REMARKS ON A PAPER OF L. CESARI ON FUNCTIONAL ANALYSIS AND NONLINEAR DIFFERENTIAL EQUATIONS

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A well-known technique in dealing with oscillations of weakly nonlinear systems, that is, systems of the form

$$\dot{x} = Ax + \varepsilon f(t, x) \tag{1}$$

(ε a small parameter), is to set up certain transcendental equations (determining equations or bifurcation equations), the solutions of which are related to the periodic solutions of (1). One way of establishing such relations has been developed in recent years by Cesari, Hale and others; a comprehensive account of this method has just been published [4] (see also [2; Section 8.5]). The reader will find in the introduction of [4] references to literature and also a survey comparing the different ways which are known to define bifurcation equations.

If the matrix A in (1) is 0, the formal aspect of the Cesari-Hale method becomes quite transparent, so that the question arises whether a suitable generalization will allow elimination of the condition that ε be small. Cesari has in fact devised such a generalization and has obtained [1] a finite system of determining equations for arbitrary systems

$$\dot{x} = f(x, t),$$

where f is defined and satisfies certain smoothness-conditions in a finite region X of the (x, t)-space. The number of these determining equations depends upon such quantities as the bound for $|f|$ and Lipschitz constants.

In the case of a single differential equation, Cesari's method can roughly be described as follows (a more thorough report is given in Chapter 11 of [4]; another summary is given in Chapter 11 of [2]):

One associates with the differential equation $\dot{x} = f(x, t)$ a certain operator F, which depends on a finite number of parameters $a_1, \cdots, a_N$. The operator F is in some respect a modification of the operator $\xi \to \int^t f(\xi(t), t)dt$ used in existence and uniqueness proofs. The modification achieves two purposes: (1) F has the contraction property on a conveniently chosen function space over a prescribed interval [0, T] (whereas in the usual case the interval has to be chosen properly); (2) a periodic function $\xi(t)$ in the space (that is, a function with $\xi(0) = \xi(T)$) is mapped into a periodic function.

As a contraction operator F has a unique fixed element $\xi(a, t)$, which depends upon the parameters $a = (a_1, \cdots, a_N)$ and which can be constructed in the usual way by iteration. It turns out that $\xi(a, t)$ is periodic and satisfies a modified differential equation of the form

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$$\dot{x} = f(x, t) + \sum_{\nu=1}^{N} A_\nu(a)c_\nu(t),$$

where the $A_\nu(a)$ are functions of the parameters a_i. The equations $A_\nu(a) = 0$ ($\nu = 1, \cdots, N$) are then the determining equations.

As one would expect, the contraction property of F follows from the assumption that f satisfies a Lipschitz condition with respect to x. The mere existence of the fixed element $\xi(a, t)$ is a consequence of a somewhat weaker hypothesis, but then ξ, and hence the A_ν, may not be uniquely determined by the parameter a.

In the following considerations we restrict ourselves to differential equations $\dot{x} = f(x, t)$, where f satisfies a Lipschitz condition with respect to x, and we establish Cesari's result in an independent way (Sections 1-3). Successively, we obtain (Section 4) properties of the coefficients A_ν, which are important in applications. First, we prove under smoothness conditions on the differential system that the A_ν are differentiable functions of the parameters $a_1, \cdots, a_N$. Finally, keeping in mind that f, a_ρ, and A_ν are actually vectors with n components f_i, $a_{i\rho}$, and $A_{j\nu}$ (n being the order of $\dot{x} = f(x, t)$), we consider the Jacobian matrix $J = (\partial A_{j\nu}/\partial a_{i\rho})$, and we give a condition for det J to be non-vanishing. This condition is analogous to a classical result [3, pp. 348-350]. The condition is that det J is certainly non-vanishing if a certain auxiliary linear system $\dot{y} = y \cdot f_x(\xi(a, t), t)$ has no non-trivial solution of period T. In our approach the uniform topology is used instead of the L^2-norm. This is a simplification, since we then need not work with two norms at the same time (as Cesari does). Furthermore, it is then possible to use standard analytic techniques for a closer investigation of the A_ν. Our results hold even in cases where X is an infinite region in (x, t)-space. This situation occurs if the function f(x, t) is continuous and bounded for all (x, t) and satisfies a global Lipschitz condition. This case is of interest because the (highly transcendental) functions A_ν then have a somewhat simple asymptotic behavior for large a, a fact which can be exploited in the discussion of the determining equations. In [5], using Miranda's version of Brouwer's fixed point theorem, we show, for example, that these equations have a solution if the following additional condition is satisfied: There exists a permutation $i \to j(i)$ such that for each i, $f_{j(i)}(x, t)$ has on half-spaces of $R^n \times I$ of the form $x_i < S$, $x_i > T$ ($x = x_1, \cdots, x_n$, $f = f_1, \cdots, f_n$) opposite constant sign. The proof will be given as a lemma at the beginning of [5], where we present existence theorems and an approximation scheme for periodic solutions of certain types of strongly nonlinear, nonautonomous, second-order differential equations. To demonstrate the kind of result which we obtain there we quote the following one.

THEOREM. *Let* $\alpha(t) \leq \beta(t)$ *be two functions, sufficiently smooth and periodic of period* T. *Assume that* f(x, y, t) *is continuous and satisfies a local Lipschitz condition in the cylindrical region*

$$\Omega = \{(x, y, t): 0 \leq t \leq T, \alpha(t) \leq x \leq \beta(t)\}.$$

Furthermore, assume that $|f|$ *does not grow more rapidly than* y^2 *for* $|y| \to \infty$. *Finally, suppose that*

$$-\ddot{\alpha} + f(\alpha, \dot{\alpha}, t) \leq 0, \quad -\ddot{\beta} + f(\beta, \dot{\beta}, t) \geq 0 \tag{2}$$

for all t. *Then the differential equation* $\ddot{x} = f(x, \dot{x}, t)$ *has a periodic solution* ξ *(of period* T) *with* $\alpha \leq \xi \leq \beta$ *for all* t.

There are immediate applications of this theorem to nonautonomous differential equations of the Liénard type, as the example $\ddot{x} = -a\dot{x}|\dot{x}| - \sin x - \sin t$ shows. This equation satisfies all conditions of the theorem, if one chooses $\alpha = \pi/2$, $\beta = 3\pi/2$. It is clear from this theorem and the example that f is not required to satisfy a uniform Lipschitz condition in (x, t)-space. The elimination of this condition in the proofs in [5] is achieved by means of an analytic procedure of changing and smoothing the right-hand members of a system of differential equations without changing its periodic solutions. The theorem above is only an example of the type of qualitative result which can be obtained by combination of Cesari's method and the analytical and topological arguments just alluded to.

1. A LEMMA ON FOURIER SERIES

The difference between a periodic function $\phi(t)$ and a partial sum of the corresponding Fourier series can be expressed as a Dirichlet integral. For the purposes of this paper we use a very slight sharpening of a standard estimate of this integral remainder.

LEMMA 1. *Consider a periodic function* $\phi(t)$ *with period* 2π *and continuous in* $(-\infty, \infty)$ *that satisfies the Hölder condition*

$$|\phi(t_1) - \phi(t_2)| \leq C|t_1 - t_2|^\alpha$$

for all $t_1, t_2 \in [0, 2\pi]$ *and for some constants* C *and* α $(0 < \alpha \leq 1)$. *Then the following is true:*

$$\left| \int_{-\pi}^{\pi} \frac{\phi(t+x) - \phi(x)}{\sin \frac{t}{2}} \sin\left(\left(m + \frac{1}{2}\right)t\right) dt \right| \leq \begin{cases} c_\alpha C & \text{for } m = 0, 1, \\ c_\alpha C \dfrac{\log m}{m^\alpha} & \text{for } m = 2, 3, \cdots, \end{cases}$$

where c_α *is a constant which is independent of* x, m, *and* $\phi(t)$.

Proof. Let

$$\Phi_x(t) = \frac{1}{2}\{\phi(x+t) + \phi(x-t) - 2\phi(x)\}.$$

According to our assumption,

$$|\phi(t_1) - \phi(t_2)| \leq C|t_1 - t_2|^\alpha$$

if t_1, t_2 are both in an interval $[2\pi\mu, 2\pi(\mu+1)]$, μ integral. Hence there exist absolute constants $c_i > 0$ such that

$$|\Phi_x(t)| \leq c_1 C|t|^\alpha, \quad |\Phi_x(t+\eta) - \Phi_x(t)| \leq c_2 C|\eta|^\alpha$$

for all $x \in [0, 2\pi]$, $t \in [0, 2\pi]$ and all η with $|\eta| \leq \pi$. From these estimates the desired conclusion now follows from a standard result [4; Th. 10-8, Chapter II].

2. DEFINITION AND BASIC PROPERTIES OF THE OPERATOR H_m

In this section we introduce in some function spaces a certain linear operator H_m which maps an arbitrary function into a periodic function and which besides has some kind of contraction property.

Let T be a positive number to be regarded as fixed throughout this section, and let I be the interval $0 \leq t \leq T$. The following system of trigonometric polynomials is then orthonormal on I:

$$c_0(t) = \sqrt{\frac{1}{T}}, \quad c_{2k}(t) = \sqrt{\frac{2}{T}} \sin\left(\frac{2\pi k}{T} t\right), \quad c_{2k-1}(t) = \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi k}{T} t\right)$$

$$(k = 1, 2, \cdots).$$

Let Σ be the space of all n-tuples $\xi(t) = (\xi_1(t), \cdots, \xi_n(t))$, where the $\xi_i(t)$ are integrable functions over I. We denote the m-th partial sum of the Fourier expansion of $\xi(t)$ by $P_m(\xi)$:

$$P_m(\xi) = \sum_{\nu=0}^{2m} c_\nu(t) \int_0^T c_\nu(\tau)\xi(\tau)d\tau.$$

A function $\xi \in \Sigma$ will be called *periodic*, if its components $\xi_i(t)$ are continuous on I and if $\xi_i(0) = \xi_i(T)$ for $i = 1, \cdots, n$. We define for a periodic ξ its norm $\|\xi\|$ to be $\max |\xi_i(t)|$ (the maximum is taken over all $i = 1, \cdots, n$ and all $t \in I$). Convergence of a sequence of periodic ξ's means uniform convergence in all components. By $\mathscr{P}_A$, for any number $A > 0$ or $A = \infty$, we denote the subspace of all periodic ξ with $\|\xi\| \leq A$. The space $\mathscr{P}_A$ is complete (with respect to the norm defined before) and, if $A < \infty$, it is also bounded.

Let us now, for every integer $m \geq 0$ and every $\xi \in \Sigma$, consider the n-tuple $H_m(\xi)$ of functions on I given by

$$\text{(2.1)} \qquad H_m(\xi) = \int_0^t \xi(\tau)d\tau - \frac{t}{T}\int_0^T \xi(\tau)\,d\tau - P_m\left(\int_0^t \xi(\tau)d\tau - \frac{t}{T}\int_0^T \xi(\tau)d\tau\right).$$

It is obvious that

$$\text{(2.2)} \qquad H_m(\xi) \in \mathscr{P}_\infty \text{ and } P_m(H_m(\xi)) = 0 \text{ for all } \xi \in \Sigma.$$

Hence, $\xi \to H_m(\xi)$ is a linear mapping of Σ into $\mathscr{P}_\infty$.

For the following we need a certain estimate of $\|H_m(\xi)\|$, which we next establish. To this purpose let us write for the moment $\eta(t) = (\eta_1(t), \cdots, \eta_n(t))$ instead of $H_m(\xi)$. Then $\eta_i(T/2\pi\, t)$ is a periodic function of period 2π without trigonometric terms of degree less than or equal to m in its Fourier expansion; hence, it can be written as a Dirichlet integral in the form

$$\text{(2.3)} \qquad \eta_i\left(\frac{T}{2\pi} t\right) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{\phi_i(t) - \phi_i(t+\tau)}{\sin\frac{\tau}{2}} \sin\left(\left(m + \frac{1}{2}\right)\tau\right)d\tau$$

with

$$\phi_i(t) = \int_0^{\frac{T}{2\pi}t} \xi_i(\tau)d\tau - \frac{t}{2\pi}\int_0^T \xi_i(\tau)d\tau .$$

In (2.3) the functions ϕ_i are considered to be defined (by periodicity) in $(-\infty, \infty)$, and they are continuous on $(-\infty, \infty)$.

Since

$$|\phi_i(t_2) - \phi_i(t_1)| \leq \left| \int_{\frac{T}{2\pi}t_1}^{\frac{T}{2\pi}t_2} \xi_i(\tau)d\tau \right| + \frac{|t_2 - t_1|}{2\pi} \left| \int_0^T \xi_i(\tau)d\tau \right| ,$$

we obtain easily from Lemma 1 the following lemma.

LEMMA 2. *If a Hölder condition of the form*

$$\left| \int_{t_1}^{t_2} \xi_i(\tau)d\tau \right| \leq C|t_1 - t_2|^\alpha$$

is satisfied for $i = 1, \cdots, n$ *and all* $t_1, t_2 \in [0, T]$ *with* $0 < \alpha \leq 1$, *then*

$$\| H_m(\xi) \| \leq \chi(m, \alpha)C .$$

Here $\chi(m, \alpha)$ *is a certain function of* α, m *and* T *that satisfies an inequality of the form*

$$\chi(m, \alpha) \leq Q(\alpha, T)\frac{\log m}{m^\alpha} \quad (m \geq 2),$$

where Q *depends only on* α *and* T.

Finally, we observe from (2.1) that $\eta = H_m(\xi)$ is differentiable almost everywhere and that $\dot{\eta} = \xi + \xi_m$, where ξ_m is a trigonometric polynomial ξ_m of degree no greater than m. On the other hand, $P_m(\eta) = 0$, and the periodicity of η implies that $P_m(\dot{\eta}) = 0$. Therefore, $\xi_m = -P_m(\xi)$ or

$$\frac{d}{dt} H_m(\xi) = \xi - P_m(\xi), \tag{2.4}$$

almost everywhere.

3. PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS AND DETERMINING FUNCTIONS

We are now ready to define the operator F that we mentioned in the introduction.

Let $f(x, t) = (f_1(x, t), \cdots, f_n(x, t))$ be a vector function defined almost everywhere in a region

$$X = \{(x, t): |x| \leq A, \ 0 \leq t \leq T\},$$

where A is a positive constant or infinity. Here x stands for the n-tuple $(x_1, \cdots, x_n)$ of variables x_i and $|x|$ for $\max_i |x_i|$. We will assume that the following conditions are fulfilled.

(i) $$f(\xi(t), t) \in \Sigma \text{ for every } \xi(t) \in \mathscr{P}_A.$$

(ii) There exist non-negative functions $K_0(t)$, $K_1(t)$, $L(t)$ defined and integrable over I such that for all $(x, t) \in X$, $(y, t) \in X$

$$|f(x, t)| \leq K_0(t)|x| + K_1(t), \tag{3.1}$$

$$|f(x, t) - f(y, t)| \leq L(t)|x - y|. \tag{3.2}$$

Furthermore, there exist constants ℓ, α, k_0, and k_1 with $(0 < \alpha \leq 1)$ such that for all $t_1, t_2 \in [0, T]$

$$\left| \int_{t_1}^{t_2} K_i(\tau)d\tau \right| \leq k_i |t_1 - t_2|^\alpha \quad (i = 0, 1) \tag{3.3}$$

and

$$\left| \int_{t_1}^{t_2} L(\tau)d\tau \right| \leq \ell |t_1 - t_2|^\alpha. \tag{3.4}$$

In view of (i), the operator H_m defined in Section 2 can be applied to all functions of the form $f(\xi(t), t)$ with $\xi(t) \in \mathscr{P}_A$. We put now $F_m(\xi) = H_m(f(\xi(t), t))$, $\xi(t) \in \mathscr{P}_A$, and have thus defined an operator $F_m: \mathscr{P}_A \to \mathscr{P}_\infty$. It has the following properties:

$$P_m(F_m(\xi)) = 0, \tag{3.5}$$

$$\| F_m(\xi) \| \leq \chi(m, \alpha)(k_0 \| \xi \| + k_1), \tag{3.6}$$

$$\| F_m(\xi_1) - F_m(\xi_2) \| \leq \chi(m, \alpha)\ell \| \xi_1 - \xi_2 \|, \tag{3.7}$$

$$\frac{d}{dt} F_m(\xi) = f(\xi(t), t) - P_m(f(\xi(t), t)). \tag{3.8}$$

The properties (3.5) and (3.8) follow immediately from the definition of F_m and from (2.2), (2.4). According to (3.1),

$$|f(\xi(t), t)| \leq K_0(t) \| \xi \| + K_1(t);$$

hence, by (3.3), we see that

$$\left| \int_{t_1}^{t_2} f(\xi(\tau), \tau)d\tau \right| \leq (k_0 \| \xi \| + k_1)|t_1 - t_2|^\alpha.$$

Applying Lemma 2, we then obtain (3.6). Finally, we prove (3.7) in a similar way:

$$|f(\xi_1(t), t) - f(\xi_2(t), t)| \le L(t) \|\xi_1 - \xi_2\|$$

according to (3.2);

$$\int_{t_1}^{t_2} |f(\xi_1(\tau), \tau) - f(\xi_2(\tau), \tau)| d\tau \le \ell \|\xi_1 - \xi_2\| \cdot |t_1 - t_2|^\alpha$$

follows from (3.4); and finally we again apply Lemma 2 (using the linearity of the operator H_m).

For the remainder of this section, we choose the number m so large, that $\chi(m, \alpha)\ell < 1$, $\chi(m, \alpha)(k_0 A + k_1) < A$, if $A < \infty$ and $\chi(m, \alpha)k_0 < 1$ if $A = \infty$, α being the exponent in (3.3), (3.4). This is always possible, since $\lim_{m \to \infty} \chi(m, \alpha) = 0$ (see Lemma 2). We will now keep m fixed and omit the subscript m in P_m, H_m, F_m.

Let a be a system

$$\{a_{i\mu}\} \quad (i = 1, \cdots, n, \ \mu = 0, \cdots, 2m)$$

of $n(2m + 1)$ real numbers and let $\lambda(a, t) = (\lambda_1(a, t), \cdots, \lambda_n(a, t))$ be the n-tuple of the trigonometric polynomials

$$\lambda_i(a, t) = \sum_{\mu=0}^{2m} a_{i\mu} c_\mu(t) .$$

In the case $A < \infty$ we assume

$$\|\lambda(a, t)\| \le A - \chi(m, \alpha)(k_0 A + k_1) . \tag{3.9}$$

We consider then the set $\mathscr{P}_A(a)$ of all $\xi \in \mathscr{P}_A$ that satisfy the conditions

$$P(\xi) = \lambda(a, t) . \tag{3.10}$$

$\mathscr{P}_A(a)$ is closed in $\mathscr{P}_A$, and the transformation

$$\xi \to \lambda(a, t) + F(\xi)$$

maps $\mathscr{P}_A(a)$ into itself, as can be seen immediately from (3.5), (3.6) and (3.9). We claim now that there exists one and only one $\xi = \xi(a, t)$ in $\mathscr{P}_A(a)$ such that

$$\xi(a, t) = \lambda(a, t) + F(\xi(a, t)) . \tag{3.11}$$

First, since $\chi(m, \alpha)\ell < 1$, which together with (3.7) implies that the map is a contraction, it is clear that there exists at most one such $\xi(a, t)$. If $A < \infty$, $\mathscr{P}_A(a)$ is a bounded complete space, and the existence of $\xi(a, t)$ is guaranteed by Banach's fixed-point theorem. If $A = \infty$, we have to proceed as follows. We choose $A_0 > 0$ so large that

$$\|\lambda(a, t)\| \le A_0 - \chi(m, \alpha)(k_0 A_0 + k_1) . \tag{3.12}$$

This is possible since $\chi(m, \alpha)k_0 < 1$. The bounded subspace $\mathscr{P}_{A_0}$ is then invariant with respect to the mapping in question and hence has a fixed point. If we differentiate both sides of the relation (3.11) with respect to t and make use of (3.8), we find that

$$\dot{\xi}(a, t) = f(\xi(a, t), t) + \Delta(a, t)$$

with

$$\Delta(a, t) = \dot{\lambda}(a, t) - P(f(\xi(a, t), t)). \tag{3.13}$$

$\Delta(a, t)$ is a trigonometric polynomial of degree no greater than m. It represents the error, so to speak, up to which $\xi(a, t)$ satisfies the differential equation

$$\dot{x} = f(x, t). \tag{3.14}$$

We are now ready to formulate a theorem.

THEOREM 1. *Let* a *be a system of real numbers*

$$\{a_{i\mu}\} \quad (i = 1, \cdots, n, \ \mu = 0, \cdots, 2m),$$

let $\lambda(a, t)$, $\xi(a, t)$, $\Delta(a, t)$ *be defined as above and by* (3.11), (3.13), *respectively, and let* (3.9) *be satisfied. Then* $\xi(a, t)$ *is a periodic solution of the differential equation* (3.14) *if and only if* $\Delta(a, t) = 0$. *Conversely, if* $\xi \in \mathscr{P}_A(a)$ *is a solution of* (3.14), *then* $\xi = \xi(a, t)$ *and* $\Delta(a, t) = 0$.

Proof. The first part of the theorem is clear. As to the second part, let $\xi \in \mathscr{P}_A(a)$ be a solution of (3.14). We see then that (see the definition of F_m)

$$F(\xi) = H(f(\xi(t), t)) = H(\dot{\xi}) = \xi(t) - \xi(0) - P(\xi(t) - \xi(0)) = \xi + \tilde{\Delta},$$

where $\tilde{\Delta}$ is a certain trigonometric polynomial of degree no greater than m. Since $P(F(\xi)) = 0$, it follows that $\tilde{\Delta} = -P(\xi) = -\lambda(a, t)$ and therefore ξ is a solution of (3.11). Furthermore, the two identities $\dot{\xi} = f(\xi(t), t)$ and $P(\dot{\xi}) = \dot{\lambda}$ imply $\dot{\lambda} = P(f(\xi(t), t))$, that is $\Delta(a, t) = 0$. This completes the proof.

It follows from (3.13), with $\Delta(a, t) = (\Delta_1(a, t), \cdots, \Delta_n(a, t))$, that

$$\Delta_i(a, t) = \dot{\lambda}_i(a, t) - \sum_{\mu=0}^{2m} c_\mu(t) \int_0^T c_\mu(\tau) f_i(\xi(a, \tau), \tau)\, d\tau. \tag{3.15}$$

We also observe that $\dot{c}_0(t) = 0$ and that $\dot{c}_\mu(t)$ is a constant multiple of $c_{\mu\pm 1}$ for $\mu > 0$. It follows then from the definition of λ_i that $\dot{\lambda}_i$ can be written in the form

$$\dot{\lambda}_i(a, t) = \sum_{\mu=1}^{2m} p_\mu\, a_{i\mu^*}\, c_\mu(t),$$

where $p_\mu \neq 0$ is a constant and $\mu \to \mu^*$ is a certain permutation of the numbers $1, \cdots, 2m$. Therefore $\Delta_i(a, t)$ is a trigonometric polynomial $\Sigma_{\mu=0}^{2m} A_{i\mu}(a) c_\mu(t)$, and its coefficients are given by

$$A_{i0}(a) = -\sqrt{\frac{1}{T}} \int_0^T f_i(\xi(a, \tau), \tau) d\tau,$$

(3.16)

$$A_{i\mu}(a) = \mathfrak{p}_\mu a_{i\mu*} - \int_0^T f_i(\xi(a, \tau), \tau) c_\mu(\tau) d\tau \quad (\mu > 0)$$

The $A_{i\mu}(a)$ are called determining functions of the differential equation (3.14). In case $A = \infty$, the $A_{i\mu}$ are defined for all $a \in R^{n(2m+1)}$; in case $A < \infty$, they are defined only for those a for which (3.12) holds. By Theorem 1, the problem of finding periodic solutions of the differential equation is equivalent to the problem of finding common zeros of the determining functions.

We finally observe that the $\xi(a, t)$ are continuous functions of t and a. This follows, in the same way as in Cesari's paper [1], from the inequality

$$\|\xi(a, t) - \xi(b, t)\| \leq \|\lambda(a, t) - \lambda(b, t)\| + \ell\chi(m, \alpha)\|\xi(a, t) - \xi(b, t)\|$$

or

$$\|\xi(a, t) - \xi(b, t)\| \leq \frac{1}{1 - \ell\chi(m, \alpha)} \|\lambda(a, t) - \lambda(b, t)\|,$$

which is a consequence of (3.7) and the definition of $\xi(a, t)$. Hence, in view of (3.16), the $A_{i\mu}(a)$ are also continuous functions of a if the f_i satisfy suitable conditions: for example, the interval $[0, T]$ can be divided into a finite number of subintervals $[T_i, T_{i+1}]$ such that $f(x, t)$ is a bounded continuous function in each subregion $X_i = \{(x, t) \in X,\ T_i < t < T_{i+1}\}$ of X.

4. FURTHER INVESTIGATION OF THE DETERMINING EQUATIONS

The functions $A_{i\mu}(a)$ depend continuously on the $a_{j\rho}$ as we have seen in Section 3. We shall now discuss whether they are differentiable functions, and we shall give a sufficient condition for the non-vanishing of their Jacobian.

We start with some general remarks. Let $\xi(u, t)$ be an n-tuple of functions of u and t, defined for $t \in [0, T]$ and $u \in U$, where U is a certain open interval. Let us assume that $\xi(u, t) \in \Sigma$ for each u and that $\partial\xi(u, t)/\partial u$ exists and is continuous in both variables. Then it follows immediately that $H_m(\xi(u, t))$ and $P_m(\xi(u, t))$ have one continuous partial derivative with respect to u and that

$$\frac{\partial}{\partial u} P_m(\xi(u, t)) = P_m\left(\frac{\partial}{\partial u}\xi(u, t)\right),$$

(4.1)

$$\frac{\partial}{\partial u} H_m(\xi(u, t)) = H_m\left(\frac{\partial}{\partial u}\xi(u, t)\right).$$

Furthermore, if the functions $f(x, t)$ have continuous partial derivatives, we obtain from (4.1) the relation

$$\frac{\partial}{\partial u} F_m(\xi(u, t)) = H_m\left(\frac{\partial}{\partial u}\xi(u, t) \cdot f_x(\xi(u, t), t)\right).$$

Here $f_x(x, t)$ stands for the Jacobian matrix $(\partial f_i/\partial x_j)$; $F_m(\xi(u, t))$, $\partial\xi(u, t)/\partial u$ have to be regarded as $1\times n$-matrices. We shall assume from now on that $f_x(x, t)$ exists and is continuous for all $(x, t) \in X$.

Let $\lambda(a(u), t) = \lambda$ be a trigonometric polynomial of degree no greater than m, the coefficients $a_{j\rho}(u)$ of which are continuously differentiable functions of a parameter u $(u \in U)$. We assume that the condition (3.9) is satisfied for all $u \in U$. The relation

$$\xi = \lambda + F_m(\xi)$$

defines uniquely a continuous function ξ of u and t, as we have seen in Section 3. We wish to show that ξ has a continuous partial derivative with respect to u. To this purpose we make the following assumptions.

(i) The partial derivatives $\partial f_i/\partial x_j$ are bounded for all $(x, t) \in X$ $(i, j = 1, \cdots, n)$, that is,

$$\left|\frac{\partial f_i}{\partial x_j}\right| \leq D; \tag{4.2}$$

(ii) $nD\chi(m, 1) < 1$.

According to Lemma 2, the second condition can always be enforced by taking m sufficiently large.

Let us now consider the linear homogeneous system of differential equations

$$\dot{y} = y \cdot f_x(\xi, t), \tag{4.3}$$

where y stands for the unknowns and the matrix of the coefficients is given by the Jacobian $f_x(x, t)$ with $x = \xi$. The differential equation (4.3) can be treated along the lines of the previous section: From (4.2) we see that

$$|y \cdot f_x(\xi, t)| \leq |y| nD;$$

hence, the conditions (3.1), (3.2) are satisfied by taking $K_1 = 0$, $K_0 = L = nD$, $A = \infty$. The condition to be imposed on m in this case turns out to be nothing else than (4.2). Therefore, the sequence η_ν, which is defined by the relations

$$\eta_0 = \frac{\partial}{\partial u}\lambda, \qquad \eta_{\nu+1} = \frac{\partial}{\partial u}\lambda + H_m(\eta_\nu \cdot f_x(\xi, t)) \tag{4.4}$$

converges uniformly with respect to t and u (uniformity with respect to u is a consequence of the fact that the ratio of contraction for the underlying mapping is independent of u).

On the other hand, $\xi = \lim_{\nu\to\infty} \xi_\nu$ with

$$\xi_0 = \lambda,$$

$$\xi_{\nu+1} = \lambda + H_m(f(\xi_\nu, t)).$$

It follows from these recursion relations that ξ_ν has a continuous derivative with respect to u and that

$$\frac{\partial}{\partial u}\xi_{\nu+1} = \frac{\partial}{\partial u}\lambda + H_m\left(\frac{\partial}{\partial u}\xi_\nu \cdot f_x(\xi_\nu, t)\right). \tag{4.5}$$

Taking norms on both sides and using Lemma 2 (with $\alpha = 1$), we conclude that

$$\left\|\frac{\partial}{\partial u}\xi_{\nu+1}\right\| \leq \left\|\frac{\partial}{\partial u}\lambda\right\| + nD\chi(m, 1)\cdot\left\|\frac{\partial}{\partial u}\xi_\nu\right\| \quad (\nu = 0, 1, \cdots).$$

This leads to the estimate

$$\left\|\frac{\partial}{\partial u}\xi_\nu\right\| \leq \left\|\frac{\partial}{\partial u}\lambda\right\| \sum_{k=0}^{\nu}(nD\chi(m, 1))^k;$$

hence, in view of (4.2),

$$\left\|\frac{\partial}{\partial u}\xi_\nu\right\| < \left\|\frac{\partial}{\partial u}\lambda\right\| \frac{1}{1 - nD\chi(m, 1)}.$$

In other words: The $\partial\xi_\nu/\partial u$ are uniformly bounded with respect to t and u. We shall now show that this sequence is in fact uniformly convergent. We do this by proving that $\eta_\nu - \partial\xi_\nu/\partial u$ converges uniformly to zero.

Let $\varepsilon > 0$ be given. Since the elements of the matrix $f_x(x, t)$ are continuous in x, we can find a number $N = N(\varepsilon)$ such that

$$\left\|H_m\left(\frac{\partial}{\partial u}\xi_\nu\cdot[f_x(\xi_\nu, t) - f_x(\xi, t)]\right)\right\| < \varepsilon \quad (\nu \geq N).$$

We then conclude from (4.5) that

$$\frac{\partial}{\partial u}\xi_{\nu+1} = \frac{\partial}{\partial u}\lambda + H_m\left(\frac{\partial}{\partial u}\xi_\nu\cdot f_x(\xi, t)\right) + r_\nu$$

with $\|r_\nu\| < \varepsilon$ if $\nu \geq N$. Subtracting the last relation from (4.4), we obtain the inequality

$$\left\|\eta_{\nu+1} - \frac{\partial}{\partial u}\xi_{\nu+1}\right\| \leq nD\chi(m, 1)\left\|\eta_\nu - \frac{\partial}{\partial u}\xi_\nu\right\| + \|r_\nu\|,$$

and thus we finally see that for $\omega = 0, 1, \cdots$,

$$\left\|\eta_{N+\omega} - \frac{\partial}{\partial u}\xi_{N+\omega}\right\| \leq \varepsilon\sum_{k=0}^{\omega-1}(nD\chi(m, 1))^k + (nD\chi(m, 1))^\omega\left\|\eta_N - \frac{\partial}{\partial u}\xi_N\right\|,$$

and

$$\left\| \eta_{N+\omega} - \frac{\partial}{\partial u} \xi_{N+\omega} \right\| < \varepsilon \frac{1}{1 - nD\chi(m, 1)} + (nD\chi(m, 1))^{\omega} \left\| \eta_N - \frac{\partial}{\partial u} \xi_N \right\|.$$

Our statement follows now immediately from this inequality.

We thus arrive at the conclusion that our assumptions on $f_x(x, t)$ and the condition (4.2) guarantee the existence and continuity of $\partial \xi(u, t)/\partial u$. In particular, they guarantee the existence and continuity of the derivatives $\partial \xi(a, t)/\partial a_{j\rho}$. Here a stands for the $n(2m + 1)$-tuple

$$\{a_{j\rho}\} \quad (j = 1, \cdots, n, \ \rho = 0, \cdots, 2m),$$

and $\xi(a, t)$ has the same meaning as in Section 3. By the condition (3.9), a has to be restricted to a certain open set, which we denote henceforth by $\mathfrak{U}$.

In view of (3.16) and our assumption on $f_x(x, t)$, the functions $A_{i\mu}(a)$ have continuous partial derivatives with respect to $a_{j\rho}$ if $\xi(a, t)$ has. Therefore we can formulate the following result.

THEOREM 2. *Assume that the functions* $f_i(x, t)$ *have bounded continuous partial derivatives with respect to* x *for all* $(x, t) \in X$. *Let* m *be chosen such that the conditions of Section* 3 *are satisfied and* (4.2) *holds in addition. Then* $a_{i\mu}(a)$ *has continuous partial derivatives with respect to every* $a_{j\rho}$ *if* $a \in \mathfrak{U}$.

We conclude this section with a remark on the Jacobian matrix

$$J = \left(\frac{\partial A_{i\mu}}{\partial a_{j\rho}} \right)$$

of the determining functions.

THEOREM 3. *Let the assumption on* f(x, t) *be the same as in Theorem* 2, *and let* $a \in \mathfrak{U}$. *If the linear homogeneous system*

$$\dot{y} = y \cdot f_x(\xi(a, t), t)$$

has no non-trivial periodic solution of period T, *then* $\det(J) \neq 0$.

Proof. We begin with an identity that is an immediate consequence of the definition of the $A_{i\mu}$:

$$\frac{\partial}{\partial t} \xi_i(a, t) = f_i(\xi(a, t), t) + \sum_{\mu=0}^{2m} A_{i\mu}(a) c_{\mu}(t),$$

or, in matrix notation,

$$\frac{\partial}{\partial t} \xi(a, t) = f(\xi(a, t), t) + \sum_{\mu=0}^{2m} A_{\mu}(a) c_{\mu}(t),$$

A_μ being the $1 \times n$ matrix $(A_{1\mu}, \cdots, A_{n\mu})$. We have already proved that $\xi(a, t)$ has continuous first partial derivatives with respect to the $a_{j\rho}$. Thus it follows from our assumptions on f and from the last relation that $\partial \xi(a, t)/\partial a_{j\rho}$ is a solution of the linear equation

$$\dot{y} = y \cdot f_x(\xi(a, t), t) + \sum_{\mu=0}^{2m} \frac{\partial}{\partial a_{j\rho}} A_\mu(a)\, c_\mu(t).$$

Let $\psi(t)$ be a fundamental matrix of the corresponding homogeneous equation

$$\dot{y} = y \cdot f_x(\xi(a, t)t), \tag{4.6}$$

and let $\psi_1(t), \cdots, \psi_n(t)$ be the columns of ψ. We may then write a representation of the form

$$\frac{\partial}{\partial a_{j\rho}} \xi = \sum_{\mu=0}^{2m} \frac{\partial}{\partial a_{j\rho}} A_\mu \cdot \left(\int_0^t \psi^{-1}(\tau) c_\mu(\tau) d\tau \right) \cdot \psi(t) + \mathfrak{p}_{j\rho}\psi(t), \tag{4.7}$$

where $\mathfrak{p}_{j\rho}$ is a certain constant $1\times n$-matrix. Since $\partial\xi/\partial a_{j\rho}$ is periodic in t, $\mathfrak{p}_{j\rho}$ must satisfy the relation

$$\mathfrak{p}_{j\rho} \cdot (\psi(0) - \psi(T)) = \sum_{\mu=0}^{2m} \frac{\partial}{\partial a_{j\rho}} A_\mu \cdot \left(\int_0^T \psi^{-1}(\tau) c_\mu(\tau) d\tau \right) \cdot \psi(T).$$

Now the assumption on the periodic solutions of (4.6) implies that

$$\det(\psi(0) - \psi(T)) \neq 0,$$

and we can solve the last equation with respect to $\mathfrak{p}_{j\rho}$. We can thus change (4.7) to

$$\begin{aligned} \frac{\partial}{\partial a_{j\rho}} \xi_i(a, t) = \sum_{\mu=0}^{2m} \frac{\partial}{\partial a_{j\rho}} A_\mu \cdot \Bigg\{ & \int_0^t \psi^{-1}(\tau) c_\mu(\tau) d\tau \\ & + \left(\int_0^T \psi^{-1}(\tau) c_\mu(\tau) d\tau \right) \cdot (\psi(0) - \psi(T))^{-1} \Bigg\} \cdot \psi_i(t) \end{aligned} \tag{4.8}$$

for $i = 1, \cdots, n$. Next we multiply the last relation by $c_\sigma(t)$ $(\sigma = 0, \cdots, 2m)$ and integrate with respect to t from 0 to T. Since

$$P_m(\xi_i(a, t)) = \sum_{\mu=0}^{2m} a_{i\mu} c_\mu(t),$$

$$P_m\left(\frac{\partial}{\partial a_{j\rho}} \xi_i(a, t)\right) = \begin{cases} 0 & \text{for } i \neq j, \\ c_\rho(t) & \text{for } i = j, \end{cases}$$

and

$$\int_0^T c_\sigma(t) \frac{\partial}{\partial a_{j\rho}} \xi_i(a, t) dt = \delta_{ij}\delta_{\rho\sigma}.$$

So we finally arrive at the following set of $(n(2m+1))^2$ relations

$$\delta_{ij}\delta_{\rho\sigma} = \sum_{\mu=0}^{2m} \frac{\partial}{\partial a_{j\rho}} A_\mu \cdot \int_0^T c_\sigma(t) \left\{ \int_0^t \psi^{-1}(\tau)c_\mu(\tau)d\tau + \left(\int_0^T \psi^{-1}(\tau)c_\mu(\tau)d\tau \right) \cdot (\psi(0) - \psi(T))^{-1} \right\} \cdot \psi_i(t)dt. \tag{4.9}$$

We now arrange the $n\times 1$-matrices

$$\int_0^T c_\sigma(t) \left\{ \int_0^t \psi^{-1}(\tau)c_\mu(\tau)d\tau + \left(\int_0^T \psi^{-1}(\tau)c_\mu(\tau)d\tau \right) \cdot (\psi(0) - \psi(T))^{-1} \right\} \cdot \psi_i(t)dt$$

in a rectangular scheme such that μ marks the rows and the pair (i, ρ) marks the columns $(i = 1, \cdots, n;\ \rho,\ \mu = 0, \cdots, 2m)$. We thus obtain an $n(2m+1)$-square-matrix Ξ. In a similar way we arrange the $1\times n$-matrices $\partial A_\mu/\partial a_{j\rho}$ in a rectangular scheme, using (j, ρ) as row index and μ as column index. This matrix is obviously the Jacobian matrix J, and from (4.9) we obtain the matrix equation

$$I = J \cdot \Xi,$$

where I is the $n(2m+1)$-unit-matrix. Hence Theorem 3 is proved.

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