INCOMPLETE ORTHOGONAL FAMILIES AND A RELATED QUESTION ON ORTHOGONAL MATRICES

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1. A QUESTION POSED BY W. KAPLAN

Wilfred Kaplan has asked the author the following question: If we are given an incomplete orthogonal family of *continuous* functions on some interval, is there necessarily a nonnull *continuous* function orthogonal to all the given functions?

In this note we shall show that the answer is negative, even when restrictions stronger than continuity are imposed on the given functions. For definiteness we shall construct our counterexample on the circle group Γ, that is, on the real numbers modulo 2π; however, our method of construction is perfectly general. (Throughout the paper, all functions and numbers are assumed to be real-valued; but our results are also valid in the complex case.)

THEOREM 1. *Corresponding to any* $f \in L^2(\Gamma)$ *of norm* 1, *there exists a complete orthonormal basis in* $L^2(\Gamma)$ *that contains* f *and whose remaining elements are trigonometric polynomials. If moreover the Fourier coefficients of* f (*with respect to the usual trigonometric system*) *are* $O(n^{-1/2})$, *then the remaining basis functions are also uniformly bounded.*

The following generalization of the first statement in Theorem 1 is also true:

THEOREM 2. *Corresponding to any finite orthonormal set of functions in* $L^2(\Gamma)$, *there exists a complete orthonormal basis in* $L^2(\Gamma)$ *that contains the given functions and whose remaining elements are trigonometric polynomials.*

These theorems are consequences of a simple theorem concerning infinite matrices. We shall say that the infinite matrix $A = \|a_{ij}\|$ $(i, j = 1, 2, \cdots)$ of real numbers is *orthogonal* if its rows form an orthonormal basis for the Hilbert space ℓ^2. If A is orthogonal, so is A^T. We denote by A_{i*} and A_{*j} the i^{th} row and j^{th} column, respectively, of A.

THEOREM 3. *Let there be given* n *orthogonal unit vectors in* ℓ^2, *which we write as row vectors*

$$A_{i*} = \{a_{ij}\} \qquad (1 \leq j < \infty,\ 1 \leq i \leq n).$$

Suppose the square matrix $\|a_{ij}\|$ $(1 \leq i, j \leq n)$ *is nonsingular. Then there exists an infinite orthogonal matrix* $A = \|a_{ij}\|$ *with the given vectors as its first* n *rows and with* $a_{ij} = 0$ *for* $j > i > n$. *These conditions uniquely determine* A. *Moreover, in the case* $n = 1$, *if* $a_{1j} = O(\sqrt{j})$, *the sums* $\Sigma_{j=1}^{\infty} |a_{ij}|$ $(i = 2, 3, \cdots)$ *are uniformly bounded.*

To see that Theorems 1 and 2 follow from this, let $\phi_n(x)$ $(n = 1, 2, \cdots)$ denote the n^{th} function in the sequence $1, 2\sqrt{2}\cos x, 2\sqrt{2}\sin x, 2\sqrt{2}\cos 2x, 2\sqrt{2}\sin 2x, \cdots$. Since $\{\phi_n(x)\}$ is an orthonormal basis for $L^2(\Gamma)$, the same is true of the system $\psi_n(x) = \Sigma_1^{\infty} a_{nj}\phi_j(x)$ whenever $\|a_{nj}\|$ is an orthogonal matrix. Thus, to prove

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Theorem 1, one simply applies Theorem 3 to the row-vector $A_{1*} = \{a_{1j}\}$, where $a_{1j} = (f, \phi_j)$; Theorem 2 follows similarly. The only difficulty occurs if $a_{11} = 0$ (or, in the case of Theorem 2, if the $n \times n$ matrix in question is singular). Since we can however always find some $a_{1j} \neq 0$ (respectively, some nonvanishing n-rowed minor), we have in this case only to renumber finitely many of the ϕ_j and then proceed in the same way.

Proof of Theorem 3. We wish to adjoin rows so as to obtain an orthogonal matrix subject to the restriction $a_{ij} = 0$ for $j > i > n$. We verify first that this is possible in at most one way. Indeed, since the matrix $\|a_{ij}\|$ $(1 \leq i \leq n,\ 1 \leq j \leq n+1)$ has by hypothesis rank n, $A_{(n+1)*}$ is uniquely determined by orthogonality and normality. Once $A_{(n+1)*}$ has been determined, the matrix $\|a_{ij}\|$ $(1 \leq i \leq n+1,$ $1 \leq j \leq n+2)$ has rank $n+1$ and thus, by orthogonality and normality, $A_{(n+2)*}$ is uniquely determined; and so forth. It remains only to show that the totality of the A_{i*} is complete in ℓ^2. For this, observe first that $a_{mm} \neq 0$ for $m > n$. Otherwise it would follow that

$$\sum_{j=1}^{m-1} a_{ij}a_{mj} = 0 \qquad (i = 1, \cdots, m-1),$$

and this would contradict the nonvanishing of the $(m-1)$-rowed principal minor of A. Suppose now b is a vector orthogonal to all A_{i*}. We can write $b = b' + b''$, where b' is spanned by $A_{1*}, \cdots, A_{n*}$ and $b''_m = 0$ for $m \leq n$ (subscripts denote the components of b''). Since $a_{mm} \neq 0$, the orthogonality relations show in turn that $b''_{n+1} = b''_{n+2} = \cdots = 0$. Hence $b'' = 0$. Since $b' = b$ is orthogonal to $A_{1*}, \cdots, A_{n*}$ and the matrix $\|a_{ij}\|$ $(1 \leq i, j \leq n)$ is nonsingular, we see also that $b' = 0$ and the theorem is proved, except for the extra details in the case $n = 1$. For this one has only to carry out the orthogonalization explicitly; since the calculation is routine, we state only the result. (For the sake of easy notation we gave the matrix without normalization.)

THEOREM 4. *Let* $a_1 > 0$, $\Sigma_1^\infty a_n^2 < \infty$. *Then the rows of the matrix*

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \cdots \\ -a_1a_2 & a_1^2 & 0 & 0 & \cdots \\ -a_1a_3 & -a_2a_3 & a_1^2+a_2^2 & 0 & \cdots \\ -a_1a_4 & -a_2a_4 & -a_3a_4 & a_1^2+a_2^2+a_3^2 & \cdots \\ -a_1a_5 & -a_2a_5 & -a_3a_5 & -a_4a_5 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

are mutually orthogonal, and complete in ℓ^2. *If the first row is multiplied by* $(\Sigma_1^\infty a_n^2)^{-1/2}$, *and the* m^{th} *row* $(m \geq 2)$ *is multiplied by*

$$\left[\left(\sum_1^{m-1} a_n^2\right)\left(\sum_1^{m} a_n^2\right)\right]^{-1/2},$$

the resulting matrix is orthogonal.

For $m \geq 2$, the elements of the m^{th} row in the above array are

$$-a_1 a_m, -a_2 a_m, \cdots, -a_{m-1} a_m, \sum_1^{m-1} a_n^2, 0, 0, \cdots .$$

Once we have arrived at the above matrix, it is of course a simple matter to prove directly that it is orthogonal. Note that the sum of the absolute values of the elements in the m^{th} row $(m \geq 2)$ is

$$|a_m| \sum_1^{m-1} |a_n| + \sum_1^{m-1} a_n^2 \leq M|a_m| (m-1)^{1/2} + M^2 ,$$

where $M^2 = \Sigma_1^\infty a_n$. Since the normalization constants remain between fixed positive limits, the proof of Theorem 3 is complete.

2. A REMARK CONCERNING BESSEL'S INEQUALITY

If $\{g_n(x)\}$ is an orthonormal set of functions in $L^2(\Gamma)$, then by Bessel's inequality $\Sigma (f, g_n)^2 \leq (f, f)$ for every $f \in L^2(\Gamma)$. One may ask whether, in case both f and the g_n have very high smoothness, it is possible to assert anything more of the Fourier coefficients of f than that they be square-summable. The results of the preceding section provide a negative answer to this question.

THEOREM 5. *Corresponding to any numbers* c_n *with* $\Sigma_1^\infty c_n^2 < 1$, *there exists an orthonormal set* $\{g_n\}$ *of trigonometric polynomials such that* $(1, g_n) = c_n$. *If* $c_n = O(n^{-1/2})$, *there exist uniformly bounded* g_n *with this property.*

Proof. In the matrix of Theorem 4, choose as the first row $a_1 = 1$, and for $n \geq 1$,

$$a_{n+1} = -(1 - c_1^2 - \cdots - c_{n-1}^2)^{-1/2} (1 - c_1^2 - \cdots - c_n^2)^{-1/2} c_n \qquad (c_0 = 0) . \tag{2.1}$$

By an easy calculation, we get from (2.1)

$$(-a_1 a_m)\left[\left(\sum_1^{m-1} a_i^2\right)\left(\sum_1^{m} a_i^2\right)\right]^{-1/2} = c_{m-1} \qquad (m \geq 2) . \tag{2.2}$$

Note that (2.1) and the inequality $\Sigma_1^\infty c_i^2 < 1$ imply that $\Sigma_1^\infty a_n^2 < \infty$ and also that $a_n = O(n^{-1/2})$. The orthogonal matrix of Theorem 4 therefore has as its first column (by (2.2))

$$t, c_1, c_2, c_3, \cdots, \qquad \text{where } t = \left(1 - \sum_1^\infty c_n^2\right)^{1/2} .$$

Taking for g_n the trigonometric polynomial determined by the $(n+1)^{st}$ row of this matrix (see the preceding section), we obtain the result.

The above construction has incidentally demonstrated the following proposition.

THEOREM 6. *Corresponding to any unit vector* A_{*1} *with* $a_{11} \neq 0$ *there exists an orthogonal matrix* A *with* A_{*1} *as its first column, and with* $a_{ij} = 0$ *for* $j > i \geq 2$. *The matrix* A *is uniquely determined.*

3. SOME OPEN QUESTIONS

a) Theorem 5 is, in a way, not convincing, because the set $\{g_n\}$ is not complete; indeed, it has deficiency one, and to complete it we must adjoin a function that has in general no smoothness properties. One can therefore ask: given $\Sigma_1^\infty c_n^2 = 1$, can we find a *complete* orthonormal set $\{g_n\}$ and a function f of norm 1, such that f and all g_n are in $C^\infty(\Gamma)$ and $(f, g_n) = c_n$? The approach used in Section 2 leads then to the question: given a unit vector A_{*1}, does there exist a *row-finite* orthogonal matrix A, [that is, a matrix with $a_{ij} = 0$ for $j > J(i)$] having A_{*1} as its first column? We have not been able to answer this.

b) In Theorem 2, can we complete the given set of functions by adjoining *uniformly bounded* functions, say, of class C^∞? Even in Theorem 1, can we do this without imposing any restriction on the Fourier coefficients of f?

c) Can we find a complete orthonormal basis $\{g_n\}$ for $L^2(\Gamma)$, and an $f \in L^2(\Gamma)$, such that all functions are C^∞ and the Fourier series of f with respect to the system $\{g_n\}$ diverges at a point?

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