FREEDOM IN POLYADIC ALGEBRAS AND TWO THEOREMS OF BETH AND CRAIG

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INTRODUCTION

In [6] we gave an algebraic version, in the context of polyadic algebras, of the well-known theorem of Beth [1] in the theory of definition (see Theorem 2.9 below). In that paper we saw how a generalization of Beth's theorem due to Svenonius [13] is embodied in the following polyadic statement reminiscent of elementary field theory: *every simple polyadic algebra admits a simple normal rich extension* [6, Corollary 4.6].

In the present paper we shall obtain a further polyadic version of Beth's theorem suggestive of yet another chapter of abstract algebra: the theory of free products of groups with amalgamation. It is well-known [11, p. 32] that in a free product of groups with amalgamation the intersection of the factors is precisely the amalgamated subgroup. Our second algebraic version of Beth's theorem is obtained by replacing the word "group" in this statement by "polyadic algebra." The proof of this and the definitions of the free product of polyadic algebras with or without amalgamation occupy the first two sections of the paper.

Beth's theorem is often presented in conjunction with Craig's *Interpolation* theorem [2], [3]. In Sections 3 and 4 we give an algebraic treatment of Craig's theorem. In its algebraic setting the Interpolation theorem becomes a statement about *free* polyadic algebras.

For unexplained notations and elementary facts concerning polyadic algebras, we refer to Halmos's publications [7] to [10].

We are indebted to the referee for his simplification of our proof of Lemma 4.2.

1. FREE PRODUCTS OF POLYADIC ALGEBRAS

We assume that the reader is familiar with the concepts of (operational) system or algebra, type of a system, subsystem, quotient system, direct product of a family of systems of the same type, homomorphism of a system into one of the same type, and so forth.

Let A^* be a class of systems of a fixed type. All systems considered are assumed to be members of A^*. The following definition is well established [11].

(1.1) *Definition.* Let $B^* = \{ B_h | \; h \in H\}$ be a family of systems. The A^**-free product* of B^* is a family of monomorphisms f_h: $B_h \to B$ such that

(1.2) B is generated by the union of the $f_h(B_h)$, and

(1.3) for any family of homomorphisms g_h: $B_h \to C$, there exists a (necessarily unique) homomorphism g: $B \to C$ such that $g_h = gf_h$.

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The results of this paper were contained in the author's doctoral dissertation [5] and have, in part, been announced in [4].

The homomorphism g is said to *extend* the g_h. When there is no ambiguity, B itself is referred to as the free product of B^*. The unicity of the free product up to equivalence is obvious. However, the free product may fail to exist for certain A^* and B^*. We shall need the following existence theorem, which the reader should compare with (viii) and (ix) in [12]. Indeed, since the class of I-polyadic algebras is equational (for a fixed I), it satisfies the hypotheses of (viii), and we could have invoked (viii) instead of Theorem (1.4) below. However, our theorem is stronger than (viii) (because it does not require A^* to be closed under the formation of homomorphic images), and we give a direct proof of it.

(1.4) THEOREM. *For the* A^**-free product of* B^* *to exist it is sufficient that the following two conditions be satisfied:*

(1.5) A^* *is closed under the extraction of subsystems and the formation of direct products; and*

(1.6) *there exists a* $C \in A^*$ *in which all members of* B^* *can be imbedded.*

Proof. Consider a set of representatives from all equivalence classes of families of homomorphisms $g_h: B_h \to C$ such that C is generated by the union of the $g_h(B_h)$. Let $\{ g_{hk}: B_h \to C_k \mid h \in H\}$ be a typical representative, k varying over some (presumably large) set K. Set $D = \Pi C_k$ (the direct product system), and define $f_h: B_h \to D$ by the equation $f_h(p) = \{ g_{hk}(p)\}$, where $p \in B_h$. Let B be the subalgebra of D generated by the union of the $f_h(B_h)$. From (1.5) it follows that $B \in A^*$, and from (1.6) it follows that the f_h are monomorphisms. It is clear that the family of monomorphisms $f_h: B_h \to B$ is the A^*-free product of B^*; for any family of homomorphisms $g_h: B_h \to C$ must be equivalent to a family of homomorphisms $g_{hk}: B_h \to C_k$, for some k, and can therefore be identified with it. If g denotes the restriction to B of the natural projection of D onto C_k, then $g_h = gf_h$ for all h, as desired. ∎

For a fixed set I of variables, the class of all I-polyadic algebras satisfies (1.5). If I is infinite and B^* consists of locally finite algebras, then (1.6) is precisely Theorem 2.6 of [6]. Therefore, if we assume I to be infinite, the free product of any family B^* of locally finite I-algebras with respect to the class of all I-algebras exists. It is obviously also a free product with respect to the class of all locally finite I-polyadic algebras.

2. FREE PRODUCTS WITH AMALGAMATION

Let $M^* = \{ m_h: M \to B_h \mid h \in H\}$ be a family of monomorphisms. As before, all systems considered are assumed to be in A^*.

(2.1) *Definition.* A family of homomorphisms $u_h: B_h \to A$ is said to *amalgamate* M^* if the homomorphism $u_h m_h$ is independent of h.

(2.2) *Definition.* The A^**-free product of* $B^* = \{ B_h \mid h \in H\}$ *with amalgamation of* M^* is a family of monomorphisms $v_h: B_h \to \hat{B}$ satisfying the conditions

(2.3) $\hat{B}$ is generated by the union of the $v_h(B_h)$,

(2.4) the family amalgamates M^*, and

(2.5) for any family of homomorphisms $u_h: B_h \to A$ that amalgamates M^*, there exists a (necessarily unique) homomorphism $u: \hat{B} \to A$ such that $u_h = uv_h$.

By "polyadic algebra" we shall henceforth mean "locally finite I-polyadic algebra of infinite degree." The following theorem was proved in [6] (Theorem 2.7).

(2.6) THEOREM. *Let* $m_i: M \to B_i$ $(i = 1, 2)$ *be monomorphisms of polyadic algebras. Then there exist monomorphisms* $u_i: B_i \to A$ *that amalgamate the* m_i.

From this theorem and the fact that the class of locally finite I-polyadic algebras is closed under the formation of unions of increasing chains, one easily obtains (by using Zorn's lemma) the following proposition.

(2.7) THEOREM. *Let* $\{m_h: M \to B_h \mid h \in H\}$ *be a family of monomorphisms of polyadic algebras. Then there exists a family of monomorphisms* $u_h: B_h \to A$ *that amalgamates the* m_h.

Henceforth we assume that A^* is the class of all polyadic algebras, and accordingly that B^* is a family of polyadic algebras and M^* a family of polyadic monomorphisms.

(2.8) THEOREM. *The* A^**-free product* $\hat{B}$ *of* B^* *with amalgamation of* M^* *exists.*

Proof. Let $\{f_h: B_h \to B$ be the free product of B^*. Let N be the intersection of the kernels of the "extensions" to B of all families of homomorphisms $u_h: B_h \to A$ that amalgamate M^*. By (2.7), $N \cap f_h(B_h) = \{0\}$ for each h. From this it follows that if we set $v_h = nf_h$ (where $n: B \to \hat{B}$ is the natural mapping), then $\{v_h: B_h \to \hat{B}\}$ is the A^*-free product of B^* with amalgamation of M^*. ■

In [6, Theorem 4.2] we gave an algebraic interpretation of Beth's theorem; it reads as follows.

(2.9) THEOREM. *Let* A_1 *be any extension of a polyadic algebra* A, *and suppose* $p \in A_1 - A$. *Then there exist two homomorphisms* e_1 *and* e_2 *of* A_1 *into some algebra* D *such that* $e_1(p) \neq e_2(p)$ *and* $e_1 \mid A = e_2 \mid A$.

We now propose another algebraic version of the same theorem.

(2.10) THEOREM. *In the free product* $\hat{B}$ *of the family* B^* *of polyadic algebras with amalgamation of* M^*, *the intersection of any two factors is precisely the amalgamated part.* (More precisely, for any pair of distinct elements h_1 and h_2 of H, the intersection of $v_{h_1}(B_{h_1})$ with $v_{h_2}(B_{h_2})$ is $v_h m_h(M)$, which is independent of h).

We shall show that (2.9) and (2.10) are interdeducible. First we prove that (2.9) implies (2.10). If the set

$$[v_{h_1}(B_{h_1}) \cap v_{h_2}(B_{h_2})] - [v_h m_h(M)]$$

is nonvoid, let p be an element of it, and apply (2.9) (with $A = v_h m_h(M)$ and $A_1 = \hat{B}$) to obtain two homomorphisms $e_i: \hat{B} \to D$ $(i = 1, 2)$ such that $e_1 \mid A = e_2 \mid A$ and $e_1(p) \neq e_2(p)$. Set $u_h = e_1 v_h$ if $h \neq h_2$, and set $u_{h_2} = e_2 v_{h_2}$. The family $\{u_h\}$ amalgamates M^*. Let $u: \hat{B} \to D$ be the extension of the u_h. Choose $p_i \in B_{h_i}$ $(i = 1, 2)$ such that $v_{h_i}(p_i) = p$. Then

$$e_1(p) = e_1 v_{h_1}(p_1) = u_{h_1}(p_1) = uv_{h_1}(p_1) = u(p).$$

Similarly, $e_2(p) = u(p)$. Therefore $e_1(p) = e_2(p)$, contradicting $e_1(p) \neq e_2(p)$. ■

To show that (2.10) implies (2.9), we let $g_i: A_1 \to B_i$ $(i = 1, 2)$ be isomorphisms. Let

$$\{v_i: B_i \to \hat{B} \mid i = 1, 2\}$$

be the free product of the B_i with amalgamation of $\{g_1 \mid A, g_2 \mid A\}$. Finally, let $e_i = v_i g_i$ $(i = 1, 2)$. Since $p \notin A$, (2.10) shows that $e_1(p) \neq e_2(p)$. ■

We now let A^* be the class of all (locally finite) polyadic algebras with equality (and with a fixed infinite set of variables). In this context one must understand "homomorphism" to mean "equality homomorphism." If $B^* = \{B_h \mid h \in H\}$ is a family of members of A^*, the A^*-free product of B^* does not always exist. An obvious necessary condition for its existence is that the subalgebra A_h of B_h generated by (the range of) the equality of B_h be independent of h to within isomorphism. This condition is also sufficient. To show this we use Lemma 3.1 of [6], which can be stated as follows.

(2.11) LEMMA. *Let* $\{C_h \mid h \in H\}$ *be a family of subalgebras of a polyadic algebra* C. *If the union of this family generates* C, *and if* E *is a common equality for the* C_h, *then* E *is an equality for* C.

From this lemma it follows that if, for each h, $m_h: A \to A_h$ is an isomorphism, then the A^*-free product of B^* is the free product $\hat{B}$ of B^* (with respect to the class of all polyadic algebras) with amalgamation of the m_h.

From these considerations there follow (equivalent) obvious equality versions of (2.9) and (2.10).

3. FREE POLYADIC ALGEBRAS

In the remaining two sections we shall confine ourselves to equality algebras, in order to avoid certain technicalities connected with operations and constants. Accordingly, all polyadic homomorphisms considered preserve equality.

Let $\sigma: A \to \bar{A}$ be an homomorphism, P an n-ary predicate of A, T an n-ary operation of A, and c a constant of A (n a positive integer). In [6, Section 1] we defined the images σP, σT and σc by the equations

$$(3.1) \qquad (\sigma P)(i_1, \cdots, i_n) = \sigma[P(i_1, \cdots, i_n)],$$

$$(3.2) \qquad \sigma[S(T(i_1, \cdots, i_n)/J)p] = S[(\sigma T)(i_1, \cdots, i_n)/J]\sigma p,$$

$$(3.3) \qquad \sigma[S(c/J)p] = S(\sigma c/J)\sigma p,$$

in which $J \subset I$, $p \in A$, and $(i_1, \cdots, i_n) \in I^n$.

Let V be a set of predicates and W a set of operations and constants of a (polyadic) algebra A. Assume that A is generated by V together with W, in other words, that A has no proper (equality) subalgebra containing the ranges of all predicates in V and closed under all operations and constants in W.

(3.4) *Definition.* A *mapping* of the union of V and W into an algebra B is a function that assigns to an n-ary predicate in V an n-ary predicate of B; to an n-ary operation in W, an n-ary operation of B; and to a constant in W, a constant of B.

(3.5) *Definition.* The union of V and W is said to *generate* A *freely* if any mapping of it into any polyadic algebra B is induced by an homomorphism of A into B (in the sense of (3.1), (3.2) and (3.3)). The algebra A is said to be *free* if it admits a set of predicates, operations, and constants that generates it freely.

The existence of free algebras can be established by adapting an argument that works more generally for classes of operational systems closed under the formation of subsystems and direct products. For notational convenience we state and prove the existence theorem only for a simple special case. The unicity is obvious.

(3.6) THEOREM. *There exists an algebra* A *freely generated by a monadic predicate* P, *a binary operation* T, *and a constant* c.

Proof. Let K be a set indexing a set of representatives of all equivalence classes of quadruples (A_1, P_1, T_1, c_1), where A_1 is an algebra generated by a monadic predicate P_1, a binary operation T_1, and a constant c_1. (Two quadruples (A_1, P_1, T_1, c_1) and (A_2, P_2, T_2, c_2) are declared equivalent if and only if there exists an isomorphism $\sigma\colon A_1 \to A_2$ such that $\sigma P_1 = P_2$, $\sigma T_1 = T_2$ and $\sigma c_1 = c_2$.) Let (A_k, P_k, T_k, c_k) be such a typical representative. Let (A, P, T, c) be obtained as follows: A is the subalgebra of the direct product of all A_k generated by P, T and c; and P, T and c are defined in an obvious fashion (for instance, T is defined by the equation

$$S[T(i_1, i_2)/J]p = \{S[T_k(i_1, i_2)/J]p_k\},$$

where $p = \{p_k\}$ is in the direct product, $(i_1, i_2) \in I^2$, and $J \subset I$). It is easy to show that A is freely generated by P, T and c. Indeed, let f be a mapping from $\{P, T, c\}$ into an algebra B. To define the homomorphism $\sigma\colon A \to B$ such that $\sigma P = fP$, and so forth, we may assume that B is generated by fP, fT, and fc, and we may therefore identify (B, fP, fT, fc) with some (A_k, P_k, T_k, c_k). The desired σ is simply the projection homomorphism $A \to A_k$. ■

4. CRAIG'S INTERPOLATION THEOREM

The algebraic version of Craig's interpolation theorem [2, Theorem 5, p. 267] is the following statement.

(4.1) THEOREM. *Let* V *and* W *be respectively a set of predicates and a set of operations and constants freely generating an (equality) polyadic algebra* A. *Let* V_1 *and* V_2 *be two subsets of* V *whose union is* V, *and let* V_0 *be the intersection of* V_1 *and* V_2. *Finally, for* $i = 0, 1, 2$, *let* A_i *be the (equality) subalgebra of* A *generated by* V_i *and* W. *Then, if* $a_1 \leq a_2$ *with* $a_1 \in A_1$ *and* $a_2 \in A_2$, *there exists an element* $a_0 \in A_0$ *such that* $a_1 \leq a_0 \leq a_2$.

Our proof is similar to an argument in logic showing that A. Robinson's consistency lemma implies Craig's theorem; the role of Robinson's lemma is here played by Theorem (2.6).

(4.2) LEMMA. *Let* B_0, B_1, B_2, B *be Boolean algebras such that* $B_1 \subset B$, $B_2 \subset B$, *and* $B_0 \subset B_1 \cap B_2$. *Assume that* $a_1 \leq a_2$, *with* $a_1 \in B_1$ *and* $a_2 \in B_2$, *and assume there is no* $a_0 \in B_0$ *such that* $a_1 \leq a_0 \leq a_2$. *Then there exist maximal ideals* N_1 *and* N_2 *of* B_1 *and* B_2, *respectively, such that* $N_1 \cap B_0 = N_2 \cap B_0$, $a_1 \notin N_1$, *and* $a_2 \in N_2$.

Proof. Let us introduce certain proper ideals of B_0, B_1, and B_2: We denote by $N_2^{(1)}$ the ideal of B_2 generated by a_2, and we define $N_0^{(1)} = N_2^{(1)} \cap B_0$. The proof will be complete when we have established the existence of a proper maximal ideal N_1 of B_1 such that $N_0^{(1)} \subset N_1$ and $a_1' \in N_1$, and of a proper maximal ideal N_2 of B_2 including $(N_1 \cap B_0) \cup N_2^{(1)}$.

If N_1 could not be chosen proper, it would follow that $a_0 \vee a_1' = 1$ for some $a_0 \in B_0$ such that $a_0 \leq a_2$, and hence that $a_1 \leq a_0 \leq a_2$, contrary to our assumption. As for N_2, if $s \vee a_2 = 1$ for some $s \in B_0$, then $s' \leq a_2$ and hence $s' \in N_0^{(1)} \subset N_1$ and $s \notin N_1$. It follows that $(N_1 \cap B_0) \cup N_2^{(1)}$ generates a proper ideal of B_2.

Since $N_1 \cap B_0$ is a maximal ideal of B_0 included in N_2, it is obvious that $N_1 \cap B_0 = N_2 \cap B_0$. Moreover, $a_1 \notin N_1$ and $a_2 \in N_2$. ■

(4.3) LEMMA. *Let* V, W, V_i *and* A_i (i = 0, 1, 2) *be as in Theorem* (4.1). *For each* i, *let* B_i *be the Boolean algebra of closed elements of* A_i. *Then, if* $a_1 \leq a_2$, *with* $a_1 \in B_1$ *and* $a_2 \in B_2$, *there exists an* $a_0 \in B_0$ *such that* $a_1 \leq a_0 \leq a_2$.

Proof. Let B be the Boolean algebra of closed elements of A. Then B_0, B_1, B_2, B, a_1, and a_2 satisfy the hypotheses of Lemma (4.2). We may therefore let N_1 and N_2 be as in the conclusion of (4.2) and set $N_0 = N_1 \cap B_0$. We apply (2.6), letting m_i (i = 1, 2) be the "natural" monomorphism $A_0/N_0 \to A_i/N_i$. (Division of a polyadic algebra by an ideal of its algebra of closed elements means the same thing as division by the generated polyadic ideal.) We obtain monomorphisms w_i: $A_i/N_i \to D$ that amalgamate the m_i. Composed with the natural homomorphisms n_i: $A_i \to A_i/N_i$, these w_i yield (because of the amalgamation) a well-defined mapping of $V \cup W$ in D. This mapping extends to an homomorphism ρ: $A \to D$ such that $\rho(a_1) = 1$ and $\rho(a_2) = 0$, contrary to the hypothesis $a_1 \leq a_2$. ■

(4.4) LEMMA. *If* A *is a free algebra, then any extension* C *obtained by fixing the new variables in a dilation of* A *is free.*

Proof. Let $V \cup W$ generate A freely, and let K be the set of fixed new variables. Then $V \cup (W \cup K)$ generates C. Let f be a mapping of $V \cup (W \cup K)$ into an algebra D, and let σ: $A \to D$ be the homomorphism that induces $f \mid (V \cup W)$. The generic element p of C is of the form $P(i_1, \cdots, i_n, k_1, \cdots, k_m)$, where P is an (n + m)-ary predicate of A, m and n are positive integers, $(i_1, \cdots, i_n) \in I^n$, and $(k_1, \cdots, k_m) \in K^m$. The homomorphism σ can be extended to an homomorphism σ: $C \to D$ that induces f by setting $\sigma p = (\sigma P)(i_1, \cdots, i_n, fk_1, \cdots, fk_n)$. ■

Proof of Theorem (4.1). Let $\{i_1, \cdots, i_n\}$ be a common support for a_1 and a_2, and let C be obtained from A by the adjunction of n new fixed variables $k_1, \cdots, k_n$. Let P_1 and P_2 be n-ary predicates of A_1 and A_2, respectively, such that $a_1 = P_1(i_1, \cdots, i_n)$ and $a_2 = P_2(i_1, \cdots, i_n)$. Then $P_1(k_1, \cdots, k_n) \leq P_2(k_1, \cdots, k_n)$. By virtue of Lemmas (4.4) and (4.3) applied to C instead of B, there exists a closed element b_0 of C_0 (the subalgebra of C generated by $V_0 \cup [W \cup \{k_1, \cdots, k_n\}]$) such that $P_1(k_1, \cdots, k_n) \leq b_0 \leq P_2(i_1, \cdots, i_n)$. The element b_0 has the form $P_0(k_1, \cdots, k_n)$, where P_0 is some n-ary predicate of A_0. It follows that $a_1 \leq a_0 \leq a_2$, where $a_0 = P_0(i_1, \cdots, i_n)$. ■

To conclude, we remark that since A is obviously the free product of A_1 and A_2 with amalgamation of A_0, $A_0 = A_1 \cap A_2$ by virtue of Theorem (2.10). Of course, this could easily be proved directly.

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