NUMERICAL RANGE AND SPECTRAL SETS

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1. Spectral sets were introduced by Von Neumann [6, p. 429] in an attempt to extend spectral theory to non-normal operators. The point at issue was to get an inequality of the type $\|f(T)\| \leq \|f\|_\infty$, where T is an operator, f is a "suitable" function and the norm is the supremum norm taken over a "suitable" set. These sets were the spectral sets. The inequality becomes trivial in the context of unitary dilations [8], and a recent result, due independently to Berger [1], Foiaş [2], and Lebow [4] displays an intimate connection between spectral sets and normal dilations. Moreover a forth-coming result of Halmos shows a relation between the normal dilation and the numerical range of an operator. The main result of this note is in the same spirit. We shall characterize (by means of normal dilations) those operators the closure of whose numerical range is a spectral set. We also discuss the equality of the convex hull of the spectrum with the closure of the numerical range, in relation to the spectrality of the latter set. We conclude with a peculiar result about the numerical range of certain operators.

I am indebted to P. R. Halmos for raising the subject, and to him, A. Brown, and S. K. Berberian for many interesting conversations at Ann Arbor in the summer of 1962.

2. The numerical range $W(T)$ of a bounded operator T on complex Hilbert space H is $\{(Tx, x): \|x\| = 1\}$. The Toeplitz-Hausdorff theorem [11, p. 234] states that $W(T)$ is convex, and if T is normal, then $\overline{W(T)}$ is the closed convex hull $\mathscr{C}\sigma(T)$ of the spectrum $\sigma(T)$ of T [7] (throughout we shall use the bar symbol for closure, ∂ for boundary, and $\mathscr{C}$ for convex hull). This is true rather more generally, as a consequence of the Berger-Foiaş-Lebow result, which we state in a slightly modified form.

THEOREM 1. *If* S *is a compact convex spectral set for an operator* T, *then there exists a normal operator* N *defined on a larger Hilbert space* $K \supset H$, *such that*

(i) $\sigma(N) \subset \partial S$

(ii) $T^n x = PN^n x \qquad (x \in H,\ n = 0, 1, 2, \cdots)$,

where P *is the orthogonal projection of* K *onto* H.

A (normal) operator N defined on $K \supset H$ such that $Tx = PNx$ for $x \in H$ is called a (normal) *dilation* of T. If also (ii) holds, it is called a *strong dilation*. Thus the theorem states that under the hypothesis on S, there exists a strong normal dilation of T with spectrum on the boundary of S.

A closed proper subset E of the complex plane is a *spectral set* for an operator T if

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$$\|u(T)\| \leq \sup\{\,|u(\lambda)|: \lambda \in E\}$$

for every rational function u of z having no poles in E. In order for u(T) to be defined, it must be that $\sigma(T) \subset E$, and it is immediate that any closed proper subset of the plane which contains a spectral set for T is itself spectral for T. See [6] for an account of spectral sets. We shall also need to know that $\mathscr{C}\sigma(T) \subset \overline{W(T)}$ for any operator T [11, p. 245]. Thus if $\sigma(T)$ is spectral so is $\mathscr{C}\sigma(T)$, and if $\mathscr{C}\sigma(T)$ is spectral so is $\overline{W(T)}$.

PROPOSITION 1. *If* $\mathscr{C}\sigma(T)$ *is spectral for* T, *then there exists a strong normal dilation* N *of* T *such that*

$$\mathscr{C}\sigma(T) = \overline{W(T)} = \overline{W(N)}\,.$$

Proof. By Theorem 1 there exists a normal operator N satisfying (i) and (ii) above, with $S = \mathscr{C}\sigma(T)$. Then

$$\overline{W(T)} = \{\overline{(Nx,\,x):\ \|x\| = 1,\ x \in H}\} \subset \overline{W(N)} = \mathscr{C}\sigma(N)$$

by the known result for normal operators, and

$$\mathscr{C}\sigma(N) \subset \mathscr{C}\partial\,\mathscr{C}\sigma(T) = \mathscr{C}\sigma(T) \subset \overline{W(T)}\,.$$

This completes the proof.

PROPOSITION 2. *If there exists a strong normal dilation* N *of* T *such that* $\overline{W(N)} = \overline{W(T)}$, *then* $\overline{W(T)}$ *is a spectral set for* T.

Proof. This follows from the converse of Theorem 1 (see [1] and [4]). The following is a simpler version of the argument. For $|\lambda|$ large both resolvents $R_\lambda(T)$ and $R_\lambda(N)$ exist and are analytic ($R_\lambda(A) = (A - \lambda I)^{-1}$ for any A), and it is immediate from their Neumann series expansions [6, p. 406] that $R_\lambda(T)x = PR_\lambda(N)x$, $x \in H$, with P and H as in (ii) above. This is equivalent to the statement

$$(R_\lambda(T)x,\,y) = (R_\lambda(N)x,\,y), \qquad (x,\,y \in H)\,.$$

Under the hypothesis $\overline{W(T)} = \overline{W(N)}$, this equality is in fact valid for all $\lambda \notin \overline{W(T)}$, because both resolvents are analytic there, and if u is a rational function with no poles in $\overline{W(T)}$, then

$$(u(T)x,\,y) = (u(N)x,\,y), \qquad (x,\,y \in H)\,,$$

or, equivalently, $u(T)x = Pu(N)x$, $x \in H$. Hence

$$\|u(T)\| \leq \|u(N)\| = \sup\{\,|u(z)|: z \in \sigma(N)\}$$

$$\leq \sup\{\,|u(z)|: z \in \overline{W(N)} = \overline{W(T)}\} \qquad \text{Q.E.D.}$$

To summarize, $\overline{W(T)}$ *is spectral for* T *if and only if there exists a strong normal dilation* N *of* T *with* $\overline{W(N)} = \overline{W(T)}$, *and a sufficient condition for this is that* $\mathscr{C}\sigma(T)$ *be spectral for* T, *which also implies* $\overline{W(T)} = \mathscr{C}\sigma(T)$.

3. It is not generally true that $\mathscr{C}\sigma(T) = \overline{W(T)}$ of course (for instance, consider any 2 by 2 non-zero nilpotent matrix), but it is true for a large class of non-normal operators, as we shall now show. Let L^2, L^∞ refer to that unit circle $\{\,|z| = 1\}$

with Lebesgue measure $(1/2\pi)\,d\theta$, let H^2 denote the closed subspace of L^2 consisting of those functions whose Fourier coefficients vanish for negative index, and let P be the orthogonal projection of L^2 onto H^2. We shall consider the operators $M_\phi: L^2 \to L^2$ of multiplication by $\phi \in L^\infty$, and their "compressions" $T_\phi: H^2 \to H^2$ defined by the relation

$$T_\phi f = PM_\phi f = P\phi f, \qquad (f \in H^2).$$

In the terminology of Section 2 above, M_ϕ is a dilation of T_ϕ, but in general not a strong dilation.

PROPOSITION 3. $\mathscr{C}\sigma(T_\phi) = \overline{W(T_\phi)}$.

Proof. If $f \in H^2$, then

$$(T_\phi f, f) = (P\phi f, f) = (\phi f, f) = \frac{1}{2\pi}\int_0^{2\pi} \phi |f|^2 \, d\theta.$$

Choose a simple function $s = \Sigma_{k=1}^n c_k X\sigma_k$ (here $X\sigma$ is the characteristic function of the set σ) which is uniformly close to ϕ. Note that the numbers c_k belong to the essential range of ϕ. We see that

$$\frac{1}{2\pi}\int_0^{2\pi} s|f|^2 \, d\theta = \sum_{k=1}^n c_k \int_{\sigma_k} |f|^2 \, d\theta,$$

a convex linear combination of the numbers c_k. Now it is well known (and clear) that $\sigma(M_\phi)$ is the closure of the essential range of ϕ, and by a theorem of Hartman and Wintner [3] $\sigma(M_\phi) \subset \sigma(T_\phi)$. Hence $\frac{1}{2\pi}\int_0^{2\pi} s|f|^2 \, d\theta$ is a convex linear combination of points in $\sigma(T_\phi)$. But $\frac{1}{2\pi}\int_0^{2\pi} s|f|^2 d\theta$ is close to

$$\frac{1}{2\pi}\int_0^{2\pi} \phi|f|^2 \, d\theta = (T_\phi f, f),$$

so that $W(T_\phi) \subset \mathscr{C}\sigma(T_\phi)$, and always $\mathscr{C}\sigma(T_\phi) \subset \overline{W(T_\phi)}$. This completes the proof.

An attempt to characterize those operators T for which $\mathscr{C}\sigma(T) = \overline{W(T)}$ was made by Wintner [11]. He called T *normaloid* if

$$\|T\| = \sup\{\,|(Tx, x)|;\ \|x\| = 1\},$$

and he presents an argument to prove that if T is normaloid, then $\mathscr{C}\sigma(T) = \overline{W(T)}$. That this argument is faulty is shown by the following example due to P. R. Halmos. Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

let I be the one-dimensional identity operator, and let $T = A \oplus I$. An elementary calculation, first performed by Toeplitz [9], shows that $\overline{W(A)} = \{ |z| \leq 1/2 \}$, and it is easy to show that $\overline{W(T)}$ is the convex hull of this set and the point $\{1\}$. Hence

$$\sup \{ |(Tx, x)| : \|x\| = 1 \} \geq 1 = \|T\|;$$

and, of course, $|(Tx, x)| \leq \|T\|$ for $\|x\| = 1$. Thus T is normaloid, but $\mathscr{C}\sigma(T) = [0, 1] \neq \overline{W(T)}$.

Notwithstanding the result of Section 2, it is difficult to tell whether $\overline{W(T)}$ is spectral for a given T. For instance, we do not know whether it is the case for the operator in the example above, or for the operators T_ϕ. We can, however, show $\overline{W(T_\phi)}$ to be spectral for a special class of functions ϕ, namely, those which are the boundary values on $\{ |z| = 1 \}$ of functions that are analytic and bounded for $|z| < 1$ (such analytic functions are known to have non-tangential L^∞ boundary values defined a.e. on $\{ |z| = 1 \}$; see, for instance, [5]).

PROPOSITION 4. *If ϕ is the boundary value of a function that is analytic and bounded for $|z| < 1$, then $\sigma(T_\phi)$ is a spectral set for T_ϕ.*

(*Remark.* With this hypothesis T_ϕ is in fact a subnormal operator, and it is known that the spectra of subnormal operators are spectral sets. See [4]. In the present circumstances it is trivial to give an independent proof, and we shall therefore do so.)

Proof. We consider the related operator M_ϕ on L^2 and note that $M_\phi(H^2) \subset H^2$, so that M_ϕ is in fact an extension of T_ϕ. Since $\sigma(M_\phi) \subset \sigma(T_\phi)$ [3],

$$(R_\lambda(M_\phi)f, g) = (R_\lambda(T_\phi)f, g)$$

for $f, g \in H^2$ and all λ in the complement of $\sigma(T_\phi)$. Hence $u(T_\phi)f = Pu(M_\phi)f$, $f \in H^2$, for any rational function u having no poles in $\sigma(T_\phi)$. The proof is completed by the same estimation of norms as in the proof of Proposition 2.

For these operators $\overline{W(T_\phi)}$ is therefore spectral, and of course equal to $\mathscr{C}\sigma(T)$. One is led by this discussion to the question whether the equality $\overline{W(T)} = \mathscr{C}\sigma(T)$ and the spectrality of $\overline{W(T)}$ always occur together. There is also the following open question. *Is* $W(T_\phi)$ *spectral for general* $\phi \in L^\infty$? We conjecture the affirmative.

In this connection, we are able to show, by an operator of the form T_ϕ, that

(α) *neither the spectrality of* $\overline{W(T)}$ *nor the equality* $\overline{W(T)} = \mathscr{C}\sigma(T)$ *implies the spectrality of* $\sigma(T)$, and

(β) *the spectrality of* $\mathscr{C}\sigma(T)$ *does not imply that of* $\sigma(T)$. The example is the following. Let

$$\phi(\theta) = \begin{cases} e^{2i\theta} & (0 \leq \theta < \pi), \\ e^{-2i\theta} & (\pi \leq \theta < 2\pi), \end{cases}$$

and consider the operator T_ϕ. By a theorem of Widom [10], $\sigma(T_\phi) = \{z: |z| = 1\}$. By a theorem of Von Neumann [6, p. 434], this set cannot be a spectral set for T_ϕ. But

$$\mathscr{C}\sigma(T_\phi) = \overline{W(T_\phi)} = \{z: |z| \leq \|T_\phi\| = 1\}$$

is spectral for T_ϕ because the circle of radius $\|A\|$ is spectral for every A (von Neumann [6, p. 431]).

4. In conclusion we exhibit a peculiar fact about $\overline{W(T)}$ for certain T. For normal T the convexity of W(T) can be shown as follows. Suppose first that T can be represented as multiplication by z on $L_2(X, \mu)$, where X is a subset of the complex plane and μ is a finite positive measure on X. For unit vectors f, g in L_2, and a given t on [0, 1], there exists an $h \in L_2$ such that for almost all $z \in X$,

$$t|f(z)|^2 + (1 - t)|g(z)|^2 = |h(z)|^2 \tag{1}$$

and $\|h\| = t + 1 - t = 1$. Now if we multiply both members of (1) by z and integrate, we get the equality $t(Tf, f) + (1 - t)(Tg, g) = (Th, h)$, which is the desired convexity relation. If we multiply both numbers of (1) by z^n $(n = 1, 2, \cdots)$ and integrate, we get our peculiar fact, which is that *given unit vectors* $f, g \in L_2$ *and a* t *on* [0, 1], *there exists a unit vector* $h \in L_2$ *such that*

$$t(T^n f, f) + (1 - t)(T^n g, g) = (T^n h, h) \tag{2}$$

for all positive integers n. Since every normal operator is the direct sum of operators of the special type considered, (2) is true for all normal operators. It is also true for all the non-normal operators T_ϕ of Section 3. Here the crucial fact is a theorem of F and M. Riesz [5], which is that *for* $f \in L_1$, $f \geq 0$, *there exists a* $g \in H^2$ *with* $|g|^2 = f$ a.e. *if and only if*

$$\int_0^{2\pi} \log f \, d\theta > -\infty .$$

Suppose we now choose unit vectors f, g in H^2 and a t on [0, 1]. Then $t|f|^2 + (1 - t)|g|^2$ is a non-negative L_1 function whose logarithm is integrable because the same is true of $t|f|^2$. Hence, by the Riesz theorem, there exists an $h \in H^2$ such that

$$t|f|^2 + (1 - t)|g|^2 = |h|^2 \quad \text{a.e.}$$

and, as before, $\|h\| = 1$. If we now multiply both members of this relation by ϕ^n and integrate, we get the equation (2) for T_ϕ because

$$\int_0^{2\pi} \phi^n |f|^2 d\theta = (\phi^n f, f) = (T_\phi^n f, f) \quad \text{for any } f \in H^2 .$$

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