WEAKLY INVERTIBLE ELEMENTS IN CERTAIN FUNCTION SPACES, AND GENERATORS IN ℓ_1

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By ℓ_1 we denote the Banach algebra of complex sequences $a = \{a_n\}$ $(n = 0, 1, 2, \cdots)$ for which $\|a\| = \Sigma_{n=0}^{\infty} |a_n|$ is finite, multiplication being defined by convolution: the product of $\{a_n\}$ and $\{b_n\}$ is $\{c_n\}$, where $c_n = \Sigma_{k=0}^{n} a_k b_{n-k}$. We say a is a *generator* of ℓ_1 if the polynomials in a are dense in ℓ_1. In [1] the following theorem was proved (wherever the range of an index of summation is not specified, it will be from 0 to ∞; for terminology and notation, the reader is referred to [1]):

THEOREM A. *Let* $f(z) = \Sigma a_n z^n$ *be univalent in* $|z| < 1$ *and map* $|z| < 1$ *onto a Jordan domain* Ω *whose boundary is rectifiable.* (Under these hypotheses, $\Sigma |a_n| < \infty$). *Let* $I(z)$ *denote the normalized inner factor of* $f'(z)$. *Then* a *is a generator of* ℓ_1 *if and only if* $(IH_2)^{\perp}$ *contains no nonnull function whose Taylor coefficients are* $O(1/n)$.

In particular, if Ω is a Smirnov domain (that is, if $I = 1$), a is a generator. Although the stated condition is necessary and sufficient, D. J. Newman later showed [2] that the inner factor I has in reality no bearing on the question, by proving the following proposition.

THEOREM B. *Let* $f(z) = \Sigma a_n z^n$ *be univalent in* $|z| < 1$ *and map* $|z| < 1$ *onto a Jordan domain* Ω *whose boundary is rectifiable. Then* a *is a generator of* ℓ_1.

Inasmuch as a *necessary* condition for a to be a generator is that f(z) be univalent in $|z| < 1$ and map $|z| < 1$ onto a Jordan domain (see [1]), Theorem B settles the question of generators insofar as functions f with derivative of class H_1 are concerned; any further progress must be in the direction of either weakening this hypothesis on f or else showing by examples that it cannot be weakened.

Newman's proof of Theorem B is somewhat complicated, and the main purpose of this note is to provide a rather simple deduction of Theorem B from Theorem A. The proof offered here is essentially a modification of Newman's proof, but has perhaps some interest in view of its simplicity.

1. WEAKLY INVERTIBLE ELEMENTS IN CERTAIN SPACES

Let F denote a topological space whose elements are functions f(z) analytic in $|z| < 1$. Assume that $1 \in F$, and that $f \in F$ implies $Pf \in F$ for every polynomial P. We shall say that an $f \in F$ is *weakly invertible* if $\lim_{n \to \infty} P_n f = 1$ for some sequence of polynomials P_n (convergence being in the topology of F). Thus, for example, the weakly invertible elements of H_2 are precisely those with no inner factor (theorem of Beurling) and this is true also for H_p $(1 \le p < \infty)$. We shall work with the space B_α, defined as follows: B_α is the Hilbert space of functions $f(z) = \Sigma a_n z^n$ analytic in $|z| < 1$ such that

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$$\|f\|_\alpha^2 = \sum \frac{|a_n|^2}{(n+1)^\alpha} < \infty .$$

For $\alpha = 0$, $B_\alpha = H_2$, and for $\alpha = 1$, B_α is the "Bergman space" of $|z| < 1$. By a simple calculation, when $\alpha > 0$,

$$(2) \quad C_1(\alpha) \iint (1-r)^{\alpha-1} |f(re^{i\theta})|^2 \, dA \leq \|f\|_\alpha^2 \leq C_2(\alpha) \iint (1-r)^{\alpha-1} |f(re^{i\theta})|^2 \, dA ,$$

where the $C_i(\alpha)$ denote positive constants depending on α, the integrations are over $|z| < 1$, and dA is Lebesgue area. Hence, the integral appearing in (2) defines an equivalent norm in B_α.

LEMMA 1. *Let* $f \in B_\alpha$ $(\alpha > 0)$ *and* $\phi \in H_\infty$. *Then* $\phi f \in B_\alpha$, *and*

$$\|\phi f\|_\alpha < C_3(\alpha) \|\phi\|_\infty \|f\|_\alpha .$$

This is an immediate consequence of (2).

LEMMA 2. *Let* $f \in B_\alpha$ $(\alpha > 0)$, $\phi \in H_\infty$, *and suppose* f *and* ϕ *are weakly invertible in* B_α. *Then* ϕf *is weakly invertible.*

Proof. For any polynomials P and Q,

$$\begin{aligned} \|PQ\phi f - 1\|_\alpha &\leq \|P\phi(Qf - 1)\|_\alpha + \|P\phi - 1\|_\alpha \\ &\leq C_3(\alpha) \|P\phi\|_\infty \|Qf - 1\|_\alpha + \|P\phi - 1\|_\alpha , \end{aligned}$$

and the result follows if we first choose P so that the second term is small, and then choose Q so that the first term is small.

LEMMA 3. *If* $f \in H_\infty$ *and* $f^{-1} \in B_\alpha$ $(\alpha > 0)$, *then* f *is weakly invertible in* B_α.

Proof. $\|Pf - 1\|_\alpha = \|f(P - f^{-1})\|_\alpha \leq C_3(\alpha) \|f\|_\infty \|P - f^{-1}\|_\alpha$, and the result follows if we choose a polynomial P that approximates to f^{-1} (for example, the n-th partial sum of the Taylor series of f^{-1}, with n large enough).

LEMMA 4. *Let* $I(z)$ *be an* H_∞ *function such that*

$$(3) \qquad |I(re^{i\theta})| > C(1-r)^N$$

for some positive constants C, N. *Let* $\alpha > 0$. *Then* $I(z)$ *is weakly invertible in* B_α.

Proof. For any $\alpha > 0$, $I(z)^{-1/m} \in B_\alpha$ if $m > 2N/\alpha$, as we see from (2) and (3). Hence, for such a positive integer m, $I(z)^{1/m}$ is weakly invertible in B_α, by Lemma 3. Hence, by repeated application of Lemma 2, $I(z)$ is also weakly invertible in B_α.

LEMMA 5. *A family* $\{f\}$ *of functions of* B_α *fails to span* B_α *if and only if there exists a nonnull sequence* $\{b_n\}$ *such that*

$$\sum (n+1)^\alpha |b_n|^2 < \infty ,$$

and such that $\sum \bar{b}_n a_n = 0$ *for every* $f(z) = \sum a_n z^n$ *in* $\{f\}$.

Proof. The mapping $a_n = (n+1)^{\alpha/2} c_n$ maps B_α isometrically on ℓ_2; using the representation of the linear functionals on ℓ_2, we obtain the result.

LEMMA 6. *A function* $f(z) = \Sigma a_n z^n$ *fails to be weakly invertible in* B_α *if and only if there exists a nonnull sequence* $\{b_n\}$ *such that*

$$\sum (n+1)^\alpha |b_n|^2 < \infty \quad \text{and} \quad \sum \bar{b}_{k+n} a_n = 0 \quad (k = 0, 1, 2, \cdots).$$

Proof. Apply Lemma 5 to the family of functions $z^k f$ $(k = 0, 1, 2, \cdots)$.

THEOREM 1. *Let* $I(z)$ *be an inner function such that* (3) *holds, and let* $\varepsilon > 0$. *Then* $(IH_2)^\perp$ *contains no nonnull function with Taylor coefficients* $O(n^{-\frac{1}{2}-\varepsilon})$.

Proof. Write $I(z) = \Sigma a_n z^n$, and suppose

$$f(z) \in (IH_2)^\perp \quad \text{and} \quad f(z) = \sum b_n z^n \text{ with } b_n = O(n^{-\frac{1}{2}-\varepsilon}).$$

Then $\Sigma \bar{b}_{k+n} a_n = 0$ $(k = 0, 1, 2, \cdots)$. But $\Sigma (n+1)^\varepsilon |b_n|^2 < \infty$. Since, by Lemma 4, $I(z)$ is weakly invertible in B_ε, we conclude from Lemma 6 that $\{b_n\}$ must be null.

COROLLARY 1. *Under the hypotheses of Theorem* A, $(IH_2)^\perp$ *contains no nonnull function whose Taylor coefficients are* $O(n^{-\frac{1}{2}-\varepsilon})$. (Hence, Theorem B is true.)

Proof. We need only verify that I satisfies (3); this is indeed the case, with $N = 2$ (see [1, p. 474]).

COROLLARY 2. *Suppose* $f(z)$ *is univalent for* $|z| < 1$ *and* $f'(z) \in H_p$ *for some* $p > 0$. *Let* $I(z) = \Sigma a_n z^n$ *denote the normalized inner factor of* f'. *If* $a_n = O(n^{-\frac{1}{2}-\varepsilon})$ *for some positive* ε, *then* $I \equiv 1$.

Proof. Once again (using [3, Section 11.3]) one verifies (3). Since $(IH_2)^\perp$ contains the function $I(z) - \overline{I(0)}^{-1}$, whose Taylor coefficients are (except for $n = 0$) the same as those of $I(z)$, the result follows.

In view of the importance of the condition (3), it is perhaps of interest to point out an equivalent form of it. If I is any inner function, we may write

$$I(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{z + e^{it}}{z - e^{it}}\, d\rho(t)\right\},$$

where $\rho(t)$ is an increasing function for $0 \leq t < 2\pi$. From this we shall deduce the following proposition.

THEOREM 2. $I(z)$ *satisfies* (3) *if and only if the modulus of continuity* $\omega(h)$ *of* $\rho(t)$ *is* $O\left(h \log \frac{1}{h}\right)$.

Proof. The "only if" assertion follows exactly as in [1, p. 474]. Suppose now that $\omega(h) \leq A h \log \frac{1}{h}$. Using the relation

$$-\log |I(re^{i\theta})| = \frac{1}{2\pi}\int_0^{2\pi} P(r, \theta - t)\, d\rho(t),$$

where P denotes the Poisson kernel, and the estimate

$$P(r, \theta - t) \leq \frac{C(1 - r)}{(1 - r)^2 + (\theta - t)^2};$$

for $h = 2\pi/n$, we get the inequality

$$-\log |I(re^{i\theta})| \leq C_1 h \left(\log \frac{1}{h}\right) \sum_{k=0}^{n-1} \max_{kh \leq t \leq (k+1)h} \left\{ \frac{1 - r}{(1 - r)^2 + (\theta - t)^2} \right\}.$$

For each θ, with the exception of a single value of k, the maximum is attained at an end point of the interval [kh, kh + h], hence the right-hand side is bounded by

$$C_2 h \left(\log \frac{1}{h}\right) \left(\frac{1}{1 - r} + \sum_{k=0}^{n-1} \frac{1 - r}{(1 - r)^2 + k^2 h^2} \right).$$

Writing $\delta = 1 - r$, choose $n = [\delta^{-1}]$; then $C_3 \delta < h < C_4 \delta$ and the last expression is less than

$$C_5 \delta \left(\log \frac{1}{\delta}\right) \left(\delta^{-1} + \delta^{-1} \sum_{k=0}^{\infty} [1 + k^2)^{-1} \right) < C_6 \log \frac{1}{\delta},$$

and hence $|I(re^{i\theta})| > (1 - r)^{C_6}$; this completes the proof.

Remark. Obviously the same method shows that, in general,

$$|I(re^{i\theta})| > \exp \left\{ -C \frac{\omega(1 - r)}{1 - r} \right\},$$

where $\omega(h)$ is the modulus of continuity of ρ, and where C is a positive constant. This shows the quantitative interplay between the rate at which an inner function can tend to zero radially and the "evenness" with which its representing measure is distributed on the circle, as measured by $\omega(h)$. The analogous theorem for subarcs of the circle is clearly also true.

2. A RELATED RESULT ON LEFT TRANSLATIONS

In this section we apply the idea of weak invertibility to show that for any inner function I, the Taylor coefficients of an $f \in (IH_2)^{\perp}$ cannot be too small, unless they vanish from some point on. We state the result as a completeness theorem:

THEOREM 3. *Let* $\{a_n\}$ $(n = 0, 1, \cdots)$ *be a complex sequence in* ℓ_2 *such that*

(i) $a_n = O(e^{-An})$ *for every* $A > 0$, *or, equivalently,* $|a_n|^{1/n} \to 0$.

(ii) $a_n \neq 0$ *for infinitely many* n.

Then the left translates of $\{a_n\}$ *are complete in* ℓ_2, *that is, the sequences* $(a_n, a_{n+1}, a_{n+2}, \cdots)$ $(n = 0, 1, 2, \cdots)$ *are complete in* ℓ_2.

Proof. If the sequences $(a_n, a_{n+1}, \cdots)$ fail to be complete, then the function $f(z) = \Sigma a_n z^n$ is orthogonal to gH_2 for some $g \in H_2$, $g \neq 0$, and hence $f \in (IH_2)^{\perp}$, where I is the normalized inner factor of g. Let $I = z^m I_0$, where m is a nonnegative integer and $I_0(0) \neq 0$. Let $f_0(z) = \Sigma a_{k+m} z^k$. Then $f_0 \not\equiv 0$ because of (ii), and $f_0 \in (I_0 H_2)^{\perp}$. Let p be a positive number $(p < 1)$ such that $I_0(z) \neq 0$ for $|z| \leq p$. Let G_p denote the Hilbert space of functions $g(z) = \Sigma b_n z^n$ normed by $\|g\|^2 = \Sigma p^{2n} |b_n|^2$. To complete the proof it suffices to show that $I_0(z)$ is weakly invertible in G_p, since f_0 belongs to the dual space of G_p, that is,

$$\sum p^{-2n} |a_{n+m}|^2 < \infty$$

because of (i). To achieve this, note first that if f and g denote any functions in G_p, and $|f(z)| \leq M$ for $|z| < p$, then

$$\|fg\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(pe^{i\theta})|^2 |g(pe^{i\theta})|^2 d\theta \leq M^2 \|g\|^2,$$

the norms being taken in G_p. Hence the analogue of Lemma 3 is true also for the space G_p, and since $I_0(z)^{-1} \in G_p$, I_0 is weakly invertible in G_p. The proof is complete.

In conclusion, we remark that all the results of this paper can be extended to functions on the half-line $0 \leq t < \infty$.

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