AN ELEMENTARY PROOF OF KATĚTOV'S THEOREM CONCERNING Q-SPACES

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We recall here that a completely regular (Hausdorff) space X is called a Q-space [1] provided that every homomorphism ϕ of the ring C(X) of all real continuous functions defined on X into the ring of real numbers which does not vanish identically on C(X) is of the form

$$\phi(f) = f(p_0) \qquad \text{for all } f \text{ in } C(X),$$

where p_0 is a fixed point of X. Q-spaces can be characterized in a purely topological manner; for instance, it is shown in [5] that a completely regular space X is a Q-space if and only if it satisfies the following condition.

(Q): *for every point* p_0 *from* $\beta X \setminus X$, *there exists a function* $f: \beta X \to I$ *such that* $f(p_0) = 0$ *and* $f(p) > 0$ *for* p *in* X.

I denotes here the closed unit interval [0, 1]; $f: \beta X \to I$ means that f is a continuous function which maps βX into I.

In [3] Katětov has proved the following theorem.

THEOREM. *If* X *is a paracompact space and every closed discrete subspace of* X *is a* Q-*space, then* X *is also a* Q-*space.*

(This theorem is a particular case of Shirota's result [6]: *if a space* X *admits a complete uniformity and every closed discrete subspace of* X *is a* Q-*space, then* X *is a* Q-*space.* Indeed, every paracompact space admits a complete uniformity.)

We shall give here another, more elementary proof of Katětov's theorem. We begin with the following remarks.

Clearly, the problem whether a discrete space is a Q-space depends only upon the cardinality of the space. Moreover, if R is a discrete space and R_0 is an arbitrary subspace of R, then $\overline{R}_0 = \beta R_0$, where $\overline{R}_0$ denotes the closure of R_0 in βR. Hence, using the condition (Q), one can easily infer that if R is a Q-space, then R_0 is also. Therefore we can state:

(i) *if* R_1 *and* R_2 *are discrete spaces with* $\overline{\overline{R}}_1 \leq \overline{\overline{R}}_2$, *and* R_2 *is a* Q-*space, then* R_1 *is also a* Q-*space.*

Denote as m_0 the least cardinal such that the discrete space of the cardinality m_0 is not a Q-space. (It can be shown [2], that m_0 is the so-called *first measurable cardinal;* this fact, however, will not be used in our reasonings. In particular, the non-existence of such cardinal would only simplify the proof.) According to (i), it follows that:

(ii) *a discrete space* R *is a* Q-*space if and only if* $\overline{\overline{R}} < m_0$.

Notice that if $\{F_\xi: \xi \in \Xi\}$ is a discrete system of subsets of a space (we recall here that a system of subsets of a space is said to be *discrete* provided that each point of the space has a neighbourhood which intersects at most one member of the

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the system) and $p_\xi \in F_\xi$ for each ξ in Ξ, then $\{p_\xi : \xi \in \Xi\}$ is a closed discrete subset of the space. Therefore, according to (ii), the Katětov theorem can be formulated as follows.

If X *is a paracompact space having the property that*

(A): *every discrete system of subsets of* X *is of the cardinality less than* m_0, *then* X *is a* Q-*space.*

We shall prove this statement.

Proof. Assume that X is a paracompact space having the property (A). Suppose that p_0 is a point from $\beta X \setminus X$. For any $A \subset X$, we denote by Cl(A) the closure of A in $X \cup \{p_0\}$. Let

$$\mathfrak{A} = \{X \setminus \overline{G} : G \text{ is a neighborhood of } p_0 \text{ in } X \cup \{p_0\}\}$$

($\overline{G}$ denotes the closure of G in X). Since X is paracompact and $\mathfrak{A}$ is an open covering of X, one can find a σ-discrete closed covering $\mathfrak{F}$ of X which is a refinement of $\mathfrak{A}$ (see, for instance, [4; Th. 28, p. 156]); in other words, members of $\mathfrak{F}$ are are closed in X, $p_0 \notin \mathrm{Cl}(F)$ for any F in $\mathfrak{F}$ (since $\mathfrak{F}$ is a refinement of $\mathfrak{A}$); moreover, $\mathfrak{F} = \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup, \cdots$, where the systems $\mathfrak{F}_n$ (n = 1, 2, ⋯) are discrete in X.

Let $S_n = \bigcup \mathfrak{F}_n$. We shall distinguish two cases:

Case 1. $p_0 \notin \mathrm{Cl}(S_n)$ for every n. In this case there exist functions $f_n \in I^{\beta X}$ such that $f_n(p_0) = 0$ and $f_n(p) = 1$ for $p \in S_n$. Setting

$$f(p) = \sum_{n=1}^{\infty} 2^{-n} f_n(p) \quad \text{for } p \text{ in } \beta X,$$

we find that $f: \beta X \to I$, $f(p_0) = 0$, and $f(p) > 0$ for p in X.

Case 2. $p_0 \in \mathrm{Cl}(S_{n_0})$ for some n_0. According to (A), $\overline{\overline{\mathfrak{F}}}_{n_0} < m_0$. Moreover, since $\mathfrak{F}_{n_0}$ is a discrete system, members of $\mathfrak{F}_{n_0}$ are open in S_{n_0} and therefore open also in $S_{n_0} \cup \{p_0\}$. We consider the collection $\mathfrak{F}_{n_0} \cup \{p_0\}$ as a decomposition space of $S_{n_0} \cup \{p_0\}$; let ϕ be the projection of $S_{n_0} \cup \{p_0\}$ onto $\mathfrak{F}_{n_0} \cup \{p_0\}$. It follows from the preceding that members of $\mathfrak{F}_{n_0}$ are isolated points of the space $\mathfrak{F}_{n_0} \cup \{p_0\}$; hence $\mathfrak{F}_{n_0}$ is a discrete subspace of $\mathfrak{F}_{n_0} \cup \{p_0\}$. Since all members of the decomposition $\mathfrak{F}_{n_0} \cup \{p_0\}$ are closed in $S_{n_0} \cup \{p_0\}$, $\mathfrak{F}_{n_0} \cup \{p_0\}$ is a T_1-space. Consequently, $\mathfrak{F}_{n_0} \cup \{p_0\}$, as a T_1-space having only one non-isolated point, is completely regular (in fact, normal).

If g is a bounded real-valued continuous function defined on $\mathfrak{F}_{n_0}$, then the function $f = g \circ \phi$ is a continuous function defined on S_{n_0}. Since S_{n_0} is a closed subset of X, f admits a continuous bounded extension over X and, in turn, it admits a continuous extension f^* over $X \cup \{p_0\}$. Setting $g^*(p) = g(p)$ for p in $\mathfrak{F}_{n_0}$ and $g^*(p_0) = f^*(p_0)$, we see that the equality $f^* = g^* \circ \phi$ still holds, and therefore g^* is a continuous function on $\mathfrak{F}_{n_0} \cup \{p_0\}$. In other words, every bounded real-valued continuous function defined on $\mathfrak{F}_{n_0}$ admits a continuous extension over $\mathfrak{F}_{n_0} \cup \{p_0\}$, and it means that p_0 can be considered as a point from $\beta\mathfrak{F}_{n_0} \setminus \mathfrak{F}_{n_0}$.

Since $\mathfrak{F}_{n_0}$ is a Q-space, there exists a function $g_0: \beta F_{n_0} \to I$ such that $g_0(p_0) = 0$ and $g_0(p) > 0$ for p in $\mathfrak{F}_{n_0}$. Let us set $f_0 = g_0 \circ \phi$. Then f_0 is a continuous function on $S_{n_0} \cup \{p_0\}$. The restriction $f_0 | S_{n_0}$ admits a bounded continuous extension f_1 over X such that $f_1(p) > 0$ for p in X. (If f_1 vanishes on X, then let $B = f_1^{-1}(0)$, and replace f_1 by the function $f_1'(p) = \max\{f_1(p), g(p)\}$ for p in X, where $g: X \to I$ is a function such that $g(p) = 0$ for $p \in S_{n_0}$ and $g(p) = 1$ for p in B.) In turn, f admits a continuous extension f over βX. Clearly, $f(p) > 0$ for p in X and $f(p_0) = f_0(p_0) = 0$ (f and f_0 agree on S_{n_0}, and therefore they agree on every point from $Cl(S_{n_0})$). This shows that X is a Q-space.

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