DISTORTION IN CERTAIN CONFORMAL MAPPINGS OF AN ANNULUS

P. L. Duren

1. INTRODUCTION

Let $\mathscr{F}_0$ denote the class of functions f(z) which are analytic and univalent in a fixed annulus R: $r < |z| < 1$, and which have the following properties:

(i) $|f(z)| < 1$ for $z \in R$;

(ii) $|f(z)| = 1$ for $|z| = 1$;

(iii) $f(z) \neq 0$ for $z \in R$.

In a previous paper [4], M. Schiffer and the author developed a method of variation within $\mathscr{F}_0$ and applied it to solve several extremal problems. In the normalized class $\mathscr{F} \subset \mathscr{F}_0$ for which $f(1) = 1$, the quantity

$$\mu = \sup_{f \in \mathscr{F}, z \in R} |f(z) - z| \tag{1}$$

was found in terms of a certain elliptic integral, and it was shown that $\mu < 8r$ for small r, the constant 8 being best possible. D. Gaier and F. Huckemann [5] completed this work by proving $\mu < 8r$ for all r. Huckemann [8] subsequently gave another solution to (1) by the method of extremal length. Independently of these developments, F. W. Gehring and G. af Hällström [6] also solved the distortion problem (1) and obtained an explicit alternating series representation from which the estimate $\mu < 8r$ follows at once.

It is the purpose of the present paper to consider another type of distortion, measured by the modulus of the derivative of the mapping function. For fixed $z = b$, $r < |b| < 1$, the specific extremal problems to be studied are

$$\min_{f \in \mathscr{F}_0} |f'(b)| \tag{2}$$

and

$$\max_{f \in \mathscr{F}_0} |f'(b)|. \tag{3}$$

We attack these problems by the variational method developed in [4]. The results are as follows. For every b, the solution to the minimum problem (2) is a mapping f of R onto the unit disk minus a radial slit from 0 toward f(b). For every b sufficiently far from the inner boundary $|z| = r$, the maximum problem (3) is solved by a mapping f of R onto the unit disk slit radially from 0 toward -f(b). As $|b|$ tends to r, however, there is a certain value b* at which the extremal

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function for (3) changes character: the radial slit sprouts a fork at its tip. The exact value of $b^* = b^*(r)$ is given in Theorem 2. The forked curve which arises in the case $|b| < b^*$ is governed by a certain differential equation. The direction field of this differential equation is investigated in Section 5 to give a description of the forked curve. In Section 6 actual representations of the radial slit mappings, in terms of Jacobian elliptic functions, are used to obtain simple estimates for $|f'(z)|$. (See Theorems 3 and 4.) In the final section (Section 7), the behavior of $|f'(z)|$ is studied as r tends to zero.

There are two related families of functions in which the extremal problems (2) and (3) have previously been studied. Grötzsch [7] considers the class of functions which map R onto a domain contained in $r < |w| < \infty$, and which preserve the inner boundary $|z| = R$. Within this class he considers the extremal problems (2) and (3) and finds the extremal functions are always radial slit mappings. Although this class of functions is related to our family $\mathscr{F}_0$ by an inversion, the derivatives are related in a more complicated way, so that the respective extremal problems are not equivalent.

Secondly, let D be a domain of arbitrary connectivity, having at least two boundary points. Distinguish a point $b \in D$, and consider the class of all functions $f(z)$ analytic and univalent in D, for which $|f(z)| \leq 1$ and $f(b) = 0$. Within this class, pose the extremal problems (2) and (3). The solutions are classical (see for example [9, Ch. VII]): the minimum is given by a radial slit mapping, the maximum by a circular slit mapping.

2. DIFFERENTIAL EQUATION FOR THE EXTREMAL CONTINUUM

Each function $f(z) \in \mathscr{F}_0$ maps the annulus R onto the unit disk minus a certain continuum C containing the origin. It was shown in [4] that for each $w_0 \in C$, $w_0 \neq 0$, there exists a family of variation functions $V(w) = V_\rho^{(0)}(w)$, depending upon the small positive parameter ρ, which operate on f in the sense that $V(f(z)) \in \mathscr{F}_0$ for all ρ sufficiently small. Furthermore [4, p. 263], $V(w)$ has the form

$$V(w) = w\left[1 + \frac{a\rho^2}{(w - w_0)w_0} - \frac{\bar{a}\rho^2 w}{(1 - w\bar{w}_0)\bar{w}_0}\right] + 0(\rho^3), \tag{4}$$

where $a = a(\rho)$ is a certain bounded function of ρ. The error term $0(\rho^3)$ depends on w but is uniform in each closed subdomain of the range of f.

Now fix a number $b \in R$, and assume without restriction of generality that $r < b < 1$. Consider first the maximum problem (3). Existence of an extremal function $f \in \mathscr{F}_0$ is assured by the fact that $\mathscr{F}_0$ is a compact normal family. By the extremal property of f, the function $f_1(z) = V(f(z))$ cannot have a larger derivative at b:

$$|f_1'(b)| = |V'(f(b))f'(b)| \leq |f'(b)|.$$

Thus

$$\Re\{\log V'(B)\} \leq 0 \qquad (B = f(b)). \tag{5}$$

From formula (4) one finds

(6) $$\log V'(w) = -\frac{a\rho^2}{(w - w_0)^2} - \frac{\bar{a}\rho^2 w(2 - \overline{w}_0 w)}{\overline{w}_0(1 - w\overline{w}_0)^2} + 0(\rho^3).$$

We shall assume (as we may) that $B = f(b)$ is real and positive. From (5), (6), and the general identity $\Re\{\overline{\alpha}\} = \Re\{\alpha\}$, the expression

(7) $$\Re\left\{a\rho^2\left[\frac{1}{(w_0 - B)^2} + \frac{B(2 - Bw_0)}{w_0(1 - Bw_0)^2}\right] + 0(\rho^3)\right\} \geq 0$$

then results. We now invoke Schiffer's fundamental lemma [11] to conclude from (7) that the continuum C is in fact a system of analytic arcs $w = w(t)$ satisfying the differential equation (or, strictly speaking, differential inequality)

(8) $$\left(\frac{dw}{dt}\right)^2\left[\frac{1}{(w - B)^2} + \frac{B(2 - Bw)}{w(1 - Bw)^2}\right] < 0.$$

For the minimum problem (2), the conclusion is the same, except that the inequality in (8) is reversed. It is convenient also to express (8) in the equivalent form

(9) $$\left(\frac{dw}{dt}\right)^2 \frac{(w - w_1)(w - w_2)}{w(w - B)^2(1 - Bw)^2} < 0,$$

where w_1 and w_2 are the solutions to the quadratic equation

(10) $$2B^3 w^2 + (1 - 4B^2 - B^4)w + 2B^3 = 0.$$

The discriminant of (10) is

$$(1 - 4B^2 - B^4)^2 - 16B^6 = (B^2 - 1)^2(B^4 - 6B^2 + 1),$$

and so w_1 and w_2 are real if and only if

(11) $$B \leq \sqrt{2} - 1.$$

Let us note the relations

(12) $$w_1 w_2 = 1; \qquad 2B^3(w_1 + w_2) = B^4 + 4B^2 - 1.$$

For $B < \sqrt{2} - 1$, we shall assume w_1 and w_2 are chosen so that $w_2 < -1 < w_1 < 0$.

3. MINIMUM DISTORTION

The minimum problem (2) turns out to be the easier, so it will be treated first.

Observe that for the rectilinear slit $w = Bt$ $(0 < t < 1)$ the expression (8) is positive. Since the continuum C must contain the origin, this is indeed the *only* curve for which (8) is positive.

To prove uniqueness, let the differential equation for C be transformed by $w = \omega^2$ into

$$\left(\frac{d\omega}{dt}\right)^2 \frac{(\omega^2 - w_1)(\omega^2 - w_2)}{(\omega^2 - B)^2(1 - B\omega^2)^2} = 1,$$

after proper choice of the parameter t. From the standard uniqueness theorem for ordinary differential equations [2, Chapter 1], it follows that there is at most one solution $\omega(t)$ for which $\omega(t_0) = 0$. In the w-plane, this implies that there is at most one arc C (parametrizable in many ways) which passes through the origin and makes the expression (8) positive. This argument even shows that the origin must be an end-point of the arc.

Therefore, each extremal function for (2) must, under the assumption $B = f(b) > 0$, map R onto $|w| < 1$ slit along the segment $0 \leq t \leq M < B$. The length $M = M(r)$ of the segment depends, of course, upon the modulus of the ring; the precise dependence is exhibited in Section 6, formula (22). Let $w = \psi(z)$ be the function which effects the radial slit mapping just described.

THEOREM 1. *Let* b *be a fixed number* $(r < b < 1)$. *For all functions* $f(z) \in \mathscr{F}_0$, *the inequality* $|f'(b)| \geq |\psi'(b)|$ *holds, where* $\psi(z)$ *is the radial slit mapping defined above. Equality occurs only if* f *is a rotation of* ψ.

4. MAXIMUM DISTORTION

As we have seen, the differential equation (9) governs the system of arcs C omitted by an extremal mapping $w = f(z)$ for the maximum problem (3). Since C must contain the origin, we verify as before that C is necessarily the segment $-M(r) \leq w \leq 0$ of the real axis, *provided* that this segment does not contain the point w_1 in its interior. (Recall our choice $w_2 < -1 < w_1 < 0$ if the roots are real.) If, on the contrary, $-M < w_1$, then C has a fork at w_1; we shall study this case in Section 5.

Let $w = \phi(z) = -\psi(-z)$ denote the function which maps the annulus $r < |z| < 1$ onto the disk $|w| < 1$ minus the segment $-M(r) \leq w \leq 0$, sending $b > 0$ into $B = \phi(b) > 0$. For all values of b except those for which w_1 is real and greater than $-M(r)$, $\phi(z)$ is the solution to the maximum problem (3). Referring to relation (11), we see that ϕ is certainly the extremal function if $B = \phi(b) > \sqrt{2} - 1$, since this makes w_1 non-real. Our result may be stated as follows.

THEOREM 2. *Let* $w = \phi(z)$ *be the function which maps the annulus* R *onto the unit disk minus the segment* $-M(r) \leq w \leq 0$, *and which maps the segment* $r < z < 1$ *onto* $0 < w < 1$. *Let* $B^* = B^*(r)$ *be the unique positive value of* B *for which*

$$\frac{1}{4}B^{-3}\{(1 - 4B^2 - B^4) - [(1 - 4B^2 - B^4)^2 - 16B^6]^{1/2}\} = M(r), \tag{13}$$

and let $\phi(b^*) = B^*$. *Then for fixed* b $(r < b < 1)$, $|f'(b)| \leq |\phi'(b)|$ *for all functions* f *in the family* $\mathscr{F}_0$, *if and only if* $b > b^*$. *If* $b \geq b^*$, *then* $|f'(b)| < |\phi'(b)|$ *unless* f *is a rotation of* ϕ.

Remarks. The expression (13) is simply $-w_1$, as given by the quadratic formula. The positive square root is taken when the quantity under the radical is positive. As noted in Section 2, this is the case if and only if $B < \sqrt{2} - 1$. A simple analysis based on relations (12) shows that as B decreases from $\sqrt{2} - 1$ to 0, $(-w_1)$ decreases monotonically from 1 to 0. Hence the equation (13) uniquely determines B^*. It also follows that $b^* = b^*(r)$ tends to zero as r does.

COROLLARY. *If* b *is such that* $\phi(b) \geq \sqrt{2} - 1$, *then* ϕ *is the extremal function, unique up to a rotation, for the maximum problem (3).*

5. THE FORKED CURVE

Although ϕ is the solution to the maximum problem (3) for all $b \geq b^*$, the extremal function is of a different character for $r < b < b^*$. This is to be expected, since as b approaches r, $\phi(b)$ tends to the tip of the slit, and so $\phi'(b)$ approaches zero. It will become evident that in the range $r < b < b^*$ the extremal function *varies* with b.

By construction, $b < b^*$ implies $w_1 > -M(r)$. Hence the extremal continuum C consists partly of the slit $w_1 \leq w \leq 0$, but it forks at w_1. To study the behavior near w_1, let us make the transformation

$$\omega = (w - w_1)^{3/2}. \tag{14}$$

The differential equation (9) then becomes

$$\left(\frac{d\omega}{dt}\right)^2 \left[\frac{w - w_2}{w(1 - Bw)^2(B - w)^2}\right] < 0.$$

The expression in brackets is negative for $w = w_1$, so

$$\left(\frac{d\omega}{dt}\right)^2 > 0 \quad \text{for } \omega = 0.$$

Thus the curve in the ω-plane leaves $\omega = 0$ either along the positive or along the negative real axis. This tells us, by (14), that as w approaches w_1 along C, there are three possibilities:

$$\lim_{w \to w_1} \arg(w - w_1) = 0, \frac{2\pi}{3}, \frac{4\pi}{3}.$$

Therefore, C consists of at most three analytic arcs joined together at w_1 at angles of 120°. One of these arcs is the real segment $w_1 \leq w \leq 0$. At first glance, one might suspect that the other two arcs of C are also line segments, but a calculation shows this to be false.

In order to describe the curve C more precisely, we shall consider the entire locus of points Γ generated by the differential equation (9) under the requirement that Γ contains the origin. This trajectory Γ is what Schaeffer and Spencer [10] called the "Γ-structure." Of course, $C \subset \Gamma$.

Since (9) has real coefficients, it is obvious that Γ is symmetric with respect to the real axis. Less apparent (although irrelevant for our purposes) is the fact that the solution curves are symmetric with respect to the unit circle $|w| = 1$. This is seen by observing that the expression (9) is invariant under the transformation $W = 1/w$. (Recall $w_1 w_2 = 1$.)

It follows from results of Schaeffer and Spencer [10, Chapter III] that Γ can branch only at w_1 and w_2, and here there must be three arcs meeting at equal angles (as we have already confirmed). Furthermore, Γ can terminate only at 0 and ∞. At every other point, Γ is locally a line segment. In particular, Γ must cross the real axis and the unit circle (if it does) at right angles.

We shall now show that the direction field of the differential equation (9) is everywhere *tangent* to the unit circle, so that Γ must lie entirely in $|w| < 1$. By proper choice of t, (9) may be written

$$\left(\frac{dw}{dt}\right)^2 = -\frac{w(w - B)^2 (w - 1/B)^2}{(w - w_1)(w - w_2)}. \tag{15}$$

Letting θ denote $\arg \frac{dw}{dt}$, we find

$$\begin{aligned} \theta = {} & \frac{1}{2}\arg w + \arg(w - B) + \arg(w - 1/B) \\ & - \frac{1}{2}[\arg(w - w_1) + \arg(w - w_2)] - \frac{3\pi}{2}. \end{aligned} \tag{16}$$

Here it is understood that

$$\arg w_1 = \arg(w_1 - B) = \arg(w_1 - 1/B) = \pi,$$

while $\arg(w_1 - w_2) = 0$. As w approaches w_1 along the ray $\arg(w - w_1) = 2\pi/3$, θ tends to $2\pi/3$. Now let $w = e^{i\alpha}$ $(-\pi < \alpha \leq \pi)$ be fixed. It is a simple exercise in trigonometry to prove

$$\arg(e^{i\alpha} - B) + \arg(e^{i\alpha} - 1/B) = \alpha + \pi \tag{17}$$

for all B, $0 \leq B \leq 1$. Perhaps the easiest proof is to show that the tangent of (17) is identically constant. Similarly, since $w_1 w_2 = 1$,

$$\arg(e^{i\alpha} - w_1) + \arg(e^{i\alpha} - w_2) = \alpha . \tag{18}$$

Combination of (16), (17), and (18) shows that for $w = e^{i\alpha}$, $\theta = \alpha - \pi/2$. This proves our assertion that the direction field of (15) is tangent to the unit circle.

Next let us study the behavior of θ as w varies along a fixed ray

$$w = w_1 + se^{i\beta} \qquad (s > 0)$$

from w_1 in the direction $e^{i\beta}$, $0 < \beta < \pi$. As s increases, it is geometrically obvious that $\arg(w - w_2)$ increases, while $\arg w$, $\arg(w - B)$, and $\arg(w - 1/B)$ decrease monotonically. Hence θ decreases as s increases. As s tends to 0, (16) shows that θ approaches $\pi - \beta/2$; therefore, $\theta \leq \pi - \beta/2$ for all w on the given ray. In particular, $\theta < \beta$ for all s if $2\pi/3 < \beta < \pi$.

This last result proves that Γ, which leaves w_1 along the ray $\arg(w - w_1) = 2\pi/3$, cannot enter the sector $2\pi/3 < \arg(w - w_1) < \pi$. By symmetry we deduce: *the curve* Γ *lies entirely within the sector* $|\arg(w - w_1)| < 2\pi/3$. Roughly speaking, this means Γ bends back toward B.

A more careful study of the direction field of (15) reveals that Γ has the form illustrated in the diagram. Other solution curves are also shown. We remark that the general structure of the solution curves is independent of the relation between B and w_1.

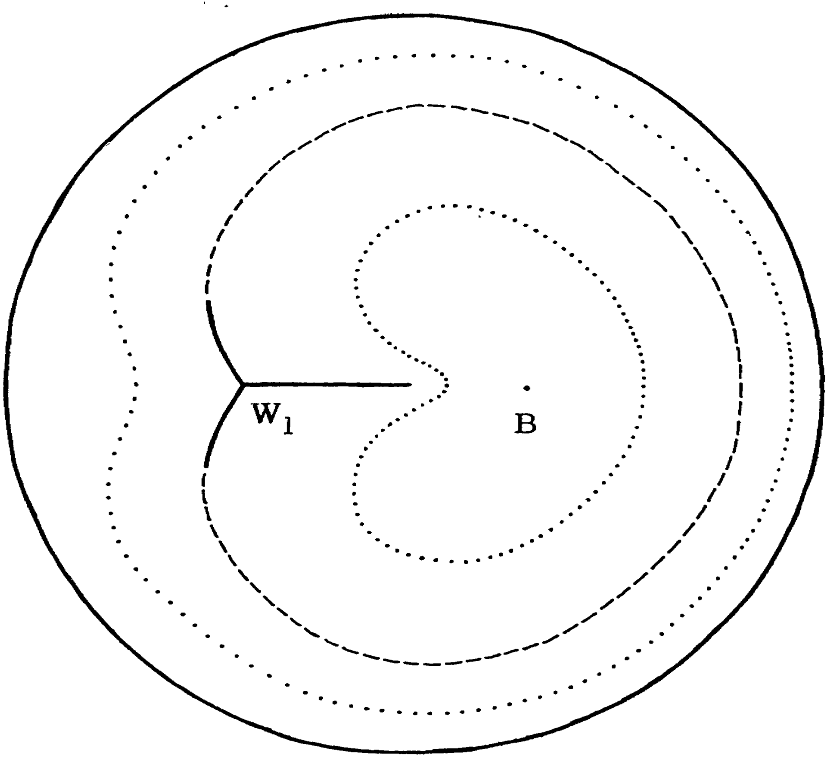

Figure 1.

We know that for $b < b^*$ the curve C omitted by the extremal function occupies some of the curved portion of Γ, as well as the segment $w_1 \leq w \leq 0$. However, we have not been able to show (although it seems likely) that C is symmetric with respect to the real axis. We do not even know that three branches actually emanate from w_1.

6. ESTIMATES THROUGH ELLIPTIC FUNCTIONS

In order to obtain quantitative information on the distortion, we need an actual expression for the function $w = \phi(z)$ which maps the annulus $r < |z| < 1$ onto the disk $|w| < 1$ slit along $-M \leq w \leq 0$. Gehring and af Hällström [6] have pointed out that such a representation can be given in the remarkably simple form

$$\phi(z) = \frac{1 + \operatorname{sn}(a \log z, k)}{1 - \operatorname{sn}(a \log z, k)}, \tag{19}$$

where

$$k = \frac{1 - M}{1 + M}; \quad a = -\frac{K}{\log r} = \frac{K'}{\pi}. \tag{20}$$

Here $\operatorname{sn}(u, k)$ is the fundamental Jacobian elliptic function with modulus k, and

$$\begin{aligned} K = K(k) &= \int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{-1/2}\, d\theta; \\ K' = K(k'), \quad k' &= (1 - k^2)^{1/2}. \end{aligned} \tag{21}$$

The length M of the slit can be expressed in the form [9, p. 295]

$$M = \frac{2L}{1 + L^2}, \quad L = 2r \prod_{n=1}^{\infty} \left(\frac{1 + r^{8n}}{1 + r^{8n-4}}\right)^2. \tag{22}$$

Differentiation of (19) and use of the relation sn' = (cn)(dn) yields the result

$$\phi'(z) = \frac{2a\,\mathrm{cn}(a \log z, k)\,\mathrm{dn}(a \log z, k)}{z[1 - \mathrm{sn}(a \log z, k)]^2}. \tag{23}$$

It is interesting to study the behavior of $\phi'(b)$ as b tends to 1. From (23) and [1, #907.01, #907.05], we obtain the asymptotic formula

$$\phi'(b) = \frac{a}{b}[2 + 4a \log b + a^2(5 - k^2) \log^2 b + 0(\log^3 b)] \quad (b \to 1). \tag{24}$$

In particular, the maximum distortion on the outer boundary of the annulus is $2a = 2K'/\pi$.

According to Theorem 1, the minimum in $\mathscr{F}_0$ of $|f'(b)|$ occurs for $f(z) = \psi(z) = -\phi(-z)$. A short calculation based on (23), (20), and [1, #122.07] yields

$$\psi'(b) = k\left[\frac{1 - \mathrm{sn}(a \log b, k)}{1 - k\,\mathrm{sn}(a \log b, k)}\right]^2 \phi'(b). \tag{25}$$

Setting b = 1, we see that the minimum distortion on the outer boundary is $2kK'/\pi$. Moreover, we find from (25) and (24)

$$\psi'(b) = \frac{ka}{b}[2 + 4ka \log b + (5k^2 - 1)a^2 \log^2 b + 0(\log^3 b)] \quad (b \to 1). \tag{26}$$

It is also possible to obtain an asymptotic expression for $\psi'(b)$ as b approaches r. For this purpose we set

$$u = K[1 - \log b/\log r] = K + a \log b \tag{27}$$

and make use of [1, #122.05] to conclude

$$\psi'(b) = \frac{-2kK(1 - k^2)}{b \log r} \cdot \frac{\mathrm{sn}(u, k)}{[\mathrm{dn}(u, k) + k\,\mathrm{cn}(u, k)]^2}. \tag{28}$$

From (28) we find

$$\psi'(b) = \frac{a k(1 - k)u}{3b(1 + k)}[6 - (k^2 - 6k + 1)u^2 + 0(u^4)] \quad (u \to 0). \tag{29}$$

This asymptotic result and the value of $\psi'(1)$ suggest the following global estimate.

THEOREM 3. *For each function* f(z) *in the family* $\mathscr{F}_0$ *and for all* z *in the annulus* $r < |z| < 1$, *the inequality*

$$|f'(z)| \geq \frac{4k(1 - k)K^2 \log(|z|/r)}{\pi(\log r)^2(1 + k)|z|}$$

holds. Here k *and* K *are constants determined by* r *according to formulas* (22), (20), *and* (21).

Proof. For $|z| = b$, we know by Theorem 1 that $|f'(z)| \geq \psi'(b)$. But we have an explicit formula (28) for $\psi'(b)$, with u defined by (27). As b ranges from r to 1, u ranges from 0 to K. The key to our proof is the inequality

$$\frac{\operatorname{sn} u}{(\operatorname{dn} u + k \operatorname{cn} u)^2} \geq \frac{2u}{\pi(1+k)^2} \quad (0 \leq u \leq K). \tag{30}$$

Because (30) appears to be of independent interest, a proof is published elsewhere [3]. If (30) is applied to estimate (28), Theorem 3 follows at once.

In similar fashion, an upper estimate for $|f'(z)|$ may be deduced from Theorem 2.

THEOREM 4. *For each* $f(z) \in \mathscr{F}_0$ *and for all* z *such that* $b^* \leq |z| < 1$, *the inequality*

$$|f'(z)| \leq \frac{2K \log(|z|/r)}{(\log r)^2 |z|}$$

holds. Here the number $b^* = b^*(r) > r$ *is as defined in Theorem* 2.

Proof. Introduce (27) into (23) and use [1, #122.05], with the result

$$\phi'(b) = \frac{2a(1-k^2)}{b} \frac{\operatorname{sn}(u, k)}{[\operatorname{cn}(u, k) + \operatorname{dn}(u, k)]^2}. \tag{31}$$

The key inequality is now

$$\frac{\operatorname{sn} u}{[\operatorname{dn} u + \operatorname{cn} u]^2} \leq \frac{u}{K(1-k)^2} \quad (0 \leq u \leq K), \tag{32}$$

a proof of which appears in [3]. The estimate of Theorem 4 follows directly from (31), (32), (27), and Theorem 2.

7. ASYMPTOTIC EXPRESSIONS FOR SMALL r

It is easy to verify that for $r = 0$, the only functions of class $\mathscr{F}_0$ are the rotations $f(z) = e^{i\alpha} z$. Because of this, it is reasonable to expect the maximum and minimum of $|f'(b)|$ to approach 1 as r tends to 0. We shall see that this is in fact true, and we shall obtain some asymptotic information on the rate of approach.

Many of the expansions for elliptic functions [1, p. 303 ff.] are given in terms of Jacobi's nome

$$q = q(k) = \exp\{-\pi K'/K\}.$$

For this reason, it is advantageous to transform the elliptic functions in (23) to depend upon the complementary modulus k', since by (20)

$$q' = q(k') = \exp\{-\pi K/K'\} = r. \tag{33}$$

Such a transformation is accomplished using [1, #161.01], and the result is

$$\phi'(z) = \frac{2a}{z}\, \mathrm{dn}(-ia\log z, k')\exp\{2i\,\mathrm{am}(-ia\log z, k')\}\,. \tag{34}$$

Now hold b fixed ($0 < b < 1$), and let r tend to zero. One finds by (34) and [1, #908.00, #908.03] the asymptotic formula

$$\phi'(b) = 1 + 4br + (9b^2 - b^{-2} - 4)r^2 + O(r^3) \quad (r \to 0)\,. \tag{35}$$

To obtain a similar asymptotic formula for $\psi'(b)$, first transform (25) by [1, #161.01] to obtain

$$\psi'(b) = k\left[\frac{1 + i\,\mathrm{tn}(ia\log b, k')}{1 + ik\,\mathrm{tn}(ia\log b, k')}\right]^2 \phi'(b)\,. \tag{36}$$

A rather laborious calculation based on (36), [1, #908.11], (22), (20) [1, #900.00], and (35) yields, finally, that

$$\psi'(b) = 1 - 4br + (9b^2 - b^{-2} - 4)r^2 + 0(r^3) \quad (r \to 0)\,. \tag{37}$$

Since $b^*(r)$ tends to zero with r, Theorems 1 and 2 allow us to deduce the following theorem from (35) and (37).

THEOREM 5. *For each* $\varepsilon > 0$ *and each fixed* z *in* $0 < |z| < 1$, *there exists an* $r_0 > 0$ *such that*

$$\big|\,|f'(z)| - 1\,\big| \leq 4|z|r + \varepsilon$$

for all $r < r_0$ *and all* $f \in \mathscr{F}_0 = \mathscr{F}_0(r)$. *The number* r_0 *depends only on* ε *and* $|z|$, *and not on* f. *The constant* 4 *is best possible.*

The coefficient $(9b^2 - b^{-2} - 4)$, which appears in both (35) and (37), changes sign at

$$b_0 = (2 + \sqrt{13})^{1/2}/3 = 0.78 \cdots.$$

Theorem 5 is false for $\varepsilon = 0$, except possibly if $|z| = b_0$. However, onesided asymptotic estimates hold for $|z| > b_0$ and for $|z| < b_0$.

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The University of Michigan