DIFFERENTIABLE FIBRE SPACES AND MAPPINGS COMPATIBLE WITH RIEMANNIAN METRICS

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1. INTRODUCTION

P. A. Griffiths and I recently [3] characterized differentiable covering spaces in terms of mappings of Riemannian manifolds satisfying a condition on shrinking of tangent vectors. Here the characterization is extended to locally trivial differentiable fibre spaces by means of Ch. Ehresmann's notion of a connection for a mapping of maximal rank. Section 2 is an extension of Ehresmann's theory, culminating in Proposition 2.3 and its corollaries. Our main results are Theorems 3.5 and 3.6.

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2. THE EHRESMANN CONNECTION FOR A MAPPING OF MAXIMAL RANK

In this section we develop a mild extension of Ch. Ehresmann's theory [1] of connections and holonomy for differentiable fibre spaces, in order to establish terminology and because it does not seem to be in print.

2.1. *The holonomy system of the Ehresmann connection.* Let $\phi: E \to B$ be differentiable of rank dim B. Given $x \in E$, consider the subspace

$$V_x = \{X \in E_x: \phi_* X = 0\}$$

of the tangent space to E at x. The subspace V_x is called the *vertical space* at x, and $\dim V_x = \dim E - \dim B$. The *vertical distribution* is $\mathscr{V} = \{V_x\}_{x \in E}$. An *Ehresmann connection for* ϕ is a differentiable distribution $\mathscr{H} = \{H_x\}_{x \in E}$ on E that is complementary to $\mathscr{V}$. Then $E_x = V_x \oplus H_x$ for every $x \in E$, and ϕ induces a linear isomorphism of H_x onto $B_{\phi(x)}$. The space H_x *is the horizontal space* at x.

We fix an Ehresmann connection $\mathscr{H}$ for $\phi: E \to B$. A tangent vector $X \in E_x$ is *horizontal* (respectively, *vertical*) if $X \in H_x$ (respectively, $X \in V_x$); a sectionally smooth curve in E is *horizontal* (respectively, *vertical*) if each of its tangent vectors is horizontal (respectively vertical). Here we make the convention that all sectionally smooth curves are parameterized so as to be regular (nowhere vanishing tangent vector) on each smooth arc.

Let $\alpha(t)$, $c \leq t \leq d$, be a sectionally smooth curve in $\phi(E) \subset B$. Given $x \in \phi^{-1}(\alpha(c))$, there is at most one sectionally smooth *horizontal* curve $\alpha_x(t)$, $c \leq t \leq d$, in E such that: (i) $\alpha_x(c) = x$, and (ii) $\phi \circ \alpha_x = \alpha$. If it exists, α_x is called the *horizontal lift of* α *to* x. If α_x exists for every $x \in \phi^{-1}(\alpha(c))$, then we say that α *has horizontal lifts.* In that case the *translation along* α is the map $\psi_\alpha: \phi^{-1}(\alpha(c)) \to \phi^{-1}(\alpha(d))$ given by $x \to \alpha_x(d)$.

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Ehresmann [1] has studied the case where $\phi(E) = B$ and every sectionally smooth curve has horizontal lifts.

Given $u, v \in \phi(E)$, let Φ_{uv} denote the collection of all translations

$$\phi^{-1}(u) \to \phi^{-1}(v)$$

along sectionally smooth curves from u to v with horizontal lifts. Φ_u denotes Φ_{uu}, and β denotes the set of all maps $\Phi_{uv} \times \Phi_{vw} \to \Phi_{uw}$ defined by conjunction of translations (defined only if neither Φ_{uv} nor Φ_{vw} is empty). The *holonomy system* Φ of $\mathscr{H}$ consists of β and all Φ_{uv}.

Each Φ_u is a semigroup under β. If each Φ_{uv} contains a bijective translation so that all Φ_u are isomorphic, then we speak of any Φ_u as the *holonomy semigroup* of $\mathscr{H}$. If, further, every element of some (and thus every) Φ_u is bijective, we speak of any Φ_u as the *holonomy group* of $\mathscr{H}$.

The following lemma, which is similar to results of Ehresmann, Reeb and others, illustrates use of the holonomy system.

LEMMA 2.2. *Let* $\phi: E \to B$ *be a differentiable map everywhere of rank* dim B. *If* $u \in \phi(E)$, *then* $\phi^{-1}(u)$ *is a closed regularly embedded submanifold of* E. *If* $\psi: \phi^{-1}(u) \to \phi^{-1}(v)$ *is a translation relative to an Ehresmann connection for* ϕ, *then* ψ *is differentiable.*

Proof. A vector field on E is vertical if and only if it annihilates the smooth functions constant on fibres. Thus [X, Y] is vertical if X and Y are vertical. Now the vertical distribution $\mathscr{V}$ is integrable, and Frobenius' Theorem shows that the arc components of the fibres are submanifolds of E. Since the fibres are closed, by continuity of ϕ, it follows that they are closed regularly embedded submanifolds.

View the Ehresmann connection as a system of ordinary differential equations. In local coordinates the coefficients are differentiable because H_x depends differentiably on x; thus the solution curve at time t depends differentiably on the initial data. This proves differentiability of ψ. *Q. e. d.*

Lemma 2.2 allows a short proof of the following results; the first two are due to Ehresmann [1].

PROPOSITION 2.3. *Let* $\phi: E \to B$ *be a differentiable map of rank* dim B, *where* E *is connected and* B *is paracompact. Suppose that there exists an Ehresmann connection* $\mathscr{H}$ *for* ϕ, *relative to which every smooth curve in* $\phi(E)$ *has horizontal lifts; let* $u \in \phi(E)$, *and let the holonomy group* Φ_u *of* $\mathscr{H}$ *be given the compact open topology for its action on* $\phi^{-1}(u)$. *Then* $\phi: E \to \phi(E)$ *is a differentiable fibre bundle with topological structure group* Φ_u.

Proof. The map $\phi(E)$ is open in B by the rank condition, so we may assume ϕ surjective in the proof. Give B a Riemannian metric, and let $u \in B$. Let U be a normal coordinate neighborhood of u. Given $v \in U$, let $v(t)$, $0 \leq t \leq 1$, be the geodesic arc in U from u to v. We now define a mapping

$$h: \phi^{-1}(u) \times U \to \phi^{-1}(U) \quad \text{by} \quad (x, v) \to v_x(1),$$

where $v_x(t)$ is the horizontal lift of $v(t)$ to x. The mapping h is surjective because $v^*(t) = v(1 - t)$ has horizontal lifts, and is injective by construction. The retraction $v_x(1) \to x$ of $\phi^{-1}(U)$ onto $\phi^{-1}(u)$ is differentiable, as is the projection

$$\phi: v_x(1) \to v(1) = v;$$

thus h^{-1} is differentiable. This proves that ϕ is differentiably locally trivial. With this local trivialization, the transition functions have values in the local system $\{\Phi_u\}$ of holonomy groups. Thus we have constructed an Ehresmann-Feldbau bundle with group Φ_u.

The product on Φ_u is continuous because we are dealing with the c-o topology. Continuity of inversion is seen by reversing the curves based at u. Thus Φ_u is a topological group, and now $\Phi_u \times \phi^{-1}(u) \to \phi^{-1}(u)$ is continuous because we are using the c-o topology. Therefore Φ_u is a topological transformation group of $\phi^{-1}(u)$. Now it follows ([2, p. 19]) that the transition functions are continuous. *Q. e. d.*

COROLLARY 2.4. *Let* ϕ: E $\to$ B *be a differentiable proper map of rank* dim B, *where* E *is connected, and where* E *and* B *are paracompact. Then* ϕ: E $\to \phi$(E) *is a locally trivial differentiable fibre space.*

Here we recall that a map is called *proper* if the inverse image of every compact set is compact.

Proof. E admits a Riemannian metric because it is paracompact, and this implies the existence of an Ehresmann connection $\mathscr{H}$ for ϕ. We must prove that every smooth curve in ϕ(E) has horizontal lifts relative to $\mathscr{H}$. Our assertion will then follow from Proposition 2.3.

Let α be a smooth curve in ϕ(E). Without loss of generality we may assume that it is given by $\alpha(t)$, $0 \leq t \leq 1$. If $0 \leq r < s \leq 1$, then $\alpha^{r,s}$ denotes the restricted curve $\alpha(t)$, $r \leq t \leq s$. Viewing $\mathscr{H}$ as a system of ordinary differential equations, one sees that there is a continuous function s(y) on $\phi^{-1}(\alpha(r))$ such that $r < s(y) \leq 1$ and $\alpha^{r,s(y)}$ has a horizontal lift to y. The function s(y) attains its minimum, say r', because $\phi^{-1}(\alpha(r))$ is compact. Now $r < r' \leq 1$ and $\alpha^{r,r'}$ has horizontal lifts. It follows that the restricted curve $\alpha(t)$, $0 \leq t < 1$, has horizontal lifts. Similarly, the curve $\beta(t) = \alpha(1 - t)$, $0 \leq t < 1$, has horizontal lifts. This proves that α has horizontal lifts. *Q. e. d.*

The following corollary summarizes our remarks on Ehresmann connections.

COROLLARY 2.5. *Let* ϕ: E $\to$ B *be a differentiable map of rank* dim B, *where* E *and* B *are connected and paracompact. Then the following statements are equivalent:*

(i) ϕ: E $\to \phi$(E) *is a locally trivial differentiable fibre space.*

(ii) *There exists an Ehresmann connection for* ϕ, *relative to which every sectionally smooth curve in* ϕ(E) *has horizontal lifts.*

(iii) *If* $\mathscr{H}$ *is any Ehresmann connection for* ϕ, *then every sectionally smooth curve in* ϕ(E) *has horizontal lifts relative to* $\mathscr{H}$.

Proof. To see that (iii) implies (ii), we must construct a connection. Choose a Riemannian metric on E, and define H_x to be the orthogonal complement of the vertical space V_x in the tangent space E_x for every $x \in E$. Now (ii) implies (i) by Proposition 2.3, so we need only verify that (i) implies (iii). This being a matter of integrating horizontal lifts of vector fields on B, it follows from local triviality. *Q. e. d.*

3. METRIC-COMPATIBLE MAPS

We next define a class of maps to which Proposition 2.3 applies.

DEFINITION 3.1. Let $f: M \to N$ be a differentiable map of *Riemannian* manifolds. Then f is *metric-compatible* if there exists a continuous function λ on N with positive real values that bounds shrinking of certain tangent vectors in the following sense: If $x \in M$ and $Y \in N_{f(x)}$, then there exists $X \in M_x$ such that (i) $f_* X = Y$ and (ii) the length $\|Y\|$ is not less than $\lambda(f(x)) \|X\|$.

We list some immediate consequences of the definition.

(3.1.1) Let $f: M \to N$ be metric-compatible. Then f is everywhere of rank $\dim N$; in particular, $\dim M \geq \dim N$.

(3.1.2) Let $f: M \to N$ be a differentiable map of Riemannian manifolds of the same dimension. Then f is metric-compatible if and only if f is complete in the sense of [3]. In that case the differential f_* is a linear isomorphism on each tangent space M_x.

(3.1.3) The main feature of the definition is that λ cannot blow up on the boundary of $f(M)$ in N. For example, if M is the unit disc in hyperbolic metric and N is the Euclidean plane, then the inclusion $M \to N$ is not metric-compatible.

DEFINITION 3.2. Let $f: M \to N$ be a differentiable map of Riemannian manifolds, everywhere of rank $\dim N$. Then the *associated Ehresmann connection* $\mathscr{H} = \{H_x\}_{x \in M}$ is defined by:

$$H_x \text{ is the orthogonal complement of } V_x = \text{Kernel } (f_* \text{ on } M_x).$$

The above definition associates an Ehresmann connection with every metric-compatible map. We define M to be *horizontally complete* relative to f if every smooth horizontal (for the associated Ehresmann connection) vector field of bounded length on M, whose image on N under f is a well-defined vector field on N, can be integrated globally. This definition is justified by the following folk lemma, which shows that completeness implies horizontal completeness.

LEMMA 3.3. *Let* X *be a differentiable vector field of bounded length on a complete Riemannian manifold* M. *Then* X *is globally integrable: given* $x \in M$, *there exists a smooth curve* $\alpha(t)$, $-\infty < t < \infty$, *in* M *such that* $\alpha(0) = x$ *and* $\alpha'(t) = X_{\alpha(t)}$.

We give a proof because we have been unable to find one in the literature. Let I be a maximal interval containing 0 on which α can be defined with $\alpha(0) = x$ and $\alpha'(t) = X_{\alpha(t)}$. The interval I is nontrivial and is open by local integrability of X. It is thus described by inequalities $a < t < b$. By hypothesis, $\|X\| \leq m$ for every $y \in M$. If $b < \infty$, it follows that $\alpha([0, b))$ is in the closed ball B of radius mb about x. The ball B is compact because M is complete; hence α extends to b by continuity. Thus $b = \infty$. Similarly, $a = -\infty$. *Q.e.d.*

PROPOSITION 3.4. *Let* $f: M \to N$ *be a metric-compatible map of connected Riemannian manifolds. If* M *is complete, then* M *is horizontally complete relative to* f. *If* M *is horizontally complete relative to* f, *then* $f(M) = N$, *and every sectionally smooth curve in* N *has horizontal lifts relative to the associated Ehresmann connection.*

Proof. The first statement is contained in Lemma 3.3. Now assume that M is horizontally complete. We shall lift curves in $f(M)$ and then prove that $f(M) = N$.

Let $\alpha(t)$, $a \leq t \leq b$, be a sectionally smooth curve in f(M). We may assume α to be smooth, for we need only lift its smooth arcs. If $a < t < b$ (respectively, $a = t$; respectively, $t = b$), then there exist $a \leq c < t < d \leq b$ (respectively, $a = t < d \leq b$; respectively, $a \leq c < t = b$) such that α is one-to-one on [c, d] (respectively, [a, d]; respectively [c, b]; thus we may assume α has no self-intersections. Now

$$\alpha([a, b]) \subset T \subset U \subset K \subset N,$$

where T is a tubular neighborhood, U is open, and K is compact. The derivative α' is extendible first to T and then to a smooth vector field Y on N that vanishes off K. The horizontal lift X of Y to M is of bounded length because f is metric-compatible and because Y is bounded by compactness of K. Further, X can be integrated globally, and the horizontal lifts of α are the appropriate restrictions of the integral curves through points of $f^{-1}(\alpha(a))$.

Now we need only prove that f(M) = N. This is done as in [3], with some minor technical changes. If f(M) ≠ N, then [3, p. 254] gives a geodesic arc $\alpha(t)$, $0 \leq t \leq 1$ (we choose t proportional to arc length) in N such that $\alpha(t) \in f(M)$ for $0 \leq t < 1$ and $\alpha(1)$ is a boundary point y of f(M) in N. As above, α' extends to a bounded vector field Y on N, and X denotes its horizontal lift. Let $\beta(t)$, $-\infty < t < \infty$, be the integral curve of X through a point of $f^{-1}(\alpha(0))$, and let $\{t_i\} \to 1$ $(0 \leq t_i < 1)$. The sequence $\{\alpha(t_i)\}$ converges to the boundary point y, and $\alpha(t_i) = f(\beta(t_i))$. But $\{\beta(t_i)\} \to \beta(1)$, so that $f(\beta(1)) = y$, which implies that y is not a boundary point. Thus f(M) = N. *Q. e. d.*

Combining Propositions 2.3 and 3.4, we obtain the following extension of the result of [3].

THEOREM 3.5. *Let* f: M → N *be a metric-compatible map of connected Riemannian manifolds. Suppose that* M *is horizontally complete relative to* f. *Let* G *be the holonomy group of the associated Ehresmann connection in the compact-open topology for its action on a fibre. Then* f(M) = N, *and* f: M → N *is a differentiable fibre bundle with topological structure group* G.

We now combine Theorem 3.5 and Corollary 2.5.

THEOREM 3.6. *Let* ϕ: E → B *be a differentiable map, where* E *and* B *are connected paracompact manifolds. Then the following conditions are equivalent:*

(i) ϕ: E → B *is a locally trivial differentiable fibre space.*

(ii) ϕ: E → B *is the fibre space underlying a differentiable fibre bundle with topological structure group.*

(iii) ϕ *is of rank* dim B, E *and* B *admit Riemannian metrics such that* E *is horizontally complete for* ϕ, *and* ϕ *is metric-compatible.*

(iv) ϕ *is of rank* dim B. *Given a complete Riemannian metric on* B, E *admits a complete Riemannian metric (which is in particular horizontally complete for* ϕ) *for which* ϕ *is metric-compatible.*

Proof. Since B admits a complete Riemannian metric, (iv) implies (iii). By Theorem 3.5, (iii) implies (ii). That (ii) implies (i) is seen by throwing away the structure group. Thus we need only prove that (i) implies (iv).

Let $d\sigma^2$ and ds_I^2 be complete Riemannian metrics on B and on E. As usual, we consider the vertical distribution $\mathscr{V} = \{V_x\}_{x \in E}$ and the Ehresmann connection

$\mathscr{H} = \{H_x\}_{x \in E}$, where H_x is the (ds_1^2)-orthogonal complement of V_x in E_x. We define a new Riemannian metric ds^2 on E by the three conditions

(a) ds^2 and ds_1^2 give the same inner product on each V_x,

(b) $V_x \perp H_x$ under ds^2, and

(c) $\phi_*: H_x \to B_{\phi(x)}$ is a linear isometry if H_x is given the inner product from ds^2 and $B_{\phi(x)}$ is given that of $d\sigma^2$.

Now ϕ: $(E, ds^2) \to (B, d\sigma^2)$ is metric-compatible by construction, and $\mathscr{H}$ is the associated Ehresmann connection by definition and (b) above.

Now assume (i). We must prove that (E, ds^2) is complete. Let $\{x_i\}$ be a Cauchy sequence in E for ds^2, and define $y_i = \phi(x_i)$. Taking vertical and horizontal components of any tangent vector X to E, we note that

$$\|X\|_{ds^2} \geq \|\phi_* X\|_{d\sigma^2}$$

so that ϕ decreases distance. Thus $\{y_i\}$ is a Cauchy sequence in B for $d\sigma^2$. Now $\{y_i\} \to y$ in B because $d\sigma^2$ is complete. By (i) and Corollary 2.5, $\mathscr{H}$ gives a local trivialization of $\phi^{-1}(U)$ for some neighborhood U of y, $x_i = (v_i, y_i)$ in product coordinates, and $\phi(v_i) = y$. The sequence $\{v_i\}$ is a Cauchy sequence in $\phi^{-1}(y)$ relative to the metric induced by ds^2 because the fibre is closed. Thus it is a Cauchy sequence relative to the metric induced by ds_1^2 because they are the same. Now $\{v_i\} \to v$ because ds_1^2 is complete. Thus $\{x_i\} \to x = (v, y)$. *Q. e. d.*

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