane P is of order m (m a cardinal number) if and only if there are $m + 1$ points on every line of P. The following consequence of the theorem of Bruck and Ryser [1] will be used below.

THEOREM B. *There exist infinitely many numbers* $m < \omega$ *for which there is no projective plane of order* m.

For example, $2 \cdot 3^{2n+1}$ is such a number, for each $n < \omega$.

The following two theorems were established in [7].

THEOREM C. *If* G *is infinite and* $I \in G$, *then* $\mathfrak{A}(G, I)$ *is representable over a group.*

THEOREM D. *If* $\overline{\overline{G}} \geq 5$, $I \in G$, *and* $\mathfrak{A}(G, I)$ *is representable, then* $G \sim \{I\}$ *is a line in some projective plane.*

Now we can give the proof of Theorem 1. Let

$$M = \{m \in \omega : 3 \leq m \text{ and there exists no projective plane of order } m\}.$$

By Theorem B, M is infinite. Let D be a nonprincipal ultrafilter over M. For each $m \in M$, let G_m be a set with $m + 2$ elements, and let $I_m \in G_m$. Then by the fundamental theorem on ultraproducts, $\Pi_{m \in M} G_m / D$ is infinite. Hence $\mathfrak{A}(\Pi_{m \in M} G_m / D, I/D)$ is representable over a group by Theorem C. On the other hand, by Theorems A and D, $\mathfrak{A}(G_m, I_m)$ is a nonrepresentable Lyndon algebra for each $m \in M$. The proof will be complete when we have shown that the natural isomorphism F of the Boolean structure of $\mathfrak{B} = \Pi_{m \in M} \mathfrak{A}(G_m, I_m)/D$ into the Boolean part of $\mathfrak{C} = \mathfrak{A}(\Pi_{m \in M} G_m / D, I/D)$ also preserves the operations ;, $^\smile$, and 1'. Let elements of $\mathfrak{B}$ be denoted by f/D, where $f \in \Pi_{m \in M} \mathfrak{A}(G_m, I_m)$, and let elements of $\Pi_{m \in M} G_m / D$ be denoted by $g//D$, where $g \in \Pi_{m \in M} G_m = E$; let sums in C be denoted by Σ. Then the basic property of F is that

$$F(b) = \sum_{f/D \leq b,\, f \in E} f//D \tag{1}$$

for all $b \in \mathfrak{B}$. From this it is obvious that $F(I/D) = I/D$. The fact that $G(b^\smile) = F(b)^\smile$ for all $b \in \mathfrak{B}$ is an easy consequence of the fundamental theorem on ultraproducts. Now, using this theorem again, we easily see that for all $f, g, h \in E$, the relation $f/D \leq (g/D) ; (h/D)$ holds if and only if one of the following four conditions is satisfied:

(i) $f/D = I/D$ and $g/D = h/D$,

(ii) $g/D = I/D$ and $f/D = h/D$,

(iii) $h/D = I/D$ and $f/D = g/D$,

(iv) f/D, g/D, h/D, and I/D are distinct.

It follows that $f/D \leq (g/D) ; (h/D)$ if and only if

$$f//D \leq (g//D) ; (h//D) \quad \text{for all } f, g, h \in E.$$

Hence by (1), F(b ; c) = F(b) ; F(c) if b and c are atoms of $\mathfrak{B}$; thus F preserves ; since ; and F are completely additive. This completes the proof of Theorem 1.

As previously mentioned, Theorem 2 follows at once from Theorem 1, and hence the main results of this note are established.

The ultraproduct construction above and the use of Theorem 5.1 of [4] may be replaced by a straightforward application of the compactness theorem for first-order logic.

In [8], the main result of this note and the prerequisite theorems of Lyndon are extended to the theory of three-dimensional cylindric algebras.

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