DIFFEOMORPHISMS OF PERIOD TWO

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1. INTRODUCTION

The purpose of this note is a study of the role of the Whitney classes of the normal bundle to the fixed point set in the Smith exact sequence of a diffeomorphism of period 2 on a closed manifold. The content of this note was suggested as an attempt to relate Smith theory to the connection between cobordism and involutions which was outlined in [3]. We wish to emphasize that we are dealing with but one aspect of the study of involutions.

For any involution (T, X) we denote the fixed point set by $F \subset X$, the quotient space by X/T, and the quotient map by $\nu: X \to X/T$. We shall use homology and cohomology with Z_2-coefficients only. If X is compact and $A \supset F$ is a closed invariant set containing the fixed points, then a Gysin exact sequence

$$\cdots \to H^j(X/T, A/T) \xrightarrow{\mu} H^{j+1}(X/T, A/T) \xrightarrow{\nu} H^{j+1}(X, A) \xrightarrow{\sigma} H^{j+1}(X/T, A/T) \to \cdots$$

may be associated with (T, (X, A)). If $A = F = \emptyset$, then the characteristic class of (T, X) is $\mu(u) = c \in H^1(X/T)$, where u is the unit cohomology class of X/T, [2]. An equivariant map f: (T, (X, A)) $\to$ (T, (Y, B)), where B contains the fixed point set of (T, Y), induces a commutative diagram joining the Gysin sequences.

We refer to [1, p. 35] for more details about the Smith theory. To each involution (T, X), there is associated the Smith exact sequence

$$\cdots \to H^j(X/T, F) \xrightarrow{J} H^j(X) \xrightarrow{I} H^j(X/T, F) + H^j(F) \xrightarrow{d} H^{j+1}(X/T, F) \to \cdots$$

where J is the composite homomorphism $H^j(X/T, F) \to H^j(X, T) \to H^j(X)$, d is the sum of $\mu: H^j(X/T, F) \to H^{j+1}(X/T, F)$ with the coboundary $\delta: H^j(F) \to H^{j+1}(X/T, F)$, and $I = \sigma + i^*$, where $i^*: H^j(X) \to H^j(F)$ is induced by inclusion and σ is the transfer homomorphism. Again, an equivariant map f: (T, X) $\to$ (T, Y) induces a commutative diagram joining the Smith sequences. With the aid of the Smith sequence, Floyd showed [1, p. 42] that if X is finite dimensional, then for any integer j,

$$\dim H^j(X/T, F) + \sum_{i \geq j} \dim H^i(F) \leq \sum_{i \geq j} \dim H^i(X),$$

where dim is that of the linear space over Z_2.

We shall be primarily interested in the homomorphisms

$$\chi_j: \sum_0^j H^i(F) \to H^{j+1}(X/T, F)$$

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defined by $\chi_j(\beta_0, \cdots, \beta_j) = \Sigma_0^j \mu^{j-1}\delta(\beta_i)$. We shall use the following elementary remark.

Remark 1.1. *If* $H^{j+i}(X) = 0$ $(r + 1 \leq i \leq k)$ *and if* $\chi_{j+k}(\beta_0, \cdots, \beta_{j+k}) = 0$, *then* $\beta_{j+i} = 0$ $(r + 1 \leq i \leq k)$ *and* β_{j+r} *lies in the image of* $H^{j+r}(X) \to H^{j+r}(F)$.

We note that all the χ_j are related by the inductive formula

$$\mu\chi_{j-1}(\beta_0, \cdots, \beta_{j-1}) + \delta\beta_j = \chi_j(\beta_0, \cdots, \beta_j).$$

Now $(\chi_{j+k-1}(\beta_0, \cdots, \beta_{j+k-1}), \beta_{j+k})$ lies in the image of

$$I: H^{j+k}(X) \to H^{j+k}(X/T, F) + H^{j+k}(F);$$

but if $H^{j+k}(X) = 0$, then $\chi_{j+k-1}(\beta_0, \cdots, \beta_{j+k-1}) = 0$ and $\beta_{j+k} = 0$. We proceed by induction to obtain (1.1). The homomorphism χ_j suggests the simplest, and most commonly employed, invariant based upon the Smith sequence. If $\beta \neq 0 \in H^i(F)$, then the *height* of β is equal to the least integer j for which $\mu^j\delta(\beta) = 0$. Under an equivariant map f: (T, X) $\to$ (T, Y), the height of $f^*(\beta) \in H^i(F)$ cannot exceed the original height of β in the Smith sequence of (T, Y).

We shall consider a diffeomorphism of period 2 on a closed manifold (T, M^{n+k}) together with a component, $V^n \subset M^{n+k}$, of the fixed point set $F \subset M^{n+k}$. This component is a closed connected regular submanifold for which the Whitney classes of the normal bundle are $v_0 = u, v_1, \cdots, v_k$. We let $\theta: H^j(V^n) \to H^{j+k}(M^{n+k})$ denote the composite homomorphism

$$H^j(V^n) \simeq H_{n-j}(V^n) \xrightarrow{i_*} H_{n-j}(M^{n+k}) \simeq H^{j+k}(M^{n+k}).$$

PROPOSITION 1.2. *For any cohomology class* $\alpha \in H^j(V^n)$,

$$I\theta(\alpha) = \left(\sum_0^{k-1} \mu^{k-1-i}\delta(v_i\alpha), v_k\alpha\right)$$

in $H^{j+k}(M^{n+k}/T, F) + H^{j+k}(F)$.

This is our principal result. It was suggested by two known results, which are that $i^*(\theta(\alpha)) = v_k\alpha$ and that $I(\beta_{n+k}) = \mu^{k-1}\delta(\alpha_n)$, where α_n and β_{n+k} are, respectively, the fundamental cohomology classes of V^n and of M^{n+k}. We shall state an immediate corollary.

COROLLARY 1.3. *If* $H^{j+i}(M^{n+k}) = 0$ $(r + 1 \leq i \leq k)$, *then for every* $\alpha \in H^j(V^n)$, $v_i\alpha = 0$, $(r + 1 \leq i \leq k)$ *and* $v_r\alpha$ *lies in the image of* $H^{j+r}(M^{n+k}) \to H^{j+r}(F)$. *If* $r + 1 = 0$, *then* $H^j(V^n) = 0$.

We note that according to Proposition 1.2, the class $(0, \cdots, 0, v_1\alpha, \cdots, v_k\alpha)$ lies in the kernel of χ_{j+k}. We apply (1.1). We can also estimate the height of $\beta \in H^i(V^n)$ in the Smith sequence of (T, M^{n+k}); in fact, the height (β) cannot exceed $k + r$, where r is the largest integer for which $\bar{v}_r\beta \neq 0$. Here $\bar{v}_r$ is the r^{th} dual Whitney class of the normal bundle. Now with this estimate and our principal theorem we can prove the following corollary.

COROLLARY 1.4. *Let* (T, M^{n+k}) *be a diffeomorphism of period* 2 *on a closed manifold which is* k-*connected* mod Z_2. *If there is a non-void component* V^n *in the fixed point set, then the fixed point set is connected and* $v_1 = v_2 = \cdots = v_k = 0$.

This last might be regarded as one possible generalization of the well-known result which asserts that the fixed point set of an involution on a mod Z_2-cohomology sphere is again a Z_2-cohomology sphere. The inequality mentioned earlier is a more immediate generalization. We shall mention one more result which reflects the influence that one component of the fixed point set may exert on the remaining components.

PROPOSITION 1.5. *If* (T, M^{n+k}) *is a diffeomorphism of period* 2 *on a closed connected manifold and if* V^n *is a component of the fixed point set for which* $i_*: H_i(V^n) \to H_i(M^{n+k})$ *is surjective for* $0 \leq i \leq r$, *then every other component of the fixed point set has dimension less than* $n + k - r$.

2. THE INVOLUTION ON A THOM SPACE

We consider a k-dimensional vector bundle over a closed connected manifold $[E, V^n, R^k, \pi; 0(k)]$. There is a natural fibre-preserving involution (T, E) which, on each fibre, agrees with the scalar matrix $-I$. We let M denote the one-point compactification of E; then M is the Thom space, and there exists an induced involution (T, M) whose fixed point set is the disjoint union $V^n \cup q = F$. We shall describe the Smith sequence of (T, M) in detail so that we may establish our result in this case. We shall then transfer back to the general case by a suitable equivariant map into (T, M).

Let N be the closed unit k-cell bundle associated with $E \to V^n$, and let $\dot{N}$ denote the boundary. Then $N/\dot{N}$ is M, and M/T is N/T with $\dot{N}/T$ collapsed to a point. The map $\pi|\dot{N} \to V^n$ induces a projective space bundle $p: \dot{N}/T \to V^n$ with fibre P^{k-1}, and N/T is the mapping cylinder of p. The properties of this associated projective space bundle were discussed in [2, Sec. 6]. In particular, if $c \in H^1(\dot{N}/T)$ is the characteristic class of the fixed point free involution $(T, \dot{N})$, then any element in $H^j(\dot{N}/T)$ can be expressed as

$$c^{k-1} p^*(\alpha_{j-k+1}) + \cdots + p^*(\alpha_j)$$

by a *unique* choice of the α_{j-k+i}. It was also noted that $c^k = \Sigma_1^k c^{k-j} p^*(v_j)$, where $v_0, v_1, \cdots, v_k$ are the Whitney classes of $E \to V^n$. In this section we shall need the formula for c^{k+r} $(r \geq 0)$, and we proceed to establish it now. We define cohomology classes on V^n by $V_{j,r} = \Sigma_0^r \bar{v}_i v_{j+r-i}$ for all $1 \leq j \leq k$. These classes satisfy the relations

(i) $V_{1,r} = \bar{v}_{r+1}$

(ii) $V_{j,r+1} = V_{j+1,r} + v_j V_{1,r}$

(iii) $V_{k,r} = \bar{v}_r v_k$.

We wish to show that $c^{k+r} = \Sigma_1^k c^{k-j} p^*(V_{j,r})$. The formula is valid for $r = 0$ since $V_{j,0} = v_j$, so we proceed inductively to write

$$c^{k+r+1} = cc^{k+r} = c\left(\sum_1^k c^{k-j}p^*(V_{j,r})\right)$$

$$= c^k p^*(V_{1,r}) + \sum_1^{k-1} c^{k-j}p^*(V_{j+1,r})$$

$$= \sum_1^{k-1} c^{k-j}p^*(v_j V_{1,r} + V_{j+1,r}) + p^*(v_k V_{1,r})$$

$$= \sum_1^k c^{k-j}p^*(V_{j,r+1}) .$$

We return now to M/T. Since N/T is the mapping cylinder of $\dot{N}/T \to V^n$, there exists a short exact sequence

$$0 \to H^j(V^n) \to H^j(\dot{N}/T) \to H^{j+1}(N/T, \dot{N}/T) \to 0 .$$

LEMMA 2.1. *For* $j > 0$ *the homomorphism* $\nu^*: H^j(M/T) \to H^j(M)$ *is trivial.*

We note that under $\nu^*: H^1(\dot{N}/T) \to H^1(\dot{N})$, $\nu^*(c) = 0$; thus

$$\nu^*(c^{k-1}p^*(\alpha_{j-k+1}) + \cdots + cp^*(\alpha_{j-1})) = 0 \in H^j(\dot{N}) .$$

If we appeal to the commutative diagram

$$\begin{array}{ccc} H^j(\dot{N}) & \to & H^{j+1}(N, \dot{N}) \\ \uparrow & & \uparrow \\ H^j(\dot{N}/T) & \to & H^{j+1}(N/T, \dot{N}/T) , \end{array}$$

we see that (2.1) follows since any class in $H^{j+1}(N/T, \dot{N}/T)$ is the image under the coboundary of a unique class of the form $c^{k-1}p^*(\alpha_{j-k+1}) + \cdots + cp^*(\alpha_{j-1})$.

LEMMA 2.2. *For* $j > 0$ *there exists a short exact sequence*

$$0 \to H^j(M) \xrightarrow{I} H^j(M/T, F) + H^j(F) \xrightarrow{d} H^{j+1}(M/T, F) \to 0 .$$

This follows from (2.1) since J is the composite homomorphism

$$H^j(M/T, F) \to H^j(M/T) \to H^j(M) .$$

In dimension 0 we note that $\tilde{H}^0(F) \simeq H^1(M/T, F) \simeq Z_2$.

We shall often wish to consider $\alpha \in H^j(V^n)$ as a cohomology class on $F = V^n \cup q$. We shall replace α by $(\alpha, 0) \in H^j(V^n) + H^j(q)$ without comment. We now introduce, corresponding to $E \to V^n$, the bundle $[B, V^n, S^{k-1} \times I, \pi; 0(k)]$ with fibre the closed annulus $S^{k-1} \times I$. There is a fibre-preserving, fixed point free involution (T, B) which on each fibre is such that $(x, t) \to (-x, t)$. The boundary of B, denoted by $\dot{B}$, is the disjoint union $\dot{N}_0 \cup \dot{N}_1$. We observe that each $\dot{N}_i$ $(i = 0, 1)$ is an equivariant deformation retract of (T, B). We denote by $p: B/T \to V^n$ the induced fibre map with fibre $P^{k-1} \times I$, and we set

$$p_0 = p|\dot{N}_0/T \to V^n, \quad p_1 = p|\dot{N}_1/T \to V^n.$$

We let

$$c \in H^1(B/T), \quad c_0 \in H^1(\dot{N}_0/T), \quad c_1 \in H^1(\dot{N}_1/T)$$

be the characteristic classes of (T, B), $(T, \dot{N}_0)$ and $(T, \dot{N}_1)$. The image under

$$i^*: H^j(B/T) \to H^j(\dot{N}_0/T) + H^j(\dot{N}_j/T)$$

of any class $c^{k-1}p^*(\alpha_{j-k+1}) + \cdots + p^*(\alpha_j)$ is

$$(c_0^{k-1}p^*(\alpha_{j-k+1}) + \cdots + p_0^*(\alpha_j),\ c_1^{k-1}p_1^*(\alpha_{j-k+1}) + \cdots + p_1^*(\alpha_j)).$$

Next we define an equivariant map $f: (T, (B, \dot{B})) \to (T, (M, F))$ by setting f equal to the identity on $B - \dot{B}$, $f|\dot{N}_1 = q$ and $f|\dot{N}_0 = \pi|\dot{N}_0 \to V^n$. Let

$$g: (B/T, \dot{B}/T) \to (M/T, F)$$

be the induced map on quotient spaces, then by excision,

$$g^*: H^j(M/T, F) \simeq H^j(B/T, \dot{B}/T).$$

Furthermore, there are commutative diagrams

$$\begin{array}{ccccccc} \cdots \to H^j(B/T, \dot{B}/T) & \xrightarrow{\mu} & H^{j+1}(B/T, \dot{B}/T) & \xrightarrow{\nu} & H^{j+1}(B, \dot{B}) \to \cdots \\ \uparrow & & \uparrow & & \uparrow \\ \cdots \to H^j(M/T, F) & \xrightarrow{\mu} & H^{j+1}(M/T, F) & \xrightarrow{\nu} & H^{j+1}(M, F) \to \cdots \end{array}$$

and

$$\begin{array}{ccc} \cdots \to H^j(\dot{N}_0/T \cup \dot{N}_1/T) & \xrightarrow{\delta} & H^{j+1}(B/T, \dot{B}/T) \\ \uparrow & & \uparrow \\ \cdots \to H^j(V^n \cup q) & \xrightarrow{\delta} & H^{j+1}(M/T, F). \end{array}$$

The map $g|\dot{B}/T$ splits into $p_0: \dot{N}_0/T \to V^n$ and $\varepsilon(\dot{N}_1/T) = q$; thus in the second diagram the first vertical homomorphism is

$$p_0^* + \varepsilon^*: H^j(V^n) + H^j(q) \to H^j(\dot{N}_0/T) + H^j(\dot{N}_1/T).$$

We shall now establish the following result.

LEMMA 2.3. *For any class* $\alpha \in H^j(V^n)$, $\Sigma_0^k \mu^{k-i}\delta(v_i\alpha) = 0$ *in* $H^{j+k+1}(M/T, F)$.

We repeat that $v_i\alpha \in H^{i+j}(F)$ is $(v_i\alpha, 0)$; thus under

$$g^*: H^{j+k+1}(M/T, F) \simeq H^{j+k+1}(B/T, \dot{B}/T)$$

we see that

$$g^*\left(\sum_0^k \mu^{k-i}\delta(v_i\alpha)\right) = \sum_0^k \mu^{k-i}\delta(p_0^*(v_i\alpha), 0).$$

We only need to show this second expression is 0. Since $\Sigma_0^k(c_0^{k-i}\, p^*(v_i\alpha), 0) = 0$ in $H^{j+k}(\dot{B}/T)$ and since $i^*(c) = c_0 + c_1$,

$$\sum_0^k c^{k-i}\delta(p_0^*(v_i\alpha), 0) = 0 \in H^{j+k+1}(B/T, \dot{B}/T).$$

Here we have used the cup product pairing of $H^*(B/T)$ with $H^*(B/T, \dot{B}/T)$ into $H^*(B/T, \dot{B}/T)$. The homomorphism $\mu\colon H^j(B/T, \dot{B}/T) \to H^{j+1}(B/T, \dot{B}/T)$ is given by forming the cup product with the characteristic class $c \in H^1(B/T)$. Thus $\Sigma_0^k \mu^{k-i}\delta(p_0^*(v_i\alpha), 0) = 0$, and (2.3) follows.

We intentionally stated (2.3) separately, and now we shall extend the result to characterize the kernel of the homomorphism $\chi_{j+k}\colon \Sigma_0^{j+k} H^i(V^n) \to H^{j+k+1}(M/T, F)$ which is defined by

$$\chi_{j+k}(\beta_0, \beta_1, \cdots, \beta_{j+k}) = \sum_0^{j+k} \mu^{j+k-i}\delta(\beta_i).$$

THEOREM 2.4. *The necessary and sufficient condition that* $\chi_{j+k}(\beta_0, \cdots, \beta_{j+k}) = 0$ *is that*

$$\sum_0^j V_{p,\, j-i}\beta_i = \beta_{j+p} \quad \textit{for } 1 \leq p \leq k.$$

We may permit $j < 0$, but the condition then implies that $\beta_0 = \cdots \beta_{j+k} = 0$. We consider the equation $\Sigma_0^{j+k}\mu^{j+k-i}\delta(\beta_i) = 0$, which, under g^*, is converted to $\Sigma_0^{j+k} c^{j+k-i} p^*(\beta_i) = 0$ on $\dot{N}/T$. We therefore write

$$0 = \sum_0^{j+k} c^{k+j-i}p^*(\beta_i) = \sum_0^{j} c^{k+j-i}p^*(\beta_i) + \sum_{j+k}^{j+k} c^{k+j-i}p^*(\beta_i)$$

$$= \sum_0^j \left(\sum_1^k c^{k-p}p^*(V_{p,\, j-i}\beta_i)\right) + \sum_1^k c^{k-p}p^*(\beta_{j+p})$$

$$= \sum_1^k c^{k-p}\left(\sum_0^j p^*(V_{p,\, j-i}\beta_i + \beta_{j+p})\right) = 0.$$

It follows, then, that

$$\sum_0^j V_{p,\, j-i}\beta_i + \beta_{j+p} = 0 \quad \text{for } 1 \leq p \leq k.$$

The cohomology of V^n is related to that of M by the Thom isomorphism $\phi\colon H^j(V^n) \simeq H^{j+k}(M)$, and M is $(k - 1)$-connected mod Z_2. To completely describe the Smith sequence of (T, M), we must give the formula for the composite homomorphism

$$I\phi\colon H^j(V^n) \to H^{j+k}(M/T, F) + H^{j+k}(F).$$

We begin with $j = n$. In this case $I\phi\colon H^n(V^n) \simeq H^{n+k}(M/T, F)$ since

$$H^{n+k}(F) = 0 = H^{n+k+1}(M/T, F).$$

Since V^n is connected, there exists a fundamental cohomology class $\alpha \in H^n(V^n)$, and we wish to show that $I\phi(\alpha) = \mu^{k-1}\delta(\alpha)$. In fact, however, we need only show that $\mu^{k-1}\delta(\alpha) \neq 0$. We turn again to our isomorphism

$$g^*\colon H^{n+k}(M/T, F) \simeq H^{n+k}(B/T, \dot{B}/T).$$

We note that

$$g^*(\mu^{k-1}\delta(\alpha)) = \mu^{k-1}\delta(p_0^*(\alpha), 0) = \delta(c_0^{k-1} p_0^*(\alpha), 0).$$

Now $c_0^{k-1} p_0^*(\alpha) \neq 0$, and so $(c_0^{k-1} p_0^*(\alpha), 0)$ cannot be in the image of

$$i^*\colon H^{j+k-1}(B/T) \to H^{j+k-1}(\dot{B}/T).$$

Thus $\mu^{k-1}\delta(p_0^*(\alpha), 0) \neq 0$, which implies that $\mu^{k-1}\delta^*(\alpha) \neq 0$.

We turn now to the case of any $\alpha \in H^j(V^n)$, $\alpha \neq 0$. By (2.3),

$$\left(\sum_0^{k-1} \mu^{k-i-1}\delta(v_i\alpha), v_k\alpha\right)$$

lies in the kernel of

$$d\colon H^{j+k}(M/T, F) + H^{j+k}(F) \to H^{j+k+1}(M/T, F);$$

therefore, there is a unique $\beta \in H^j(V^n)$ for which $I\phi(\beta) = (\Sigma_0^{k-1} \mu^{k-i-1}\delta(v_i\alpha), v_k\alpha)$. We shall assume that $\alpha \neq \beta$ and derive a contradiction. We select a homology class $h \in H_j(V^n)$ for which $\langle\alpha, h\rangle = 1$ and $\langle\beta, h\rangle = 0$. We choose a map of a closed connected j-dimensional manifold $g\colon W^j \to V^n$ for which $g_*(\sigma) = h$, where $\sigma \in H_j(W^j)$ is the fundamental homology class. Since

$$\langle g^*(\beta), \sigma\rangle = \langle\beta, g_*(\sigma)\rangle = 0,$$

$g^*(\beta) = 0$; while, on the other hand, $g^*(\alpha)$ is the fundamental cohomology class of W^j. The map g induces a bundle $E' \to W^j$ and an equivariant map $G\colon (T', M') \to (T, M)$ between the involutions on the respective Thom spaces. According to Thom, the diagram

$$\begin{array}{ccc} H^{j+k}(M) & \xrightarrow{G^*} & H^{j+k}(M') \\ \uparrow \phi & & \uparrow \phi' \\ H^j(V^n) & \xrightarrow{g^*} & H^j(W^j) \end{array}$$

is commutative. Since $g^*(\beta) = 0$, then by naturality $(\Sigma\mu^{k-i-1}\delta(v_i\alpha), v_k\alpha)$ lies in the kernel of

$$g^*: H^{j+k}(M/T, F) + H^{j+k}(F) \to H^{j+k}(M'/T', F') + H^{j+k}(F').$$

From purely dimensional considerations, the classes $g^*(v_i\alpha) = 0$ for $i > 0$. Thus, $\mu^{k-1}\delta(g^*(\alpha)) = 0 \in H^{j+k}(M'/T', F')$; but this contradicts our earlier remark that $I'\phi'(g^*(\alpha)) = \mu^{k-1}\delta(g^*(\alpha)) \neq 0$, since $g^*(\alpha)$ is the fundamental cohomology class of W^j. We conclude that $\alpha = \beta$.

THEOREM 2.5. *For any cohomology class* $\alpha \in H^j(V^n)$,

$$I\phi(\alpha) = \sum_0^{k-1} \mu^{k-i-1}\delta(v_i\alpha),\ v_k\alpha)$$

in $H^{j+k}(M/T, F) + H^{j+k}(F)$.

To close this section we compute the height in the Smith sequence of (T, M) for a class $\beta \in H^i(V^n)$.

COROLLARY 2.6. *The height of* $\beta \neq 0 \in H^i(V^n)$ *is* $k + r$, *where* r *is the largest integer for which* $\bar{v}_r\beta \neq 0$.

We note that

$$\chi_{j+k}(0, \cdots, 0, \beta, 0, \cdots, 0) = \mu^{k+j-i}\delta(\beta).$$

Now according to (2.4), if $i > j$ and if $\mu^{k+j-i}\delta(\beta) = 0$, then $\beta = 0$, so we need only concern ourselves with the case $j \geq i$. We seek the *smallest* integer s for which $\mu^{k+s-i}\delta(\beta) = 0$. The height of β is, then, $k + s - i$. We wish to show that if

$$V_{1,\, j-i}\beta = \bar{v}_{j-i+1}\beta = 0 \quad \text{for all } j \geq s,$$

then $V_{p,\, s-i}\beta = 0$ $(1 \leq p \leq k)$ so that $\mu^{k+s-i}\delta(\beta) = 0$ by (2.4). We assume inductively that $V_{p,\, j-i}\beta = 0$ for all $j \geq s$ and a fixed p. We write

$$V_{p+1,\, j-i}\beta = V_{p,\, j-i+1}\beta + v_p V_{1,\, j-i}\beta.$$

But $V_{p,\, j-i+1}\beta = 0$, and $V_{1,\, j-i}\beta = \bar{v}_{j-i+1}\beta = 0$; thus $V_{p+1,\, j-i}\beta = 0$ for all $j \geq s$. If we select the smallest s for which $\bar{v}_{j-i+1}\beta = 0$ for all $j \geq s$, then $s - i = r$ is the largest integer for which $\bar{v}_r\beta \neq 0$. The reader may show that $\mu^{k+s-i}\delta(\beta) = 0$ implies $\bar{v}_{j-i+1}\beta = 0$ for all $j \geq s$. We have computed the height of β in terms of the dual Whitney classes of $E \to V^n$. Since $(\mu^{k+r-1}\delta(\beta), 0)$ is in the kernel of

$$d: H^{i+k+r}(M/T, F) + H^{i+k+r}(F) \to H^{i+k+r+1}(M/T, F),$$

a straight forward computation shows that in fact $I\phi(\bar{v}_r\beta) = (\mu^{k+r-1}\delta(\beta), 0)$.

3. DIFFEOMORPHISMS OF PERIOD TWO

We shall examine diffeomorphisms of period 2 on closed manifolds in this section. The fixed point set of a (differentiable) involution on a closed manifold is known to be the finite disjoint union of closed connected regular submanifolds [4, p. 203]. Without loss of generality, we may assume that each involution is a Riemannian isometry. We consider a (T, M^{n+k}) on a closed $(n + k)$-manifold, and we single out a non-void component of the fixed point set $V^n \subset M^{n+k}$. We shall wish to extend a class $\alpha \in H^j(V^n)$ to a class on the full fixed point set $F \subset M^{n+k}$. We map F onto $V^n \cup q$ by a map ε that is the identity on V^n and that collapses every other component of F onto the point q. We then consider $\varepsilon^*(\alpha, 0) \in H^j(F)$, which we also write as $\alpha \in H^j(F)$.

We choose a small closed invariant tube N of geodesics normal to V^n. A natural equivariant map $f: (T, M^{n+k}) \to (T, M)$ onto the involution of the Thom space of the normal bundle to V^n is defined by collapsing the complement of the interior of N to the point q. We obtain the diagram

$$\begin{array}{ccccccc} \cdots \to H^j(M^{n+k}) & \xrightarrow{I} & H^j(M^{n+k}/T, F) + H^j(F) & \xrightarrow{d} & H^{j+1}(M^{n+k}/T, F) \to \cdots \\ \uparrow & & \uparrow \qquad\quad \uparrow \varepsilon^* & & \uparrow \\ \cdots \to H^j(M) & \to & H^j(M/T, V^n \cup q) + H^j(V^n \cup q) & \to & H^{j+1}(M/T, V^n \cup q) \to \cdots \end{array}$$

together with

$$\begin{array}{ccc} H^{j+k}(M) & \to & H^{j+k}(M^{n+k}) \\ & \nwarrow \phi \quad \theta \nearrow & \\ & H^j(V^n) . & \end{array}$$

We may first apply (2.5).

THEOREM 3.1. *For any class* $\alpha \in H^j(V^n)$

$$I\theta(\alpha) = \left(\sum_0^{k-1} \mu^{k-i-1} \delta(v_i \alpha), v_k \alpha \right)$$

in $H^{j+k}(M^{n+k}/T, F) + H^{j+k}(F)$.

In (2.4) we gave a necessary and sufficient condition that $(\beta_0, \cdots, \beta_{j+k})$ belong to the kernel of

$$\chi_{j+k}: \sum H^i(V^n) \to H^{j+k+1}(M/T, V^n \cup q) .$$

We loose the necessity of the condition in passing to (T, M^{n+k}), but if we consider

$$\chi_{j+k}: \sum_0^{j+k} H^i(F) \to H^{j+k+1}(M^{n+k}/T, F) ,$$

and if $(\beta_0, \cdots, \beta_{j+k}) \in \Sigma H^i(V^n)$ satisfies the condition $\Sigma_0^j V_{p,\, j-i} \beta_i = \beta_{j+p}$ $(1 \leq p \leq k)$, then $\chi_{j+k}(\beta_0, \cdots, \beta_{j+k}) = 0$ in $H^{j+k+1}(M^{n+k}/T, F)$. We may take this sufficient condition and combine it with (1.1) to obtain the following result.

THEOREM 3.2. *If* $H^{j+p}(M^{n+k}) = 0$ $(k - r \leq p \leq k)$, *then, for any*

$$(\beta_0, \cdots, \beta_j) \in \sum_0^j H^j(V^n), \quad \sum_0^j V_{p,\, j-i} \beta_i = 0 \quad \textit{for } k - r \leq p \leq k .$$

We merely complete $(\beta_0, \cdots, \beta_j)$ to $(\beta_0, \cdots, \beta_j, \cdots, \beta_{j+k})$ by setting

$\beta_{j+p} = \Sigma_0^j V_{p,\, j-i}\beta_i$. This class will lie in the kernel of χ_{j+k}, and we apply (1.1). This is a generalization of Corollary 1.3 in the introduction. Should $r = k$, it would follow that $H^j(V^n) = 0$.

THEOREM 3.3. *If* (T, M^{n+k}) *is an involution on a closed manifold which is* k-*connected* mod Z_2 *and if there is a non-void* n-*dimensional component in the fixed point set* V^n, *then the fixed point set is connected and* $v_1 = v_2 = \cdots = v_k = 0$.

We apply Corollary 1.3 to $v_0 = u \in H^0(V^n)$ with $r = 0$. It follows that $v_1 = \cdots = v_k = 0$, but $u = v_0$ must also be in the image of $H^0(M^{n+k}) \to H^0(F)$, and this is certainly possible only if F is connected. The vanishing of the normal classes to V^n was proved in the detailed exposition of [3] by a different approach. It also follows from [3] that, under the hypothesis of (3.3), $[V^n]_2 = 0 \in \mathscr{N}_n$, and $[M^{n+k}]_2 = 0 \in \mathscr{N}_{n+k}$.

So far we have dealt with but a single component of the fixed point set. Now we would like to examine some relations between these components. We begin with a simple enlargement of (3.3). We again consider an involution on a closed manifold (T, M^{n+k}) and the n-dimensional components of the fixed point set, but we shall only assume that M^{n+k} is $(k - 1)$-connected mod Z_2. We enumerate the n-dimensional components of F by $V_1^n, \cdots, V_m^n$. The Whitney classes of the normal bundle to V_i^n are denoted by $u_i = v_{0,\, i}, v_{1,\, i}, \cdots, v_{k,\, i}$. To each V_i^n there corresponds the Gysin cohomology homomorphism $\theta\colon H^j(V_i^n) \to H^{j+k}(M^{n+k})$. We let $\theta(u_i) = \lambda_i \in H^k(M^{n+k})$ for $1 \leq i \leq m$. We note without comment that $\lambda_i \cup \lambda_j = 0$ $(i \neq j)$.

THEOREM 3.4. *Let* (T, M^{n+k}) *be an involution on a closed manifold which is* $(k - 1)$-*connected* mod Z_2, *and let* $V_1^n, \cdots, V_m^n$ *denote the* n-*dimensional components of the fixed point set. If, for some* $p \leq m$, $\Sigma_1^p \lambda_i = 0$, *then* $p = m$, F *consists only of its* n-*dimensional components, and, for each* V_i^n, *all the characteristic classes of its normal bundle vanish.*

We are saying that the only possible linear relation in $H^k(M^{n+k})$ which can be satisfied by the classes λ_i is $\Sigma_1^m \lambda_i = 0$. If this relation does hold, then we reach some special conclusions about (T, M^{n+k}). Let us suppose that $\Sigma_1^p \lambda_i = 0$; then

$$I\left(\sum_1^p \lambda_i\right) = \left(\sum_1^p \sum_0^{k-1} \mu^{k-j-1}\,\delta(v_{j,\, i}),\quad \sum_1^p v_{k,\, i}\right) = 0\,.$$

Since $\Sigma_1^p v_{k,\, i} = 0$ and $v_{k,\, i} \in H^k(V_i^n)$, it follows that $v_{k,\, i} = 0$, $0 \leq i \leq p$. In addition, $(\Sigma_1^p \Sigma_0^{k-2} \mu^{k-j-2}\,\delta(v_{j,\, i}),\ \Sigma_1^p v_{k-1,i})$ lies in the kernel of

$$d\colon H^{k-1}(M^{n+k}/T,\, F) + H^{k-1}(F) \to H^k(M^{n+k}/T,\, F)\,.$$

Now $H^{k-1}(M^{n+k}) = 0$, so again $v_{k-1,\, i} = 0$ $(0 \leq i \leq p)$ and $\Sigma_1^p \Sigma_0^{k-2} \mu^{k-j-2}\,\delta(v_{j,\, i}) = 0$. We proceed until we have shown that $v_{j,\, i} = 0$ $(1 \leq j \leq k$ and $1 \leq i \leq p)$ and that $(0, \Sigma_1^p u_i)$ lies in the kernel of d and, hence, in the image of $H^0(M^{n+k}) \to H^0(F)$. The image of the unit cohomology class on M^{n+k} must be the unit class on F, but $\Sigma_1^p u_i$ can be the unit class of F if and only if $p = m$ and there are no components of the fixed point set other than the V_i^n. It now follows [3] that if $\Sigma_1^m \lambda_i = 0$, then $\Sigma_1^m [V_i^n]_2 = 0$ and $[M^{n+k}]_2 = 0$. We should keep in mind two important properties of the classes λ_i; namely, that $T^*(\lambda_i) = \lambda_i$ and $\lambda_i \cup \lambda_j = 0$ $(i \neq j)$.

THEOREM 3.5. *If* (T, M^{n+k}) *is an involution on a closed connected manifold and if* $V^n M^{n+k}$ *is a component of the fixed point set for which*

$$\theta: H^{n-i}(V^n) \to H^{n+k-i}(M^{n+k})$$

is surjective for $0 \leq i \leq r$, *then every other component of the fixed point set has dimension less than* $n + k - r$.

We note that our hypothesis is equivalent to the requirement that

$$H_i(V^n) \to H_i(M^{n+k})$$

be surjective for $0 \leq i \leq r$. We proceed with some more general considerations.

We fix j and r, and we suppose that $\theta: H^{j-i}(V^n) \to H^{j+k-i}(M^{n+k})$ is surjective for $0 \leq i \leq r$. We would like to examine the implications concerning the kernel of

$$\chi_{j+k-i}: \sum H^p(F) \to H^{j+k-i+1}(M^{n+k}/T, F).$$

If $\chi_{j+k-i}(\beta_0, \cdots, \beta_{j+k-i}) = 0$ for some $0 \leq i \leq r$, then there exists an $\alpha \in H^{j-i}(V^n)$ for which $\beta_{j+k-i} = v_k \alpha$ and for which

$$\chi_{j+k-i-1}(\beta_0, \cdots, \beta_{j-i}+\alpha, \beta_{j-i+1} + v_1 \alpha, \cdots, \beta_{j+k-i-1} + v_{k-1} \alpha) = 0.$$

We have simply applied (3.1). If V^m is a component of the fixed point set distinct from V^n, then β_{j-i+k} lies in the kernel of $H^{j-i+k}(F) \to H^{j-i+k}(V^m)$; and $(\beta_0, \cdots, \beta_{j+k-i-1})$ and $(\beta_0, \cdots, \beta_{j-i} + \alpha, \cdots, \beta_{j+k-i-1} + v_{k-1}\alpha)$ have the same image under $H^*(F) \to H^*(V^m)$. This suggests that by a simple inductive argument we may conclude that if $\chi_{j+k}(\beta_0, \cdots, \beta_{j+k}) = 0$, then β_{j+k-i} lies in the kernel of

$$H^{j+k-i}(F) \to H^{j+k-i}(V^m) \quad \text{for } 0 \leq i \leq r.$$

We suppose that $m + s = n + k$, and we let $w_0, w_1, \cdots, w_s$ denote the Whitney classes of the normal bundle to V^m.

LEMMA 3.6. *Let* $\theta: H^{j-i}(V^n) \to H^{j+k-i}(M^{n+k})$ *be surjective for* $0 \leq i \leq r$, *and let* V^m *be a component of the fixed point set distinct from* V^n. *If* $p + s \leq j + k$, *then for* $\beta \in H^p(V^m)$, $w_{s-i}\beta = 0$, *provided* $j + k - r \leq p + s - i$. *If* $j + k - r \leq p$, *then* $H^p(V^m) = 0$.

We note that by (3.1), $(0, \cdots, \beta, w_1\beta, \cdots, w_s\beta, 0, \cdots, 0)$ lies in the kernel of χ_{j+k}. We apply the foregoing remarks to obtain (3.6). We can derive (3.5) immediately, for if V^m is a component of F, distinct from V^n, for which $n + k - r \leq m$, then by (3.6) $H^m(V^m) = 0$, which is certainly a contradiction.

A general relationship, referred to several times in this note, between diffeomorphisms of period 2 and unoriented cobordism was described in [3]. We wished to add to the picture, as in (3.3) and (3.4), by combing the general results from cobordism considerations with the content of this note in several special cases. We might illustrate (3.5), in this spirit, as follows. We consider an involution (T, CP(2m)) on an even-dimensional complex projective space. Since the Euler characteristic of CP(2m) is equal to 1 mod Z_2, according to [3], there is a component of the fixed point set $V^n \subset CP(2m)$ of codimension k for which $v_k \neq 0$. Since $i^* \theta(u) = v_k$, k is even. If $c \in H^2(CP(2m))$ generates the cohomology ring mod Z_2, then $\theta(i^*(c^r)) = c^r \theta(u) \neq 0$ for $0 \leq 2r \leq k$. Therefore $\theta: H^{n-i}(V^n) \to H^{4m-i}(CP(2m))$ is surjective for $0 \leq i \leq n + 1$. According to (3.5), every other component of the fixed point set has dimension less than $4m - n - 1 = k - 1$. In particular, there exists a *unique* component of the fixed point set of $(T, CP(2m))$ whose dimension is at least $2m$. Similar remarks apply to involutions on quaternion projective spaces.

Now we shall discuss the height of a cohomology class on the fixed point set in the Smith sequence of a diffeomorphism of period 2. We consider (T, M^{n+k}) and an n-dimensional component of the fixed point set $V^n \subset F$. According to (2.6), the height of $\beta \in H^i(V^n)$ cannot exceed $k + r$, where r is the largest integer for which $\bar{v}_r \beta \neq 0$. In general, it is quite difficult to compute the height exactly. We should mention that $\beta \in H^i(V^n)$ has height $k + n - i$ if and only if $\bar{v}_{n-i} \beta \neq 0$. This is an extreme case, of course, which follows immediately from the facts that

$$I: H^{n+k}(M^{n+k}) \simeq H^{n+k}(M^{n+k}/T, F)$$

and that the equivariant map of (T, M^{n+k}) onto the involution of the Thom space of the normal bundle to V^n has degree 1 mod Z_2. In particular, the fundamental cohomology class on V^n has height k since $\bar{v}_0 = u$.

THEOREM 3.7. *Let* (T, M^{n+k}) *be an involution on a closed connected manifold for which* $i^*: H^j(M^{n+k}) \to H^j(F)$ *is injective if* $0 \leq j \leq k + s - 1$. *If* F *is not connected, then for each* n-*dimensional component* $V^n \subset F$ *there exists an integer* $r \geq s$ *for which* $\bar{v}_r \neq 0 \in H^r(V^n)$.

We wish to show that if $u \in H^0(V^n)$ is the unit class, then height (u) is at least $k + s$. We must assume F is not connected so that $\delta(u) \neq 0 \in H^1(M^{n+k}/T, F)$ and height (u) > 0. We let $0 < j =$ height (u) and assume $j \leq k + s - 1$; then

$$\mu^j \delta(u) = 0 \in H^{j+1}(M^{n+k}/T, F),$$

but $\mu^{j-1} \delta(u) \neq 0 \in H^j(M^{n+k}/T, F)$. Now $(\mu^{j-1} \delta(u), 0)$ must lie in the image of

$$I: H^j(M^{n+k}) \to H^j(M^{n+k}/T, F) + H^j(F),$$

but $I = \sigma + i^*$ and $i^*: H^j(M^{n+k}) \to H^j(F)$ is injective. Therefore $\mu^{j-1} \delta(u) = 0$, which is a contradiction. Hence, height (u) $\geq k + s$ and (3.7) follows. In homology our hypothesis requires that $H_j(F) \to H_j(M^{n+k})$ be surjective for $0 \leq j \leq k + s - 1$. We should compare both hypothesis and conclusion with (3.5). We might consider an involution on a complex projective space $(T, CP(m))$ for which $V^{2n} \subset CP(m)$ is a component of the fixed point set and for which $H^{2n}(CP(m)) \simeq H^{2n}(V^{2n})$. Then $H^j(CP(m)) \to H^j(V^{2n})$ is injective for $0 \leq j \leq 2n + 1$. If $2n \geq m$ and if the fixed point set is not connected, we write $2n + 1 = 2(m - n) + 2(2n - m + 1) - 1$, so that for some $r \geq 2(2n - m + 1)$, $\bar{v}_r \neq 0 \in H^r(V^{2n})$.

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