EXCEPTIONAL LIE GROUPS AND STEENROD SQUARES

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1. The mod 2 cohomology algebras of the exceptional Lie groups have been determined by Borel [3], Araki [1], and Araki and Shikata [2]. These algebras are as follows (x_i indicates a generator of degree i):

$$H^*(G_2) = Z_2[x_3]/(x_3^4) \otimes \Lambda_2(x_5),$$

$$H^*(F_4) = Z_2[x_3]/(x_3^4) \otimes \Lambda_2(x_4, x_{15}, x_{23}),$$

$$H^*(E_6) = Z_2[x_3]/(x_3^4) \otimes \Lambda_2(x_5, x_9, x_{15}, x_{17}, x_{23}),$$

$$H^*(E_7) = Z_2[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4) \otimes \Lambda_2(x_{15}, x_{17}, x_{23}, x_{27}),$$

$$H^*(E_8) = Z_2[x_3, x_5, x_9, x_{15}]/(x_3^{16}, x_5^8, x_9^4, x_{15}^4) \otimes \Lambda_2(x_{17}, x_{23}, x_{27}, x_{29}).$$

The behaviour of the Steenrod squares in these algebras has also been largely determined [3], [2]. Generators can be chosen so that

$Sq^2 x_3 = x_5$ in G_2, F_4, E_6, E_7, E_8,

$Sq^8 x_{15} = x_{23}$ in F_4, E_6, E_7, E_8,

$Sq^4 x_5 = x_9$, $Sq^8 x_9 = x_{17}$ in E_6, E_7, E_8,

$Sq^4 x_{23} = x_{27}$ in E_7, E_8,

$Sq^2 x_{27} = x_{29}$ in E_8.

The one result missing (for E_6, E_7, E_8) is the value of $Sq^2 x_{15}$. Knowledge of this value can be used in the calculation of the Atiyah-Hirzebruch K-groups for E_6, E_7, and E_8.

We shall prove the following proposition.

THEOREM 1. *In the* mod 2 *cohomology algebras of* E_6, E_7, *and* E_8, *there exists a generator* x_{15}, *of degree* 15, *such that*

$$Sq^2 x_{15} = x_{17}.$$

Let X denote a connected H-space with integral homology of finite type such that H*(X) (mod 2 coefficients) is a Z_2-module of finite rank. Our proof will use the projective plane of X, $P_2 X$ as defined in [6] and [5]. Set

$$A = H^*(X), \qquad C = H^*(P_2 X).$$

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For any graded algebra B over Z_2 such that $B_0 = Z_2$, we denote by $\overline{B}$ the ideal of elements of positive degree. As shown in [5; Section 3], there is an exact triangle of modules,

$$\begin{array}{ccc} \overline{A} & \xrightarrow{\psi} & \overline{A} \otimes \overline{A}, \\ & \iota \nwarrow \quad \swarrow \lambda & \\ & \overline{C} & \end{array}$$

where ψ has degree zero, λ has degree 2, and ι has degree -1. The map ψ is the (reduced) diagonal map induced by the group multiplication $X \times X \to X$. Thus a class $u \in \overline{H}^*(X)$ is primitive if and only if $\psi(u) = 0$. Consequently,

$$\text{Image } \iota = \text{Kernel } \psi = P,$$

where P is the subspace of $\overline{A}$ spanned by the primitive classes. Both homomorphisms λ and ι commute with the Steenrod squares, since ι is defined in terms of the suspension isomorphism and λ is the composition of two Mayer-Vietoris coboundaries. The ring structure in C is related to λ as follows: Given $u_1, u_2 \in \overline{C}$, set $x_i = \iota u_i \in \overline{A}$ (i = 1, 2). Then

$$u_1 \cup u_2 = \lambda(x_1 \otimes x_2). \tag{1.1}$$

Finally, ι annihilates all decomposable elements in C. For the details of these statements see [5].

Denote by P^- the subspace of $\overline{A}$ spanned by the primitive classes of odd dimension. Let U be any nonzero subspace of P^-, and let U^+ be a complementary summand to U in P. Since the elements of U are all indecomposable (they are odd-dimensional primitive classes), there exists a summand Q in $\overline{A}$ that is complementary to U and contains the ideal D generated by the decomposable elements. Thus,

$$P = U \oplus U^+, \qquad \overline{A} = U \oplus Q, \qquad D \subset Q.$$

Consequently, we may write

$$\overline{A} \otimes \overline{A} = (U \otimes U) \oplus R,$$

where $R = (U \otimes Q) \oplus (Q \otimes U) \oplus (Q \otimes Q)$. Choose a summand V^+ in $\overline{C}$ such that $\iota: V^+ \approx U^+$, and set

$$S = \lambda(R) + V^+ \text{ in } \overline{C}.$$

Choose classes $\{u_i\}$ in $\overline{C}$ so that the classes $\{\iota u_i\}$ form a basis for U. The following result is stated in [5; Section 5].

THEOREM 2. $H^*(P_2 X) = (B / \overline{B} \cdot \overline{B} \cdot \overline{B}) \oplus S$, *where* $B = \bigotimes_i Z_2[u_i]$.

We sketch a proof of this in Section 2.

The proof of Theorem 1 will be by contradiction. We pose the alternatives as follows:

(1.2) *Let* $X = E_6, E_7$, *or* E_8, *and set* $A = H^*(X)$. *Then either*

(i) $Sq^2 A_{15} \subset D_{17}$, *or*

(ii) *there exists an indecomposable element* $x_{15} \in A_{15}$ *such that*

$$Sq^2 x_{15} = x_{17},$$

where $x_{17} = Sq^8 Sq^4 Sq^2 x_3$.

Let y be any choice of indecomposable element in A_{15}; then we can write

$$Sq^2 y = ax_{17} + bx_3 x_5 x_9 + cx_3^4 x_5,$$

where a, b, c $\in Z_2$. (If $X = E_6$ or E_7, we take $c = 0$, since $x_3^4 = 0$.) If $a = 0$ or $b = c = 0$, the result is proved. Suppose that $a \neq 0$, $c \neq 0$, and set $y' = y + x_3^5$. Then

$$Sq^2 y' = x_{17} + bx_3 x_5 x_9;$$

moreover y' is still indecomposable. Now

$$\psi(x_3 x_5 x_9) = x_3 \otimes x_5 x_9 + x_5 x_9 \otimes x_3 + x_5 \otimes x_3 x_9$$
$$+ x_3 x_9 \otimes x_5 + x_9 \otimes x_3 x_5 + x_3 x_5 \otimes x_9.$$

If one writes out all the terms that can appear in $\psi(y')$ and then applies Sq^2 to these terms, one finds that it is not possible to obtain the terms

$$x_9 \otimes x_3 x_5 + x_3 x_5 \otimes x_9.$$

Thus b = 0, and we take $x_{15} = y'$ to complete the proof.

Proof of Theorem 1. In Theorem 2 we take $X = E_6$, E_7, or E_8 and set $A = H^*(X)$. To prove the theorem it suffices, by (1.2), to show that the assumption $Sq^2 A_{15} \subset D_{17}$ leads to a contradiction. Define U to be the subspace of P^- spanned by x_{17} (= $Sq^8 Sq^4 Sq^2 x_3$). As has been remarked, the complementary summand Q can be chosen to contain D; since $A_{16} = D_{16}$, D contains $Sq^1 A_{16}$ and also, by hypothesis, $Sq^2 A_{15}$. Thus,

$$Sq^1 Q \subset Q, \qquad Sq^2 Q \subset Q,$$

and so by Cartan's product formula,

$$Sq^1 R \subset R, \qquad Sq^2 R \subset R.$$

Since $V_{34}^+ = V_{35}^+ = 0$, and since λ commutes with Sq^i, it follows that

$$Sq^1(S) \subset S, \qquad Sq^2(S) \subset S.$$

We now obtain our contradiction. Let u be a class in $H^*(P_2 X)$ such that $\iota u = x_{17}$. By Theorem 2, $u^2 \neq 0$ and $u^2 \notin S$. But by the Adem relations,

$$u^2 = Sq^{18} u = Sq^1(Sq^{16} Sq^1) u + Sq^2 Sq^{16} u,$$

and so

$$u^2 \in Sq^1(S) + Sq^2(S),$$

which is a contradiction. Hence by (1.2), $Sq^2 x_{15} = x_{17}$.

Remark. The argument given here is very similar to the argument used in proving Theorem 2.1 of [7]. The reason Theorem 1 does not follow directly from the results obtained in [7] is that the mod 2 cohomology of E_6, E_7, and E_8 is not primitively generated.

2. *Proof of Theorem* 2. We follow the proof given by W. Browder, which uses the Bockstein spectral sequence [4]. Let $\{E_q(r), d_r\}$ $(q \geq 0,\ r \geq 1)$ denote the Bockstein spectral sequence in homology for the H-space X. In particular $E_q(1) = H_q(X)$.

LEMMA 1. *Let* u *and* v *be odd-dimensional primitive classes in* $H_*(X)$. *Then*

(1) $d_r(u) = d_r(v) = 0$ *for all* $r \geq 1$,

(2) $u^2 = 0$, $u \cdot v + v \cdot u = 0$.

Suppose either that $d_1(u) \neq 0$, or that $d_i(u) = 0$ for $1 \leq i < r$ and that $d_r(u) \neq 0$. Set $x = d_r(u)$. Then x is a nonzero even-dimensional primitive class in $E_*(r)$ such that $x \in \text{Image } d_r$. Thus by Theorem 6.1 of [4], x has infinite implications (as there defined), which is impossible since $H_*(X)$ has finite rank. Therefore $d_r(u) = d_r(v) = 0$. Now the classes u^2 and $u \cdot v + v \cdot u$ are primitive, since u and v are primitive. Each d_r is a derivation, and therefore by (1),

$$d_r(u^2) = 0 \text{ and } d_r(u \cdot v + v \cdot u) = 0 \text{ for all } r \geq 1 .$$

Thus u^2 and $u \cdot v + v \cdot u$ represent even-dimensional primitive classes in $E_*(\infty)$; hence by Corollary 4.14 of [4] they must be zero.

Using the same notation as in Section 1, we prove the following.

LEMMA 2. *Set*

$$x = \sum_i a_i u_i + \sum_{j<k} b_{jk} u_j u_k + \sum_{\ell} c_{\ell} u_{\ell}^2 ,$$

where a_i, b_{jk}, $c_\ell \in Z_2$. *If* $x \in S$, *then all the coefficients* a_i, b_{jk}, c_ℓ *are zero.*

By definition, if $x \in S$ there exist classes $y \in R$, $z \in V^+$ such that $x = \lambda(y) + z$. Now $\iota\lambda(y) = 0$ by exactness, and $\iota(u_j u_k) = \iota(u_\ell^2) = 0$, since ι annihilates decomposable elements. Therefore

$$\sum_i a_i \iota u_i = \iota z .$$

But $\iota z \in U^+$, $\Sigma_i a_i \iota u_i \in U$ and $U^+ \cap U = 0$, which shows that

$$\iota z = 0, \qquad \sum_i a_i \iota u_i = 0 .$$

By hypothesis the classes $\{\iota u_i\}$ form a basis for U, and therefore $a_i = 0$ for each i. Also, ι on V^+ is an isomorphism, which means that $z = 0$. Thus,

$$x = \sum_{j<k} b_{jk} u_j u_k + \sum_{\ell} c_\ell u_\ell^2 = \lambda(y) .$$

Set $\iota u_i = v_i$ and let $\{\bar{v}_i\}$ be a set of homology classes dual to $\{v_i\}$. That is, let

$$<v_i, \bar{v}_j> = \delta_{ij},$$

where $<\ ,\ >$ denotes the Kronecker index. Since $\bar{A} = U \oplus Q$, we may choose the classes $\{\bar{v}_i\}$ so that

$$<Q, \bar{v}_i> = 0 \text{ for all } i,$$

and hence, $<R, \bar{v}_i \otimes v_j> = 0$ for all i, j. Define

$$w = \sum_{j<k} b_{jk} v_j \otimes v_k + \sum_{\ell} c_\ell v_\ell \otimes v_\ell$$

in $U \otimes U$. By (1.1),

$$\lambda(w) = \sum_{j<k} b_{jk} u_j u_k + \sum_{\ell} c_\ell u_\ell^2,$$

and therefore $\lambda(w - y) = 0$. Thus by exactness there exists a class $f \in \bar{A}$ such that $\psi(f) = w - y$. Notice that

$$<w, \bar{v}_j \otimes \bar{v}_k> = b_{jk}, \qquad <w, \bar{v}_k \otimes \bar{v}_j> = 0 \ (j < k),$$

$$<w, \bar{v}_\ell \otimes \bar{v}_\ell> = c_\ell .$$

Thus, if we set

$$\bar{g} = \bar{v}_j \cdot \bar{v}_k + \bar{v}_k \cdot \bar{v}_j, \ \bar{h} = \bar{v}_\ell^2,$$

then

$$<f, g> = <\psi(f), \bar{v}_j \otimes \bar{v}_k + \bar{v}_k \otimes \bar{v}_j> = <w - y, \bar{v}_j \otimes \bar{v}_k + \bar{v}_k \otimes \bar{v}_j> = b_{jk};$$

and similarly,

$$<f, \bar{h}> = c_\ell .$$

Since $D \subset Q$ and $<Q, \bar{v}_i> = 0$, each class $\bar{v}_i$ is primitive, and so by Lemma 1, $\bar{g} = \bar{h} = 0$, which shows that each $b_{jk} = c_\ell = 0$.

The proof of Theorem 2 now follows from Lemma 2 exactly as given in Section 4 of [5]. We leave the details to the reader.

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