CERTAIN MANIFOLDS WITH BOUNDARY THAT ARE PRODUCTS

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There exists a 3-manifold M^3 with boundary, whose interior is topologically E^3 and whose boundary is topologically E^2, while M^3 is not topologically $E^2 \times [0, 1)$. Infinitely many such 3-manifolds exist, as was shown in [1] and [16]. We shall show that this situation is unique to dimension 3.

It is well to point out that the following can be obtained by using the results of Homma [14]. This is the approach taken by Cantrell in [6]. The respective methods of this paper and [6] have been combined to study the local embedding of n-manifolds with boundary in n-manifolds [11].

THEOREM 1. *Let* M^n *be an* n*-manifold with boundary such that* Int $M^n = E^n$ *and* Bd $M^n = E^{n-1}$. *Then, if* $n \neq 3$, $M^n = E^{n-1} \times [0, 1)$.

In the statement of the main result, Int and Bd denote the interior and boundary of M^n, respectively. Since the result is trivial for $n = 1$ or 2, we shall assume that $n \geq 4$. One obtains as corollaries the following.

COROLLARY 1. *If* $A \subset E^n$ $(n \neq 3)$ *is an arc that is locally tame except perhaps at an endpoint* p, *then* A *is tame.*

Proof. We note that by [14], A - p is a monotone union of tame arcs. Thus one can evidently swell A - p up into a set K such that $K \cup p$ is an n-cell whose boundary is locally bicollared except at p, while A can be moved by a homeomorphism h of E^n onto E^n so that $h(A) \subset \mathrm{Bd}(K \cup p)$. Then if S^n is the one-point compactification of E^n, $\overline{S^n - K} - p$ is a manifold with boundary of the type described in Theorem 1. Therefore $\overline{S^n - K}$ is a closed n-cell and K is a flat n-cell. Since by [15] each arc on Bd K is tame, A is tame.

COROLLARY 2. *Let* D^n *be a compact* n*-manifold with boundary, and let* Int $D^n = E^n$. *If* Bd $D^n = E^{n-1} \cup R$ *is a standard decomposition of* Bd D^n [9], *then* D^n/R *is* I^n, *the* n*-cell.*

The proof of Theorem 1 will entail several lemmas.

LEMMA 1. *In* E^n *let* $\{D_i\}$ *be a sequence of disjoint* (n - 1)*-cells converging to a point* p. *If for each pair of indices* j *and* k, D_j *and* D_k *can be carried to flat polyhedra by a homeomorphism of* E^n *on* E^n, *then there exists a homeomorphism* h *of* E^n *on* E^n *such that* $\{h(D_i)\}$ *is a sequence of polyhedral flat* (n - 1)*-cells.*

Proof. Since at each point of an (n - 1)-sphere in E^n one can pierce the sphere by an arc that is locally polyhedral except at the point [18, pp. 66-67], there exists an arc J having p and q as endpoints; J pierces each D_i at a single point q_i, and J is locally polyhedral except at p and $\{q_i\}$. By [7], J is a tame arc for $n \geq 4$. We assume without loss of generality that as J is traversed from q to p, the points $q_1, q_2, \cdots, q_i, \cdots$ have the same order on J as they have in their original order in $\{D_i\}$.

The tameness of J ensures the existence of a sequence of bicollared (n - 1)-spheres $\{S_i''\}$ such that $S_{i+1}'' \subset \mathrm{Int}\, S_i''$, $A_i'' = \overline{\mathrm{Int}\, S_i''} - \mathrm{Int}\, S_{i+1}''$ is a closed annulus

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containing q_i in its interior, and $\bigcup_1^\infty$ Int $S_i'' = p$. Since the disks D_i are flat, we may assume that $D_i \subset$ Int A_i'' for each i. By [10], each S_i'' may be replaced by a tame (n - 1)-sphere S_i' so that $\{D_i\}$, $\{S_i'\}$, $\{A_i'\}$ are related in the same way as $\{D_i\}$, $\{S_i''\}$, $\{A_i''\}$ above, and so that $D_i \cup$ Bd A_i' is tame. For if S is any bicollared (n - 1)-sphere in E^n and if U is an open annular neighborhood of S in E^n, then given any bicollared (n - 1)-sphere S' in E^n ($S' \cap U = \Box$), there exists in U a bicollared (n - 1)-sphere S" such that $S' \cup S''$ bounds a closed annulus in E^n and there exists a homotopy j_t in U of the identity map j: S → S such that $j_1(S) = S''$. (This change would not be necessary if the Annulus Conjecture were proved [5].)

In the interior of each A_i', place D_i on a bicollared (n - 1)-sphere S_i such that $\overline{S_i - D_i}$ is an (n - 1)-cell, while S_i bounds a closed annulus in A_i' along with each component of Bd A_i'. The sequence $\{S_i\}$ may now be carried by a homeomorphism h of E^n onto E^n so that $h(S_i)$ is the boundary of an n-simplex and $h(D_i)$ is a face of this simplex. This completes the proof of Lemma 1.

In the preceeding proof a fact of interest appears in connection with piercing properties of spheres. If S^{n-1} is a sphere in E^n and if $n \neq 3$, then S^{n-1} is pierced at each point by a tame arc [7]. The corresponding problem for E^3 is discussed in such papers as [2], [8], [12].

LEMMA 2. *Let* $C^n \subset E^n$ *be an* n*-cell such that* Bd C^n *is locally bicollared except at a point* p. *Then there exists a sequence of disjoint flat* (n - 1)*-cells* $\{D_i'\}$ *such that*

(i) Int $D_i' \cap C^n$ *is an* (n - 1)*-cell* D_i *and* $\overline{D_i' - D_i}$ *is a closed annulus,*

(ii) D_i *is a flat spanning cell of* Bd C^n *that separates* C^n *into two components* C_i *and* C_{ip} *such that* p *lies in* C_{ip} *while* $\overline{C_i}$ *and* $\overline{C_{ip}}$ *are* n*-cells meeting in* D_i,

(iii) $D_i' \cup \overline{C_i}$ *and* $D_i' \cup \overline{C}_{i+1}$ *are tame sets,*

(iv) $\overline{C}_{i+1} \supset \overline{C}_i$, *and*

(v) $\{D_i'\}$ *converges to* p.

Proof. The local bicollaredness of Bd C^n - p implies by [4] that in each open U containing C^n - p there exists a topological $E^{n-1} \times (0, 1)$, K, that lies in U - C^n and is the interior of a collar on Bd C^n - p in $E^n - C^n$. There exists a homeomorphism k from a standard simplex σ^n with a topological $E^{n-1} \times [0, 1)$ attached to its boundary less a point onto $C^n \cup K$, and such that $k(\sigma^n) = C^n$. The disks $\{D_i'\}$ meeting conditions (i) to (v) are the images under k of such disks in the standard model.

LEMMA 3. *Under the hypothesis of Lemma* 2 *there exists a homeomorphism* h *of* E^n *onto* E^n *such that* $\{h(D_i')\}$, $\{h(D_i)\}$ *are sequences of polyhedral flat* (n - 1)*-cells.*

Proof. This follows from Lemma 1.

LEMMA 4. *Let* M^n *be an* n*-manifold with boundary such that* Int $M^n = E^n$ *and* Bd $M^n = E^{n-1}$. *If* N^n *is a copy of* $E^{n-1} \times [0, 1)$ *and* Q^n *is the* n*-manifold obtained by sewing* N^n *and* M^n *together along their boundaries by homeomorphism, then* $Q^n = E^n$.

Proof. If M^n is any n-manifold with boundary, and if an open collar is attached to its boundary, the resulting manifold is homeomorphic to Int M^n.

Lemma 4 ensures that if $M^n \neq E^{n-1} \times [0, 1)$, then there exists a wild n-cell C^n in E^n, and Bd C^n is locally bicollared except at a point p. Further, if

$C^n \subset E^n \subset S^n$, then M^n can be embedded in S^n, as the set S^n - (Int $C^n \cup p$). The problem of showing that $M^n = E^{n-1} \times [0, 1)$ is then equivalent to showing that the cell C^n of Lemma 2 must be tame. In the following, C^n refers to the cell of Lemma 2, where the cells D_i' and D_i are polyhedral by virtue of Lemma 3.

LEMMA 5. *Let* A *be an arc in* C^n, *with endpoint* p, *such that* $A - p \subset \text{Int } C^n$ *and* $A \cap D_i$ *is a point* d_i *for each* i *and such that* A *pierces each* D_i *and* A *is locally polyhedral except at* p. *If* B *is an arc in* $(E^n - C^n) \cup p$ *with* p *as endpoint, and if* B *is locally polyhedral except at* p, *then the arc* $J = A \cup B$ *is a tame arc, and for each fixed* i, $J \cup D_i'$ *and* $J \cup D_i$ *are tame.*

Proof. That J is tame follows from [7]. Similarly, $J \cup D_i'$ and $J \cup D_i$ are tame, since J can be thrown onto a polygon, by a homeomorphism on E^n, without moving D_i'.

LEMMA 6. *Under the hypothesis of Lemma* 5, *let* U *be an open* n-*cell neighborhood of the point* p. *If* D_i' *lies in* U *as well as* $\overline{C}_{ip}$ (*the closure of the component of* $C^n - D_i$ *containing* p), *then there exists a bicollared* (n - 1)-*sphere* S *such that*

(i) p *lies in* Int S,

(ii) $S \subset U$,

(iii) $D_i' \subset S$, *and*

(iv) $S \cap (C^n \cup B) = D_i \cup s$, *where* s *is a point of* B.

Proof. Since $D_i' \cup J$ is tame, there exists a bicollared sphere S' meeting the conditions (i), (ii), and (iii), while $S' \cap B$ is a point and $S' \cap \overline{C}_i = D_i$. It may happen, however, that S' meets a finite number of the tame n-cells in C^n that are cut off on C^n by successive pairs of (n - 1)-cells D_j and D_{j+1}. Since D_j is flat for each j, we can certainly assume that $S' \cap D_j = \square$ if $i \neq j$.

If F_j is the closed n-cell in C^n lying between D_j and D_{j+1}, suppose $S' \cap F_j \neq \square$. Corresponding to any open set V containing both $D_j \cup D_{j+1}$ and $A \cap F_j$, there exists a homeomorphism g of E^n onto E^n that maps F_j into V and reduces to the identity on $D_j \cup D_{j+1}$, outside of V, and outside of an arbitrarily preassigned neighborhood of F_j.

One can construct g by taking sub-disks D_j'' and D_{j+1}'' of D_j' and D_{j+1}' in V and shrinking F_j, a tame cell, into the neighborhood V of $A \cap F_j$ and $D_j'' \cup D_{j+1}''$. Thus, g can be selected so that $S' \cap g(F_j) = \square$. Applying the same argument a finite number of times, we obtain a homeomorphism h that reduces to the identity, outside of U and on J, with $h(C^n) \cap S' = D_i$. Evidently the set $h^{-1}(S') = S$ meets the conditions described in the lemma.

We can now give the proof of Theorem 1. Let $\{U_j\}$ be the symmetrical n-balls with p as center and 1/j as radius. In each U_j there exists, by Lemma 6, a bicollared (n - 1)-sphere S_j such that S_j contains some (n - 1)-cell D_i', which we denote by P_j, such that p lies in the interior of S_j and $S_j \cap J$ is a pair of points one of which is $P_j \cap A$, one of the d_i in Lemma 5. This is because J as well as each $J \cup D_i'$ is tame. We next observe that without loss of generality one may assume that $S_k \cap S_m = \square$ if $k \neq m$, and from Lemma 6 it follows that

$$S_j \cap C^n = P_j \cap C^n$$

for each j.

Now let E^n be compactified by a single point so that $C^n \subset S^n$. If D is any compact set in S^n - (Int $C^n \cup p) = M^n$, then D lies in a closed n-cell L in M^n. We note that L can be so chosen that Bd L consists of $S_j - (P_j \cap C^n)$ together with a component of Bd $C^n - P_j$, for some j. Since this (n - 1)-sphere is locally bicollared by construction, it is tame. Thus by the characterization in [10], $M^n = E^{n-1} \times [0, 1)$. For completeness we state this characterization as follows: Let M^n be an n-manifold with boundary such that $M^n = \bigcup_1^\infty C_i^n$, where $C_i^n \subset C_{i+1}^n$, C_i^n is a closed n-cell for each i, Bd $M^n \cap C_i^n$ is an (n - 1)-disk D_i^{n-1}, Int $D_{i+1}^{n-1} \supset D_i^{n-1}$, and $(C_i^n - D_i^{n-1}) \subset$ Int C_{1+i}. Then $M^n = E^{n-1} \times [0, 1)$.

The fact that an n-manifold with boundary M^n, with Int $M^n = E^n$, and with Bd $M^n = E^{n-1}$ is topologically unique for $n \neq 3$ does not entail that

$$M^n = E^{n-1} \times [0, 1)$$

is a unique factorization in general even into manifolds with boundary. For any 4-simplex σ^4 in E^4, let us construct a manifold $M^4 = \text{Int}\, \sigma^4 \cup K^3$, where K^3 is an open contractible 3-manifold in Bd σ^4. Then $M^5 = M^4 \times E^1$ is a 5-manifold with boundary, and Int $M^5 = E^5$ while Bd $M^5 = E^4$ by [17].

There is an interesting consequence of Theorem 1 in connection with the property of local peripheral unknottedness for arcs (L. P. U.) [13]. An arc $A \subset E^n$ is L. P. U. at an interior point x if each neighborhood U of x contains an n-cell C such that x lies in Int C and $C \cap A$ is an arc with endpoints only on Bd C. It follows from Theorem 1 that C can always be selected a tame n-cell, when $n \neq 3$. The same is true if x is an end point.

The following corollary to Theorem 1 might be obtainable independently. Let X be a topological space such that $X = P^n \cup R$, where P^n is an open set in X which is topologically E^n.

COROLLARY 3. *If the suspension* Y *of* X *is* S^{n+1}, *then* R *is a cellular subset of* Y.

Proof. The set Y - X is a pair of disjoint open (n + 1)-cells. Let M_1 and M_2 be the closures of these cells in Y - R. Each M_i is a manifold with boundary. Since Int $M_i = E^{n+1}$ and Bd $M_i = P^n$, we see that $M_i = E^n \times [0, 1)$ (for $n \neq 2$) and $Y - R = E^{n+1}$. Thus R is point-like in Y. For n = 2, X is the 2-sphere, and the result follows.

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