

SOME REMARKS ON SET THEORY, IX.
COMBINATORIAL PROBLEMS IN MEASURE THEORY
AND SET THEORY

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To the memory of our friend and collaborator, J. Czijszer

1. INTRODUCTION

A well-known theorem of Ramsey [12; p. 264] states that if the k -tuples of an infinite set S are split into a finite number of classes, then there exists an infinite subset of S all of whose k -tuples belong to the same class. (For $k = 1$, this is trivial.)

Suppose that with each element x of an infinite set S there is associated a measurable set $F(x)$ in the interval $[0, 1]$. It is known that if the measure $m(F(x))$ of the sets $F(x)$ are bounded away from zero, then some real number c is contained in infinitely many sets $F(x)$. For the sake of completeness, we prove this.

It clearly suffices to consider the case where S is the set of natural numbers. For each t in S , let

$$G_t = \bigcup_{n=t}^{\infty} F(n) \quad \text{and} \quad G = \bigcap_{t=1}^{\infty} G_t,$$

where $m(F(n)) \geq u > 0$ for $n \in S$. Clearly, $m(G_t) \geq u$ and $G_{t+1} \subset G_t$ ($t = 1, 2, \dots$) (throughout the paper, the symbol \subset refers to inclusion in the broad sense). Thus, by a classical theorem of Lebesgue, $m(G) \geq u$. Since each c in G is contained in infinitely many sets $F(t)$, this completes the proof.

Now, in analogy to Ramsey's theorem, one might consider the following problem. Suppose that, for some $u > 0$, there is associated with each k -tuple $X = \{x_1, \dots, x_k\}$ of elements of an infinite set S a measurable set $F(X)$ of $[0, 1]$ such that $m(F(X)) \geq u$. Does there always exist an infinite subset S' of S such that the sets $F(X)$ corresponding to the k -tuples X of S' have a nonempty intersection? We study this and related questions. In the course of our investigation we are led to a surprising number of unsolved problems.

All of our results concern the case $k = 2$, but we shall state some problems for $k > 2$ as well.

Instead of choosing a measurable subset of $[0, 1]$ for every k -tuple of a set S , we could choose an abstract set having certain properties. Interesting problems of a new type then arise, which we discuss briefly in Section 4. There we investigate some purely graph-theoretical questions, and in particular we give a simple construction of graphs that contain no triangle and have arbitrarily high chromatic numbers.

2. NOTATION AND DEFINITIONS

We adopt the following notation:

cardinal numbers: a, b, m, n ;
ordinal numbers: $\alpha, \beta, \dots, \nu, \mu, \dots$;
nonnegative integers: i, j, k, ℓ, r, s, t ;
real numbers in $[0, 1]$: $c, u, v, u_1, u_2, \dots, \theta$;
abstract sets: S, X, Y ;
the cardinal number of S : \overline{S} ;
elements of sets: x, y, \dots ;
the least cardinal number greater than n : n^+ .

The symbols $[S]^a$ and $[S]^{<a}$ denote the classes of subsets of S that have cardinality a and less than a , respectively. If X and Y are disjoint sets, we write

$$[X, Y] = \{ (x, y) \mid x \in X \text{ and } y \in Y \}.$$

Let S be a set of power m , and let F denote a function that associates a measurable subset of $[0, 1]$ with each $X \in [S]^k$. For brevity, we shall say that F is a set-function on S of type k . (The symbol F will always denote a set-function.) Suppose $0 \leq u \leq 1$. If, for each $x \in [S]^k$, $m(F(X)) \geq u$ or $m(F(X)) > u$, we say that F is of order at least u or of order greater than u , respectively.

Let Z be a subset of $[S]^k$. If

$$\bigcap_{X \in Z} F(X) \neq \emptyset,$$

we say that Z possesses property \mathcal{P} (with respect to F).

With specific reference to the problems mentioned in Section 1, we introduce the following symbols.

$$(1) \quad (m, k, u) \Rightarrow n \quad \text{and} \quad (m, k, > u) \Rightarrow n$$

represent the respective statements: If $\overline{S} = m$ and if F is a set-function on S of type k and of order at least u (of order greater than u), then S has a subset S' , of cardinality n , such that $[S']^k$ possesses property \mathcal{P} . To say that a statement involving the symbol \Rightarrow is false, we replace \Rightarrow by $\not\Rightarrow$.

The symbolic statement

$$(2) \quad (m, u) \Rightarrow (n_1, n_2)$$

means that if $\overline{S} = m$ and F is a set-function on S , of type 2 and of order at least u , then there exist disjoint subsets S_1 and S_2 of S with cardinality n_1 and n_2 , respectively, such that $[S_1, S_2]$ possesses property \mathcal{P} . Instead of $(m, 2, u) \Rightarrow n$, we often write that S contains a complete graph of power n that has property \mathcal{P} (with respect to F).

The theorems in whose proofs we use the generalized continuum hypothesis are marked by an asterisk: (*).

3. THE CASE $m \leq \aleph_0$

THEOREM 1. *Suppose that $2 \leq r < \omega$. Then $(\aleph_0, 2, u) \Rightarrow r + 1$ if and only if $u > 1 - 1/r$.*

Proof. First we show the condition that $u > 1 - 1/r$ to be necessary. If $\theta \in (0, 1)$, let

$$(3) \quad \theta = \sum_{t=1}^{\infty} \frac{s_t}{r^t} \quad (0 \leq s_t < r)$$

be its r -ary expansion with infinitely many positive coefficients s_t . Let S be the set of positive integers. The desired set-function F of type 2 on S is defined as follows. If $1 \leq t_1 \neq t_2 < \omega$, then

$$(4) \quad \theta \in F(\{t_1, t_2\}) \quad \text{if and only if } s_{t_1} \neq s_{t_2}$$

in the r -ary expansion (3) of θ .

Clearly,

$$m(F(\{t_1, t_2\})) = 1 - \frac{1}{r},$$

and thus F is of order no less than $1 - 1/r$. On the other hand, S does not contain a complete graph of power $r + 1$ that has property \mathcal{P} . For if $S' = \{t_1, \dots, t_{r+1}\}$ and $[S']^2$ possesses property \mathcal{P} , then there exists a $\theta \in (0, 1)$ such that

$$\theta \in \bigcap_{t_i, t_j \in S'; i \neq j} F(\{t_i, t_j\}).$$

Therefore, by (4), the numbers $s_{t_1}, \dots, s_{t_{r+1}}$ are all different, which contradicts

(3). This establishes the necessity of our condition.

We complete the proof of the theorem by proving not only the sufficiency of our condition but a stronger result as well; namely, we prove that corresponding to each $u > 1 - 1/r$, there exists an integer k_u such that

$$(k_u, 2, u) \Rightarrow r + 1.$$

Indeed, let k denote a positive integer, let $S = \{0, 1, \dots, k - 1\}$, and let F be a set-function on S , of type 2 and of order not less than u .

There is no loss of generality in supposing that $m(F(X)) = u$ for each $X \in [S]^2$. For if $m(F(X))$ were greater than u for some of the X , we could replace each of the sets $F(X)$ by a subset $F_1(X)$, of measure u . Clearly, a subset of $[S]^2$ having property \mathcal{P} relative to F_1 would also have property \mathcal{P} relative to F .

Suppose now that every point c of $(0, 1)$ lies in fewer than $u \binom{k}{2}$ of the sets $F(X)$. Then

$$\sum_{X \in [S]^2} m(F(X)) < u \binom{k}{2},$$

contrary to the hypothesis that F has order at least u . Hence some c lies in at least $u \binom{k}{2}$ of the sets $F(X)$. That is, the graph induced by some c has at least $u \binom{k}{2}$ edges, and of course the number h of its vertices is at most k . A special case of a theorem of P. Turán [14; p. 26] asserts that a graph with h vertices and more than $\frac{1}{2}(1 + \varepsilon - 1/r)h^2$ edges contains a complete $(r + 1)$ -gon. It follows that the graph induced by c contains a complete $(r + 1)$ -gon. This completes the proof of Theorem 1.

Let S be the set of natural numbers, and let F be a set-function on S , of type 2 and of order at least u . For each subset S' of S , we write

$$\Pi(S') = \bigcap_{X \in [S']^2} F(X).$$

The "if" part of Theorem 1 asserts that if $u > 1 - 1/r$, then some set S' of $r + 1$ natural numbers has property \mathcal{P} , that is, satisfies the condition $\Pi(S') \neq \emptyset$. The question now arises as to what can be said about the measure of $\Pi(S')$. We prove the following assertion, which provides a sharpening, for the special case $r = 2$, of Theorem 1.

THEOREM 1(A). *Let S be the set of natural numbers, and let F be a set-function on S , of type 2 and of order at least u ($u > 1/2$). Then, for every $\varepsilon > 0$, there exists a set S' of three natural numbers such that $m(\Pi(S')) \geq u(2u - 1) - \varepsilon$.*

This result is best possible for some special values of u , in the following sense: If $u = 1 - 1/k$ ($k = 3, 4, \dots$), then there exist set-functions F on S , of order u and of type 2, such that $m(\Pi(X)) \leq u(2u - 1)$ for every $X \subset S$ with $\overline{X} = 3$.

Remarks. It is obvious that Theorem 1(A) is a generalization of the special case $r = 2$ of Theorem 1. We do not know whether the positive part of this result is best possible for other values of u . As to the cases $r > 2$, we conjecture that if $u > 1 - 1/r$, then there exists a subset $S' \subset S$ with $\overline{S'} = r + 1$ for which

$$m(\Pi(S')) \geq u(2u - 1)(3u - 2) \cdots (ru - (r - 1)) - \varepsilon.$$

Here we also know that the result, if true, is best possible for certain special values of u .

Before proving Theorem 1(A), we state some well-known results that we shall often use in the sequel (see [5] and [9]).

(5) *To each $\varepsilon > 0$ and each positive integer r , there corresponds an integer $s_0(\varepsilon, r)$ with the following property. If $\{A_k\}$ ($1 \leq k \leq s_0(\varepsilon, r)$) is a family of measurable subsets of $[0, 1]$ and if $m(A_k) \geq u > 0$ for all k , then there exist r integers $k_1 < k_2 < \dots < k_r \leq s_0(\varepsilon, r)$ such that*

$$m\left(\bigcap_{i=1}^r A_{k_i}\right) > u^r - \varepsilon.$$

The following is an easy corollary.

(6) Let $\{A_k\}$ ($1 \leq k < \infty$) be a sequence of measurable subsets of $[0, 1]$, let $m(A_k) \geq u > 0$, and let $\varepsilon > 0$. Then, corresponding to each positive integer r , there exists an increasing sequence $\{h_j\}$ of integers such that

$$m\left(\bigcap_{i=1}^r A_{k_i}\right) > u^r - \varepsilon$$

for every set $\{k_i\}$ ($1 \leq i \leq r$) taken from $\{h_j\}$.

Now we outline the proof of Theorem 1(A). Let S be the set of natural numbers, let F be a set-function on S satisfying the requirements of Theorem 1(A), and let $\varepsilon > 0$. Without loss of generality, we may assume that $m(F(X)) = u$ for each $X \in [S]^2$.

First we define a partition

$$[S]^3 = J_1 \cup J_2 \cup J_3 \cup J_4$$

as follows. For each $X = \{t_1, t_2, t_3\}$ ($t_1 < t_2 < t_3$) we put

$$F_1(X) = F(\{t_1, t_2\}) \cap F(\{t_1, t_3\}),$$

$$F_2(X) = F(\{t_1, t_3\}) \cap F(\{t_2, t_3\}),$$

and we write

$$(7) \quad \left\{ \begin{array}{l} X \in J_1 \text{ if } m(F_1(X)) > u^2 - \varepsilon/2 \text{ and } m(F_2(X)) > u^2 - \varepsilon/2, \\ X \in J_2 \text{ if } m(F_1(X)) > u^2 - \varepsilon/2 \text{ and } m(F_2(X)) \leq u^2 - \varepsilon/2, \\ X \in J_3 \text{ if } m(F_1(X)) \leq u^2 - \varepsilon/2 \text{ and } m(F_2(X)) > u^2 - \varepsilon/2, \\ X \in J_4 \text{ if } m(F_1(X)) \leq u^2 - \varepsilon/2 \text{ and } m(F_2(X)) \leq u^2 - \varepsilon/2. \end{array} \right.$$

If $S_1 \subset S$ and $\bar{S}_1 = \aleph_0$, then S_1 contains triplets X_1 and X_2 such that

$$m(F_1(X_1)) > u^2 - \varepsilon/2 \quad \text{and} \quad m(F_2(X_2)) > u^2 - \varepsilon/2.$$

This is so because by (5) (with $r = 2$) the set S_1 contains no infinite subset all of whose triplets belong to the classes J_i ($i = 2, 3, 4$).

From Ramsey's theorem (see the beginning of the Introduction) it follows that all triplets of some infinite subset of S belong to J_1 . Let $S' = \{t_1, t_2, t_3\}$ ($t_1 < t_2 < t_3$) be any triplet in J_1 . Then, by the assumption that $m(F(\{t_1, t_2\})) = u$ and by the first line of (7),

$$m(\Pi(S')) > u^2 - \frac{\varepsilon}{2} - \left[u - \left(u^2 - \frac{\varepsilon}{2} \right) \right] = u(2u - 1) - \varepsilon.$$

This completes the proof of the first part of Theorem 1(A).

Now we prove the "best possible" part of Theorem 1(A). Let S be the set of natural numbers, and for any $k \geq 3$, consider the k -ary expansion (3) (with k in place of r) of an arbitrary $\theta \in [0, 1]$. Using again the idea of (4), we define $F(\{t_1, t_2\})$ (for $1 \leq t_1 \neq t_2 < \omega$) by the rule

$$(8) \quad \theta \in F(\{t_1, t_2\}) \quad \text{if and only if} \quad s_{t_1} \neq s_{t_2}.$$

Clearly, $m(F(\{t_1, t_2\})) = u = 1 - 1/k$. On the other hand, suppose that $X \in [S]^3$, $X = \{t_1, t_2, t_3\}$ ($t_1 < t_2 < t_3$). From well-known properties of the expansion (3) and from (8) it follows that

$$m(\Pi(X)) = \frac{k(k-1)(k-2)}{k^3} = (1 - 1/k)(1 - 2/k) = u(2u - 1).$$

This completes the proof of Theorem 1(A).

THEOREM 1(B).

$$\left(\aleph_0, 2, > 1 - \frac{1}{r} \right) \not\Rightarrow r + 1 \quad 2 \leq r < \omega.$$

We only outline the proof. First we establish the following result.

(9) *Let S be the set of natural numbers. Corresponding to each pair t_1, t_2 ($1 \leq t_1 \neq t_2 < \omega$) and each $\varepsilon > 0$, one can define a set function $F_{\{t_1, t_2\}}$ on S , of type 2 and satisfying the following conditions:*

- (a) $\Pi(Z) = \emptyset$ for every $Z \in [S]^{r+1}$,
- (b) $m(F_{\{t_1, t_2\}}(X)) = 1 - \frac{1}{r}$ for every $X \in [S]^2$ except $X = \{t_1, t_2\}$,
- (c) $m(F_{\{t_1, t_2\}}(\{t_1, t_2\})) > 1 - \varepsilon$.

This can be proved, by a slight modification of the construction used in the proof of Theorem 1, as follows.

Let ℓ be an integer, put $k = \ell r$, and for any $\theta \in [0, 1]$, let

$$\theta = \sum_{t=1}^{\infty} \frac{\tau_t}{k^t} \quad (0 \leq \tau_t < k).$$

For $t \in S$ and $i = 0, \dots, r-1$, we now define a set $S_{t,i}$ as follows. If $t \neq t_1$ and $t \neq t_2$, then $S_{t,i}$ is the set of natural numbers s satisfying the condition $\ell i \leq s < \ell(i+1)$; for the other cases,

$$\begin{aligned} S_{t_1,0} &= \{0, 1, \dots, (\ell-1)r\}, \\ S_{t_1,i} &= \{(\ell-1)r + i\} \quad \text{for } i = 1, \dots, r-1, \\ S_{t_2,i} &= \{i\} \quad \text{for } i = 0, \dots, r-2, \end{aligned}$$

$$S_{t_2, r-1} = \{r - 1, \dots, \ell r - 1\}.$$

Now we define $F(\{t, t'\})$ for $1 \leq t \neq t' < \omega$ by the stipulation that $\theta \in F(\{t, t'\})$ if and only if s_t and $s_{t'}$ belong to sets $S_{t,i}$ and $S_{t',i'}$ with $i \neq i'$.

F clearly satisfies the requirements (a) and (b) of (9).

On the other hand,

$$m(F_{\{t_1, t_2\}}(\{t_1, t_2\})) \geq \frac{1}{k^2} (k - r)^2 \geq 1 - \frac{2}{\ell} > 1 - \varepsilon$$

if ℓ is sufficiently large.

Now let $\{X_j\}$ ($j < \omega$) be a well-ordering of type ω of the set $[S]^2$. It follows from (9) that, corresponding to every $j < \omega$, there exists a set-function F_{X_j} on S that satisfies the following conditions:

$$(10) \quad F_{X_j}(X) \subset (2^{-j-1}, 2^{-j}) \quad \text{for every } X \in [S]^2;$$

the set $\Pi(Z)$ (defined with respect to F_{X_j}) is empty for every $Z \in [S]^{r+1}$;

$$m(F_{X_j}(X)) = \left(1 - \frac{1}{r}\right) 2^{-j-1} \quad \text{for every } X \in [X]^2 \text{ except } X_j;$$

$$m(F_{X_j}(X_j)) > (1 - \varepsilon) 2^{-j-1}.$$

Next we define the set-function F on S , of type 2, by the condition

$$(11) \quad F(X) = \bigcup_{j < \omega} F_{X_j}(X) \quad \text{for every } X \in [S]^2.$$

We easily see from (10) and (11) that $\Pi(Z) = \emptyset$ for every $X \in [S]^{r+1}$, and that

$$m(F(X_j)) = 1 - \frac{1}{r} + \left(\frac{1}{r} - \varepsilon\right) 2^{-j-1} > 1 - \frac{1}{r}$$

if $\varepsilon < \frac{1}{r}$. Hence F is of order greater than $1 - \frac{1}{r}$, and this proves Theorem 1(B).

The idea of the proof is partly due to J. Czipser.

Let $m_j = m(F(X_j)) - \left(1 - \frac{1}{r}\right)$ for $j < \omega$, and write $m = \sum_{j=0}^{\infty} m_j$. In the case of the example just constructed, $m > 1/r - \varepsilon$. We do not know how far this inequality can be improved; we only have some special results which show that if m is sufficiently large for a set-function F on S , of type 2 and of order greater than $1 - 1/r$, then there always exists a complete $(r + 1)$ -gon with the property \mathcal{P} . We omit the proof of this, and we only mention that questions of this type lead to interesting problems in measure theory.

THEOREM 2. *If u is positive, then $(\aleph_0, u) \Rightarrow (r, \aleph_0)$ for each nonnegative integer r .*

Proof. We are given a set S with cardinality \aleph_0 . Without loss of generality we suppose that $S = \{t \mid t < \omega\}$. Let F be a set-function on S , of type 2 and of order at

least u . We shall prove that, in fact, to each r and u ($u > 0$), there corresponds an integer $s = s(u, r)$ with the following property. Amongst any s integers t_1, \dots, t_s , there exist r integers t_{i_1}, \dots, t_{i_r} such that an infinite subset S' of S exists for which $[\{t_{i_1}, \dots, t_{i_r}\}, S']$ possesses property \mathcal{P} .

If s is a positive integer, we let $Z = \{t_1, \dots, t_s\}$, and for some t not in Z , we consider the sets $F(\{t_i, t\})$ ($1 \leq i \leq s$). Let δ be a positive number less than u^r . It follows from (5) that if s is sufficiently large, say $s > s_0(u^r - \delta, r)$, then there exist r vertices t_{i_1}, \dots, t_{i_r} among the t_i for which

$$m = m\left(\bigcap_{j=1}^r F(\{t_{i_j}, t\})\right) > \delta.$$

Since there are infinitely many $t \notin Z$ but only $\binom{s}{r}$ possible choices of indices i_1, \dots, i_r , some set of indices, say $\{i_1, \dots, i_r\}$, corresponds to infinitely many t . Denote this set of t 's by S'' . Then S'' is a subset of S of power \aleph_0 .

Let

$$E_t = \bigcap_{j=1}^r F(\{t_{i_j}, t\}) \quad (t \in S'').$$

Since $m(E_t) > \delta$, the theorem proved in the Introduction guarantees the existence of a denumerable subset S' such that

$$\bigcap_{t \in S'} E_t \neq \emptyset.$$

But this means that $[\{t_{i_1}, \dots, t_{i_r}\}, S']$ has property \mathcal{P} . This proves Theorem 2.

The question may now be asked: if u is positive, is the statement

$$(\aleph_0, u) \Rightarrow (\aleph_0, \aleph_0)$$

true? We were not, in general, able to answer this question, which is one of the most interesting unsolved problems of our paper. We describe a simple example by means of which J. Czipser showed that the answer is negative if $u < 1/2$. Let S be the set of natural numbers, and let $2 \leq r < \omega$. Czipser defined a set-function F_r^* of type 2 on S as follows. If (t_1, t_2) is any pair with $1 \leq t_1 < t_2 < \omega$, and if $\{s_t\}$ denotes the sequence of digits in the nonterminating r -ary expansion of a number θ in $(0, 1]$, then

$$(12) \quad \theta \in F_r^*(\{t_1, t_2\}) \quad \text{if and only if} \quad s_{t_1} > s_{t_2}.$$

Clearly, $m(F_r^*(X)) = \frac{1}{2} \left(1 - \frac{1}{r}\right)$; hence F_r^* is of order at least $\frac{1}{2} \left(1 - \frac{1}{r}\right)$. Since $\frac{1}{2} \left(1 - \frac{1}{r}\right) \rightarrow \frac{1}{2}$, we only need to show that if S', S'' are disjoint infinite subsets of S , then $[S', S'']$ does not possess property \mathcal{P} with respect to F_r^* for $2 \leq r < \omega$. In

fact, if S' and S'' are disjoint infinite subsets of S , then there exists an infinite increasing sequence $\{t_k\}$ of natural numbers such that $t_k \in S'$ if k is odd and $t_k \in S''$ if k is even, and $[S', S'']$ does not possess property \mathcal{P} with respect to F_r^* since the set of edges $\{t_i, t_{i+1}\}$ ($1 \leq i \leq k$) also fails to possess property \mathcal{P} for $k \geq r$.

Czipszer's example leads to some interesting new questions. First we need some definitions.

Let S be the set of natural numbers, let $T_r = \{t_1, \dots, t_{r+1}\}$ be a sequence of $r + 1$ natural numbers, and let $T_\infty = \{t_1, \dots, t_r, \dots\}$ be an infinite sequence of different natural numbers. Put

$$J_{r+1} = \{\{t_i, t_{i+1}\}\} \quad (1 \leq i \leq r), \quad J_\infty = \{\{t_i, t_{i+1}\}\} \quad (1 \leq i \leq \omega).$$

Further, let F be a set-function defined on S , of type 2 and of order at least u . We briefly say that S contains a path J_{r+1} of length $r + 1$ (with property \mathcal{P}) or an infinite path J_∞ (with property \mathcal{P}) if there exists a T_r or a T_∞ such that the corresponding sets J_{r+1} or J_∞ possess property \mathcal{P} (with respect to F), respectively. If in addition the sequences T_r or T_∞ are increasing, we say that S contains an increasing path of length $r + 1$ or an increasing infinite path, respectively. We do not know under what conditions on u the set S contains an infinite path. Perhaps this is the simplest unsolved problem in our paper.

Now Czipszer's set-functions F_r^* show that for $u < 1/2$ the set S need not contain an infinite increasing path, and more generally, that with respect to a set-function of type 2 and order at least $\frac{1}{2} \left(1 - \frac{1}{r}\right)$, S need not contain an increasing path of length $r + 1$. The question arises whether this is best possible in u . It may be true that if $u \geq 1/2$ then there exists an infinite increasing path, or that if $u > \frac{1}{2} \left(1 - \frac{1}{r}\right)$ then there exists an increasing path of length $r + 1$, respectively. We can prove this only for $r = 2$.

The character of a problem concerning increasing paths is somewhat different from that of the problems treated so far in our paper; for the problem is meaningful only if the basic set S is an ordered set, and the answer depends not only on the power of S , but also on its order type.

Now we give our result concerning the case $r = 2$.

THEOREM 3. *Let S be the set of natural numbers, and let F be a set-function defined on S , of type 2 and of order at least u . If $u > 1/4$, then there exists an increasing path I_3 with property \mathcal{P} . For $u \leq 1/4$, this is not necessarily true.*

We do not know what happens in case F is merely required to be of order greater than $1/4$.

Proof. The negative part of our theorem is shown by the set-function F_2^* defined in (12). Consider now a set-function satisfying the requirements of Theorem 3.

For $t = 1, 2, \dots$, define

$$(13) \quad E_t = \bigcup_{t < t' < \omega} F(\{t, t'\}) \quad \text{and} \quad m_t = m(E_t),$$

and let $0 < \varepsilon < u - 1/4$. There exists a real number m and an infinite subset $S' \subset S$ such that $|m - m_t| < \varepsilon/2$ for $t \in S'$.

By (5) and (13), there exist t_1 and t_2 ($1 \leq t_1 < t_2 < \omega$, $t_1, t_2 \in S'$) such that

$$m(E_{t_1} \cap E_{t_2}) > m^2 - \varepsilon/2.$$

Now $F(\{t_1, t_2\}) \subset E_{t_1}$, and $m + \varepsilon/2 < m^2 - \varepsilon/2 + u$, since $\varepsilon < u - 1/4 < m^2 - m + u$.

Hence

$$m(F(\{t_1, t_2\}) \cap E_{t_2}) > 0,$$

and therefore

$$F(\{t_1, t_2\}) \cap E_{t_2} \neq \emptyset.$$

Thus, by (13),

$$F(\{t_1, t_2\}) \cap F(\{t_2, t_3\}) \neq \emptyset \quad \text{for some } t_3 > t_2.$$

By the definition of a path with property \mathcal{P} , this completes the proof of Theorem 3.

Theorem 3 implies immediately that each infinite subset S' of S contains an increasing path J_3 . Now there are two kinds of nonincreasing paths J_3 : either $t_2 < t_1, t_3$, or else $t_2 > t_1, t_3$. It follows from (5) that each infinite subset S' of S contains nonincreasing paths J_3 of both kinds, for each $u > 0$, and that each infinite subset S' of S contains two elements $X, Y \in [S]^2$, with $X = \{t_1, t_2\}$, $Y = \{t_3, t_4\}$, and $X \cap Y = \emptyset$, such that $F(X) \cap F(Y) \neq \emptyset$ for each prescribed ordering of t_1, t_2, t_3, t_4 . With a partition of $[S]^3$ and $[S]^4$ similar to the partition we used in the proof of Theorem 1(A), we can (by applying Ramsey's theorem) prove the following result.

THEOREM 4. *Let S be the set of natural numbers, and let F be a set-function on S , of type 2 and of order at least u with $u > 1/4$. Then there exists an infinite subset S' of S such that $F(X) \cap F(Y) \neq \emptyset$ for every pair $X, Y \in [S]^2$. The condition $u > 1/4$ is necessary.*

We omit the proof.

Here we may ask the following question. Let S again be the set of natural numbers, and let a system $Z \subset [S]^2$ of edges be called independent if $X \cap Y = \emptyset$ for every pair $X \neq Y$ ($X, Y \in Z$). Is it true that if F is a set-function on S , of type 2 and of order at least u ($u > 0$), then S contains an infinite subset S' such that each independent system $Z \subset [S']^2$ possesses property \mathcal{P} ?

We know that there always exists an infinite subset S' satisfying the weaker condition that every independent system $Z \subset [S']^2$ of edges possesses property \mathcal{P} provided $Z \leq 3$. This can be shown similarly to Theorem 4.

4. THE ABSTRACT CASE

In this section, S always denotes the set of natural numbers.

We say that F is an *abstract set-function of type 2*, provided F associates with each $X \in [S]^2$ a subset $F(X)$ of a fixed set H , and that F possesses property $\mathcal{A}(k)$ if

$$\bigcap_{i=1}^k F(X_i) \neq \emptyset$$

for every sequence $\{X_i\}$ ($1 \leq i \leq k$) in $[S]^2$.

A set-function F of type 2 and of order at least u ($u > 1 - 1/k$) obviously is an abstract set-function with property $\mathcal{A}(k)$. The following result shows that in the positive theorems proved in Section 3, the assumption that F is of order at least u ($u > 1 - 1/k$) can not be replaced by the corresponding assumption that F possesses property $\mathcal{A}(k)$. However, some weaker results hold. We state two of them without proof.

THEOREM 5. (a) *Suppose that F is an abstract set function with property $\mathcal{A}(k)$ for some k ($3 \leq k < \omega$). Then there exists an infinite subset S' of S such that each nonincreasing path $I_3 \subset [S']^2$ has property \mathcal{P} .*

(b) *There exists an abstract set-function F , possessing property $\mathcal{A}(3)$, such that no increasing path I_3 of S has property \mathcal{P} with respect to F .*

We shall now describe some graph-theoretic constructions suggested by these considerations. Let \mathcal{G} be a graph, and let G denote the set of vertices of \mathcal{G} . A subset G' of G is said to be a *free subset* of \mathcal{G} if no two vertices belonging to G' are connected by an edge in \mathcal{G} . The graph \mathcal{G} is said to have *chromatic number* n provided n is the least cardinal number such that G is the sum of n free subsets.

A well-known result of Tutte [2] states that if n is an integer, then there exists a finite graph \mathcal{G} that contains no triangle and has chromatic number n . Several other authors have constructed such graphs and have given estimates for the minimal number of vertices of \mathcal{G} (see [4, p. 346] and [11]). In our next theorem, we shall give a construction for such graphs that we believe to be simpler than the previous ones; unfortunately, it does not give a very good estimate for the minimal number of vertices of \mathcal{G} .

It is sufficient to construct a graph \mathcal{G} that has chromatic number \aleph_0 and contains no triangle, since, by a theorem of N. G. de Bruijn and P. Erdős (see [1]), if every finite subgraph of a graph \mathcal{G} is r -chromatic, then \mathcal{G} is also r -chromatic. (In place of this argument, we could also use Ramsey's theorem.)

THEOREM 6. *Let $G = [S]^2$ ($S = \{1, 2, \dots\}$), and let the graph \mathcal{G} with the set G of vertices be defined by the rule that two distinct vertices $X = \{s_1, s_2\}$ and $Y = \{t_1, t_2\}$ ($1 \leq s_1 < s_2 < \omega$; $1 \leq t_1 < t_2 < \omega$) are connected if and only if either $s_2 = t_1$ or $t_2 = s_1$. Then \mathcal{G} contains no triangle, and its chromatic number is \aleph_0 .*

Proof. The first statement is trivial. Suppose that the second is false. Then $G = G_1 \cup \dots \cup G_k$, where k is finite and G_1, \dots, G_k are free sets in \mathcal{G} . Considering that $G = [S]^2$, we see from Ramsey's theorem that there exists an $S' \subset S$ ($\overline{S'} = \aleph_0 > 3$) such that $[S']^2 \subset G_i$ for some i ($1 < i < k$). Let $t_1, t_2, t_3 \in S'$. Then $X = \{t_1, t_2\}$, $Y = \{t_2, t_3\} \in G_i$, and X and Y are connected in \mathcal{G} , contrary to the assumption that G_i is a free set. This completes the proof of Theorem 6.

Generalizing Tutte's theorem, P. Erdős and R. Rado proved [8, p. 445] that if n is an infinite cardinal number, then there exists a graph \mathcal{G} that contains no triangle and has chromatic number n . Moreover, the graph constructed by them has n vertices. Their construction is not quite simple. Using the same idea as in the proof of Theorem 6 and applying a generalization of Ramsey's theorem, we can now give a very simple proof for a part of this result. Namely, we can similarly construct a graph \mathcal{G} that contains no triangle and has chromatic number n ; but the set of vertices of this graph is of power greater than n .

P. Erdős proved [3, p. 34-35] the following generalization of Tutte's theorem. If k and n are positive integers, then there exists a graph \mathcal{G} , of chromatic number at least n , that contains no circuit of length i for $3 \leq i < k$.

One could have believed that, in analogy with Tutte's theorem, this theorem also could be generalized for $n > \aleph_0$. Surprisingly, this is not so:

If a graph \mathcal{G} contains no circuit of length 4, then its chromatic number is at most \aleph_0 .

We shall publish the proof of this theorem in a forthcoming paper in which we shall also try to determine what kinds of subgraphs a graph \mathcal{G} of chromatic number greater than \aleph_0 must contain. A typical result: \mathcal{G} must contain an infinite path and an even graph $[S_0, S_1]$, where $\overline{S}_0 = r$, $\overline{S}_1 = \aleph_1$.

On the other hand, we prove the following generalization of the theorem of Erdős and Rado cited above.

THEOREM 7. *Let k be a positive integer, and let n be an infinite cardinal number. Then there exists a graph \mathcal{G} that has chromatic number at least n and contains no circuit of length $2i + 1$ for $1 \leq i \leq k$.*

In our construction, the set of vertices of \mathcal{G} is of power greater than n . We do not know whether there exist such graphs \mathcal{G} with $\overline{G} = n$. (*Added in proof:* Recently, we proved that such graphs exist for every n .)

We only outline the proof of Theorem 7. Let m be a cardinal number greater than n , and let ϕ denote the initial number of m . To define \mathcal{G} , we put $Z = \{\nu\}$ ($\nu < \phi$) and $G = [Z]^{k+1}$, and for arbitrary different elements

$$X = \{\nu_1, \dots, \nu_{k+1}\} \quad \text{and} \quad Y = \{\mu_1, \dots, \mu_{k+1}\}$$

$$(\nu_1 < \dots < \nu_{k+1}; \mu_1 < \dots < \mu_{k+1})$$

of G , we let X and Y be connected in \mathcal{G} if and only if either

$$\nu_2 = \mu_1, \nu_3 = \mu_2, \dots, \nu_{k+1} = \mu_k$$

or

$$\mu_2 = \nu_1, \mu_3 = \nu_2, \dots, \mu_{k+1} = \nu_k.$$

The fact that \mathcal{G} contains no circuit of length $2i + 1$ for $1 \leq i \leq k$ is assured by a simple and essentially finite combinatorial theorem, which we omit.

Suppose now that the chromatic number of \mathcal{G} is less than n . Then $G = \bigcup_{\alpha < \psi} G_\alpha$, where $\overline{\psi} < n$ and where G_α is a free subset of \mathcal{G} for every $\alpha < \psi$. If m is chosen to be greater than

$$2^2 \cdot \dots \cdot 2^n \quad (k \text{ symbols } 2),$$

then, as a corollary of a generalization of Ramsey's theorem that was proved by P. Erdős and R. Rado (see [7, p. 567]), there exist a subset S' of S ($\overline{S}' = k + 2$) and an $\alpha < \phi$ such that $[S']^{k+1} \subset G_\alpha$.

Let $S' = \{\nu_1, \dots, \nu_{k+1}, \nu_{k+2}\}$ ($\nu_1 < \dots < \nu_{k+2}$). Then

$$X = \{\nu_1, \dots, \nu_{k+1}\}, Y = \{\nu_2, \dots, \nu_{k+2}\} \in G_\alpha,$$

and X, Y are connected in \mathcal{G} . This contradicts the assumption that G_α is free.

5. THE CASE $m = \aleph_1$ OR $m = 2^{\aleph_0}$

In this section we shall often refer to the partition symbol $m \rightarrow (b_\nu)_c^r$ introduced by P. Erdős and R. Rado [6, p. 428]. For the convenience of the reader we restate the definition.

Let m and c ($c \geq 2$) be cardinal numbers, let r be an integer ($r \geq 1$), let ϕ denote the initial number of c , and let (b_ν) ($\nu < \phi$) be a sequence of type ϕ of cardinal numbers.

The implication $m \rightarrow (b_\nu)_c^r$ means that if S is a set of power m and (J_ν) ($\nu < \phi$) is a partition of the set $[S]^r$ (that is, $[S]^r = \bigcup_{\nu < \phi} J_\nu$), then there exist a subset S' of S and a $\nu_0 < \phi$ such that $\overline{S}' = b_{\nu_0}$ and $[S']^r \subset J_{\nu_0}$. The expression $m \not\rightarrow (b_\nu)_c^r$ means that the above statement is false.

Several results concerning the symbolic statement $m \rightarrow (b_\nu)_c^r$ are proved in [6] and [7]. A forthcoming paper by P. Erdős, A. Hajnal, and R. Rado [5] will contain an almost complete discussion of the symbol.

Note that the problem of proving that $m \rightarrow (b_\nu)_c^r$ is a generalization of the problem settled by Ramsey's theorem. Indeed, Ramsey's theorem (see Section 1) asserts that if c is finite, then $\aleph_0 \rightarrow (\aleph_0, \dots, \aleph_0)_c^r$ (or, more precisely, that $\aleph_0 \rightarrow (b_\nu)_c^r$ provided c is finite and $b_\nu = \aleph_0$ for every $\nu < \phi$.)

Now we turn to our original problems. First we prove the following negative result.

(*) **THEOREM 8.** *If S is a set of power 2^{\aleph_0} , then there exists a set-function F on S , of type 2 and of order at least 1, such that no $Z_1 \subset [S]^2$ with cardinality greater than \aleph_0 possesses property \mathcal{P} ; that is, if $m = 2^{\aleph_0}$, there need not exist a graph that has at least \aleph_1 vertices and possesses property \mathcal{P} .*

Proof. Let $\{u_\nu\}$ ($\nu < \omega_1$) and $\{X_\nu\}$ ($\nu < \omega_1$) be well-orderings of type ω_1 of the sets $[0, 1]$ and $[S]^2$, respectively. For each $\nu < \omega_1$, we define

$$F(X_\nu) = \{u_\mu\} \quad (\nu \leq \mu < \omega_1).$$

Since each $F(X_\nu)$ has a denumerable complement, $m(F(X_\nu)) = 1$ if $\nu < \omega_1$, and thus F is of order at least 1. On the other hand, the intersection of any \aleph_1 of the sets $F(X_\nu)$ is obviously empty. This completes the proof.

A corollary of Theorem 8: if $0 \leq u \leq 1$, then $(2^{\aleph_0}, 2, u) \not\Rightarrow \aleph_1$ provided (*) is assumed. Without the generalized continuum hypothesis, we can prove only the following weaker result.

THEOREM 9. *If $u \leq 1/2$, then $(2^{\aleph_0}, 2, u) \not\Rightarrow \aleph_1$.*

Proof. Let S be a set of power 2^{\aleph_0} . By a result of Sierpiński [13], there exists a partition of $[S]^2$ such that the statements

$$[S]^2 = J_1 \cup J_2, \quad J_1 \cap J_2 = \emptyset, \quad S' \subset S, \text{ and } [S']^2 \subset J_j \quad (j = 1, 2)$$

imply that $\overline{S'} \leq \aleph_0$. (In terms of the partition relations implied by $m \rightarrow (b_\nu)_c^r$, this means that $2^{\aleph_0} \not\rightarrow (\aleph_1, \aleph_1)^2$.)

Now we define

$$F(X) = \begin{cases} (0, 1/2) & \text{if } X \in J_1 \\ (1/2, 1) & \text{if } X \in J_2. \end{cases}$$

This set-function obviously satisfies the requirement of our theorem.

If $u > 1/2$, the argument above is inconclusive. Now, the edges of a complete graph of power 2^{\aleph_0} can presumably be split into 2^{\aleph_0} disjoint classes in such a way that each subset of S of power \aleph_1 contains an edge from each class. This theorem has never been proved, not even for three classes, without the help of the generalized continuum hypothesis (*). A proof using (*) is given in [5]. If we could prove the theorem for r classes ($r > 2$) without using (*), then by following our proof of Theorem 9, we could clearly show that for each $r < \omega$,

$$(2^{\aleph_0}, 2, 1 - 1/r) \not\Rightarrow \aleph_1.$$

On the other hand, it is easy to see without using (*) that $(2^{\aleph_0}, 2, > 0) \not\Rightarrow 3$. To prove this, we let S be the interval $[0, 1]$, and we let $F(\{x, y\})$ be the open interval (x, y) . Obviously,

$$m(F(\{x, y\})) = |x - y| > 0,$$

and no triangle has the property \mathcal{P} .

Trees whose longest paths have length at most 2 are unions of stars. It is well known that every complete graph of power \aleph_1 is a countable sum of trees, in fact, a countable sum of trees that are unions of stars. Thus, if we assume the continuum hypothesis, then we can construct an $F(X)$ such that $m(F(X)) > 0$ and no graph containing a path of length 3 has property \mathcal{P} . For the sake of completeness we remark that if $c = \aleph_1$ and S is a set of power \aleph_1 , we can construct by the above remark a set-function on S , of positive order and type 2, so that no graph of power \aleph_1 and no path of length 3 has property \mathcal{P} .

Our only positive result in this section is the next theorem.

THEOREM 10. *If u is positive and $m > \aleph_0$, then $(m, 2, u) \Rightarrow \aleph_0$.*

Remark. If $u \leq 1/2$, we know by Theorem 9 that this result is best possible if $m \leq 2^{\aleph_0}$. If we assume (*), then, by Theorem 8, Theorem 10 is best possible for each $u \leq m$ and each $m \leq 2^{\aleph_0}$.

Proof. It is sufficient to prove that if $u > 0$, then $(\aleph_1, 2, u) \Rightarrow \aleph_0$.

Let S be a set of power \aleph_1 ; without loss of generality, we suppose that $S = \{\alpha\}$ ($\alpha < \omega_1$). Let F be a set-function on S , of type 2 and of order at least u . For brevity, we write

$$F(\{\alpha_1, \alpha_2\}) = F_{\alpha_1, \alpha_2} = F_{\alpha_2, \alpha_1}.$$

Let $X \circ Y$ denote the symmetric difference $X \cup Y - (X \cap Y)$ of the sets X and Y . The following theorem is well known (see [10, p. 168]).

THEOREM α . *There exists a denumerable sequence $\{E_s\}$ ($s < \omega$) of measurable subsets of $[0, 1]$ such that if E is a measurable subset of $[0, 1]$, then corresponding to each $\varepsilon > 0$ there exists an $s < \omega$ for which $m(E \circ E_s) < \varepsilon$.*

Applying Theorem α , we obtain the following result.

THEOREM β . *There exist an $\alpha_0 < \omega_1$ and a subset S_1 of S with cardinality \aleph_1 such that for each α and α' in S_1 ,*

$$m(F_{\alpha_0, \alpha} \circ F_{\alpha_0, \alpha'}) < \varepsilon.$$

Suppose now that $S_k \subset \dots \subset S_1$ and the elements $\alpha_0, \dots, \alpha_{k-1}$ are already defined for some k ($0 < k < \omega$) in such a way that S_k has power \aleph_1 . Then, if we apply Theorem α $k + 1$ times, we establish the following result.

THEOREM γ . *There exist an $\alpha_k \in S_k$ ($\alpha_k \neq \alpha_i$ if $i < k$) and a subset $S_{k+1} \subset S_k$ with cardinality \aleph_1 such that for each $i \leq k$ and for each α and α' in S_{k+1}*

$$m(F_{\alpha_i, \alpha} \circ F_{\alpha_i, \alpha'}) < \varepsilon 2^{-k-1}.$$

Thus by induction on k , α_k and S_{k+1} are defined for every $k < \omega$. Now let

$$G_k = \bigcap_{t=k+1}^{\infty} F_{\alpha_k, \alpha_t}.$$

For each $k < \omega$, it follows from Theorem β and Theorem γ that

$$m(G_k) \geq m(F_{\alpha_k, \alpha_{k+1}}) - \sum_{t=k}^{\infty} \varepsilon 2^{-k-1};$$

hence, if $0 < \varepsilon < u/2$, then

$$m(G_k) \geq u - \varepsilon > u/2.$$

Finally, the theorem proved in the Introduction enables us to conclude that there exists an infinite sequence $\{k_r\}$ ($r < \omega$) such that

$$\bigcap_{r=0}^{\infty} G_{k_r} \neq \emptyset.$$

Let $S' = \{\alpha_{k_r}\}$ ($r < \omega$). Then S' has power \aleph_0 , and $[S']^2$ possesses property \mathcal{P} . The proof of Theorem 10 is now complete.

6. THE CASE $m = \aleph_2$.

Throughout this and the next section we shall assume the generalized continuum hypothesis (*).

(*) THEOREM 11. $(\aleph_2, 2, > 0) \Rightarrow \aleph_0$.

Proof. Let S be a set of power \aleph_2 , and let F be a set-function on S , of type 2 and of positive order. We split the edges of the complete graph S into countably many classes J_t by stipulating that $X \in J_t$ if and only if for each $t < \omega$ and each $X \in [S]^2$

$$2^{-t-1} < m(F(X)) \leq 2^{-t}.$$

Since the order of $F(X)$ is positive,

$$[S]^2 = \bigcup_{t < \omega} J_t.$$

It follows from a theorem in [5] that $\aleph_2 \rightarrow (\aleph_1, \dots, \aleph_1, \dots)_{\aleph_0}^2$ (see the definition of $m \rightarrow (b_\nu)_c^x$ in Section 5). Hence, at least one of the graphs J_t contains a complete graph of power \aleph_1 ; that is, there exist a subset S' of S of power \aleph_1 and a $t_0 < \omega$ such that $[S']^2 \subset J_{t_0}$. Applying Theorem 10, with S' playing the role of S , we obtain the desired conclusion.

Theorem 11 is probably best possible. In fact, it seems likely that even if we were to assume that the order of F is at least 1, we could not deduce the existence of a complete subgraph of power \aleph_1 that has property \mathcal{P} . This question is connected with the following unsolved problem stated in [5].

Let S be a set of power \aleph_2 . Does there exist a partition of the complete graph S into disjoint sets J_ν ($\nu < \omega_1$) such that no countable union of J_ν 's contains a complete graph of power \aleph_1 ; that is, such that if S' is a subset of S with cardinality \aleph_1 , then $[S']^2 \cap J_\nu \neq \emptyset$ for at least \aleph_1 sets J_ν ?

Probably such a decomposition exists, but we have been unable to construct one. For the sake of the argument, assume that it exists. Let $\{u_\nu\}$ ($\nu < \omega_1$) be a well-ordering of type ω_1 of the interval $[0, 1]$, and define a set-function of type 2 on S by the condition

$$F(X) = \{u_\mu\} \quad (\nu < \mu < \omega_1) \quad \text{for } X \in [S]^2$$

if and only if $X \in J_\nu$ for each $\nu < \omega_1$. Obviously, F is of order at least 1. Moreover, if S' is a subset of power \aleph_1 of S , then F assumes \aleph_1 distinct values on $[S']^2$; hence, $[S']^2$ does not possess property \mathcal{P} .

7. THE CASE $m > \aleph_2$.

Under the assumption that $m > \aleph_2$, the connection between our problems and measure theory becomes tenuous, and the questions become purely set-theoretical. In this section we shall make heavy use of [5].

(*) THEOREM 12. *If $m = \aleph_{\alpha+1}$ and $\text{cf}(\alpha) > 1$, then $(m, 2, > 0) \Rightarrow \aleph_\alpha$.*

This theorem is a corollary of the following stronger proposition.

(*) THEOREM 12 (A). *If $m = \aleph_{\alpha+1}$, $\text{cf}(\alpha) > 1$; and if to each $X \in [S]^2$ there corresponds a nonempty subset of $[0, 1]$, then there exists a subset S' of S with cardinality \aleph_α and such that $[S']^2$ possesses property \mathcal{P} .*

Proof. Let $\{u_\nu\}$ ($\nu < \omega_1$) be a well-ordering of type ω_1 of $[0, 1]$. We define a partition of $[S]^2$ into sets J_ν ($\nu < \omega_1$) as follows. For each $X \in [S]^2$ and each $\nu < \omega_1$, X is in J_ν if and only if ν is the least ordinal number for which $u_\nu \in F(X)$.

A theorem of [5] states that, under the hypotheses of Theorem 12 (A),

$$\aleph_{\alpha+1} \rightarrow (\aleph_\alpha)^2 \aleph_1.$$

The next theorem implies that Theorem 12 (A) is best possible even if the complement of each set $F(X)$ consists of one element.

(*) THEOREM 13. *If $m = \aleph_\beta$ and if either β is of the first kind or β is of the second kind and $\aleph_{\text{cf}(\beta)}$ is not an inaccessible cardinal greater than \aleph_0 , then*

(a) $(\aleph_\beta, 2, 1) \not\Rightarrow \aleph_\beta$ if $\text{cf}(\beta) \neq 0$,

(b) for each $u < 1$, $(\aleph_\beta, 2, u) \not\Rightarrow \aleph_\beta$ if $\text{cf}(\beta) = 0$.

Moreover, in case (a) the desired set-function F can be chosen so that the complement of each $F(X)$ consists of exactly one element.

Proof. Let S be a set of power \aleph_β , and consider the case (a). A theorem in [5] implies that there exists a partition of $[S]^2$ into disjoint sets J_ν ($\nu < \omega_1$) such that if S' is a subset of S with cardinality \aleph_β , and if $\nu < \omega_1$, then $[S']^2 \cap J_\nu \neq \emptyset$. We let $\{u_\nu\}$ ($\nu < \omega_1$) be a well-ordering of $[0, 1]$ of type ω_1 , and for each $\nu < \omega_1$ and each $X \in [S]^2$ we define

$$F(X) = [0, 1] - \{u_\nu\} \quad \text{if } X \in J_\nu.$$

Clearly, the function F has the desired properties.

We now consider case (b). By virtue of Theorem 1, we may suppose that $\beta > 0$. Doing so, we choose an increasing sequence $\{\beta_t\}$ ($t < \omega$) of ordinal numbers less than β and cofinal with β , and for each $t < \omega$ we choose a subset S_t of S with cardinality \aleph_{β_t} so that the S_t are disjoint and

$$S = \bigcup_{t < \omega} S_t.$$

On the set $S^* = \{S_t\}$ ($t < \omega$) there exists by Theorem 1 a set-function F^* , of type 2 and of order at least u , such that if $S^{*'}$ is a subset of S^* of power \aleph_0 , then $[S^{*'}]^2$ does not possess property \mathcal{P} with respect to F^* .

For any $\{x, y\} \in [S]^2$, suppose that $x \in S_{t_1}$ and $y \in S_{t_2}$, and define $F(\{x, y\})$ as follows:

$$F(\{x, y\}) = (0, 1) \quad (t_1 = t_2),$$

$$F(\{x, y\}) = F^*(\{S_{t_1}, S_{t_2}\}) \quad (t_1 \neq t_2).$$

It is easy to verify that F has the desired properties. This completes the proof.

For the case where $\aleph_{\text{cf}(\beta)}$ is inaccessible and greater than \aleph_0 , the problem remains unsolved.

(*) THEOREM 14. *If $m = \aleph_\beta > \aleph_0$ is a limit cardinal and if $n < m$, then $(m, 2, > 0) \Rightarrow n$.*

It is a theorem in [5] that

$$m \rightarrow (n)_{\aleph_1}^2.$$

Both Theorem 12 and Theorem 14 follow from this. Moreover, just as in Theorem 12 (A), instead of assuming that F is of positive order, we can merely assume that $F(X)$ is nonempty for each $X \in [S]^2$. We omit the details.

The only cases we have not yet discussed are $m = \aleph_{\alpha+1}$, where $\alpha > 1$ and either $\text{cf}(\alpha) = 0$ or $\text{cf}(\alpha) = 1$.

(*) THEOREM 15. *If $m = \aleph_{\alpha+1}$, $\alpha > 0$, $\text{cf}(\alpha) = 0$, and $u > 0$, then*

$$(m, 2, u) \Rightarrow \aleph_\alpha.$$

Proof. Let S be a set of power $\aleph_{\alpha+1}$. Without loss of generality suppose that $S = \{\nu\}$ ($\nu < \omega_{\alpha+1}$). Let F be a set-function of S , of type 2 and of order at least u . We shall use methods employed in [5].

By the ramification method used there, we know that there exists an increasing sequence $\{\nu_\mu\}$ ($\mu < \omega_\alpha$) such that if $\mu < \mu' \leq \mu'' < \omega_\alpha$, then

$$(16) \quad F(\{\nu_\mu, \nu_{\mu'}\}) = F(\{\nu_\mu, \nu_{\mu''}\}).$$

We shall write $F_\mu = F(\{\nu_\mu, \nu_{\mu+1}\})$. Let $\{\alpha_t\}$ ($t < \omega$) be an increasing sequence of ordinal numbers less than α , cofinal with α and such that $\alpha_t > 2$ and \aleph_{α_t} is regular. Since $[0, 1]$ has only \aleph_2 subsets, it follows that corresponding to each $t < \omega$, there exist a μ_t and a set Z_t of ordinal numbers

$$\mu \quad (\omega_{\alpha_{t-1}} < \mu \leq \omega_{\alpha_t}; \alpha_{-1} = 0)$$

such that Z_t has power \aleph_{α_t} , and $F_\mu = F_{\mu_t}$ for each μ in Z_t . If $t < \omega$, then $m(F_{\mu_t}) \geq u > 0$; therefore, we conclude from the theorem proved in the Introduction that there exists an infinite subsequence $\{t_s\}$ ($s < \omega$) such that

$$\bigcap_{s < \omega} F_{\mu_{t_s}} \neq \emptyset.$$

Let

$$Z = \bigcup_{s < \omega} Z_{t_s}, \quad S' = \{\nu_\mu\} \ (\mu \in Z).$$

Clearly,

$$\overline{\overline{S'}} = \sum_{s < \omega} \aleph_{\alpha_{t_s}} = \aleph_\alpha,$$

and, by (16), $[S']^2$ possesses property \mathcal{P} . This completes the proof of the theorem.

We note that under the hypotheses of Theorem 15, $(m, 2, > 0) \not\Rightarrow \aleph_\alpha$. This is true because of a theorem in [5] which states that if $\text{cf}(\alpha) = 0$, then

$$\aleph_{\alpha+1} \not\leq (\aleph_\alpha)_{\aleph_0}^2.$$

That is, if S is a set of power $\aleph_{\alpha+1}$ ($\text{cf}(\alpha) = 0$), then there exists a partition of $[S]^2$ into disjoint sets J_t ($t < \omega$) such that if $t < \omega$, $S' \subset S$, and $[S']^2 \subset J_t$, then

$$\overline{\overline{S'}} < \aleph_\alpha.$$

For each $X \in [S]^2$ and each $t < \omega$, we define $F(X) = (2^{-t-1}, 2^{-t})$ if $X \in J_t$. For this F it is obvious that $(m, 2, > 0) \not\Rightarrow \aleph_\alpha$.

(*) THEOREM 16. *If $m = \aleph_{\alpha+1}$, $\text{cf}(\alpha) = 1$, and $\alpha > 1$, then*

(a) $(m, 2, 1) \not\Rightarrow \aleph_\alpha$,

(b) $(m, 2, > 0) \Rightarrow n$ ($n < \aleph_\alpha$).

Note that, in harmony with our remarks in the discussion of the case $m = \aleph_2$, we do not know whether or not $(m, 2, 1) \not\Rightarrow \aleph_\alpha$ is true if the condition $\alpha > 1$ is omitted.

Proof of Theorem 16. The conclusion (b) follows trivially from Theorem 14. To prove (a), we refer to the following theorem in [5]. *Let S be a set of power $\aleph_{\alpha+1}$, where $\text{cf}(\alpha) = 1$ and $\alpha > 1$. Then there exists a partition of $[S]^2$ into disjoint sets J_ν ($\nu < \omega_1$) such that if S' is a subset of S of power \aleph_α , then*

$$[S']^2 \cap J_\nu \neq \emptyset$$

for \aleph_1 sets J_ν .

We now let $\{u_\nu\}$ ($\nu < \omega_1$) be a well-ordering of type ω_1 of the interval $[0, 1]$, and we define a set-function F by the condition that for each $\nu < \omega_1$ and each $X \in [S]^2$,

$$F(X) = \{u_\mu\} \ (\nu < \mu < \omega_1) \quad \text{if } X \in J_\nu.$$

By analogy with the remark made after the proof of Theorem 11, it is easy to see that F has the desired properties.

8. THE CASE $k > 2$

We shall discuss this case only briefly. At present we cannot even settle the following question: Is it true that for each $u > 0$

$$(\aleph_1, 3, u) \Rightarrow 4?$$

Let k , m , and n be integers. It is an old problem of P. Turán's to determine the smallest integer $f(k, n, m)$ such that if

$$A_1, \dots, A_{f(k, n, m)}$$

are k -tuples formed from a set S of m elements, then there always exist n elements of S such that each k -tuple of these n elements is an A_i . As we stated earlier, Turán determined $f(2, n, m)$. For $k > 2$, the problem appears to be quite difficult. It is easy to show that

$$C_{k, n} = \lim_{m \rightarrow \infty} \frac{f(k, n, m)}{m^k}$$

exists. The results of Turán [2] imply that

$$0 < C_{k, n} < \frac{1}{k!} \quad \text{and} \quad C_{2, n} = \frac{1}{2} \left(1 - \frac{1}{n-1} \right);$$

but even the value of $C_{3, 4}$ is not known.

It is easy to deduce by the methods used to prove Theorem 1 that if $u > k! C_{k, n}$, then

$$(\aleph_0, k, u) \Rightarrow n.$$

This is no longer true if $u = k! C_{k, n}$.

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