A REMARK ON RINGS AND ALGEBRAS

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We have recently been concerned with conditions which, when imposed on a ring, render the commutator ideal of this ring to be a nil ideal [2, 3]. A standard tool for exhibiting nil ideals is the following result of Amitsur [1]: If A is a finitely-generated *algebra* over a field, and A satisfies a polynomial identity, then the (Jacobson) radical of A is a nil ideal.

This above-cited theorem of Amitsur has one shortcoming in that it has not as yet been established for rings, while the natural area of its application is to rings. For this reason we prove the metamathematical theorem, given below, which allows us to transfer a certain class of results from algebras (in which Amitsur's theorem would play a significant role) to arbitrary rings.

As applications of the principal result of this note, we shall give a simplified proof of a theorem which we have recently proved by more complicated means [3] and we complete the proof, for rings, of the result proved in [2]. In [2] we applied Amitsur's theorem too widely, namely to rings; the proof as given in [2], however, is valid only for algebras. By the theorem to be proved, it automatically then becomes valid for rings as well.

A ring R is said to be of characteristic 0 if whenever $mx = 0$ with $x \neq 0$ in R and m an integer then $m = 0$. Let R be of characteristic 0, and suppose that $M = \{(x, n) \mid x \in R, n \neq 0 \text{ any integer}\}$; in M equality is defined, as usual, componentwise. Given (x_1, n_1), (x_2, n_2) in M we define $(x_1, n_1) \sim (x_2, n_2)$ if $n_2 x_1 = n_1 x_2$. It is immediate that this defines an equivalence relation on M. Let R* be the set of equivalence classes of M; if $[x, n]$ denotes the equivalence class of (x, n), then, since R is of characteristic 0, it follows easily that addition and multiplication defined by $[x_1, n_1] + [x_2, n_2] = [n_2 x_1 + n_1 x_2, n_1 n_2]$ and $[x_1, n_1][x_2, n_2] = [x_1 x_2, n_1 n_2]$ are well defined operations in R* under which R* becomes a ring containing an isomorphic copy of R. Moreover, R* is an algebra over the rational field. We call R* the rationalization of R.

One final bit of notation: for any ring R let C(R) denote the commutator ideal of R. We proceed to our theorem.

THEOREM. *Let* P *be a property defined on rings such that:*

(1) *if* P(R) *is true, then so are* P(U) *and* P(R/U) *for any (two-sided) ideal* U *of* R.

(2) *if* R *is of characteristic* 0 *and if* P(R) *is true, then so is* P(R*).

(3) *if* A *is an algebra over a field for which* P(A) *is true, then* C(A) *is a nil ideal.*

Then if P(R) *is true for any ring* R, C(R) *must be a nil ideal.*

Proof. Let R be a ring in which P(R) is true. We claim that, without loss of generality, we may assume that R has no non-zero nil ideals. For, if N is the

Received December 21, 1962.
This work was supported by an Army research grant to Cornell University.

maximal nil ideal of R, then R/N has no non-zero nil ideals. Since P(R) is true, by (1), so must P(R/N) be true. If we could prove that C(R/N) = (0), that is, that R/N were commutative, then it would follow that $C(R) \subset N$, whence C(R) would be nil.

Thus to prove the theorem we may assume that R has no non-zero nil ideals; our objective becomes to prove that R is commutative.

Let $W = \{x \in R \mid mx = 0$ for some positive integer $m\}$. Clearly W is an ideal of R. Moreover, as is well-known, since R has no nil ideals, W also has no nil ideals. In consequence, if $x \in W$ and $mx = 0$, it follows that we may assume that m is a square-free integer. Thus W is the direct sum of W_p's where

$$W_p = \{x \in R \mid px = 0\}$$

and where p is a prime number.

Now W_p is an algebra over GF(p); and since, by property (1), $P(W_p)$ is true, invoking property (3), we see that the commutator ideal of W_p is nil. Since R is devoid of nil ideals, so is W_p, whence W_p is commutative. In consequence, W is a commutative ideal of R.

If R is any ring having no non-zero nilpotent ideals and if T is a right ideal of R which is commutative as a ring, we claim that T must be contained in Z, the center of R, and that TC(R) = (0). For, suppose that $a \in T$ and $r \in R$. Then $ar \in T$, whence $(ar)a = a(ar)$, that is

$$(*) \qquad a(ar - ra) = 0 .$$

In (*) replace r by rs; we then see that

$$0 = a(ars - rsa) = a\{(ar - ra)s + r(as - sa)\} = ar(as - sa) .$$

Thus $aR(as - sa) = (0)$ for any $s \in R$. But then $(as - sa)R(as - sa) = (0)$, and so $(as - sa)R$ is a nilpotent right ideal. From this we learn that $as - sa = 0$ for any $s \in R$, that is, that $a \in Z$. If $r \in R$ and $a \in T \subset Z$, then ar is in T, and so, in Z. Thus for any $s \in R$, $(ar)s = s(ar) = asr$ since a and ar are both in Z. Therefore, $a(rs - sr) = 0$ for any $r, s \in R$. If $x \in C(R)$ then,

$$x = \sum u_i(r_i s_i - s_i r_i)v_i ,$$

whence

$$ax = \sum au_i(r_i s_i - s_i r_i)v_i = \sum u_i a(r_i s_i - s_i r_i)v_i \quad (\text{since } a \in Z) = 0 .$$

We have shown that $T \subset Z$ and that TC(R) = (0).

Let us return to our particular situation. Since W is a commutative ideal of R, by the above discussion, $W \subset Z$ and WC(R) = (0). Now $\overline{R} = R/W$ is of characteristic 0, and since P(R) is true, by (1), $P(\overline{R})$ is also true. Therefore by (2), $P(\overline{R}^*)$ is true. However, $\overline{R}^*$ is an algebra over the field of rational numbers; hence, by property (3), $C(\overline{R}^*)$, and so $C(\overline{R})$, must be nil. Let $x \in C(R)$; then $\overline{x}$, the image of x in $\overline{R}$, is in $C(\overline{R})$. Thus $\overline{x}$ must be nilpotent so that there is an integer n such that $\overline{x}^n = 0$. This means $x^n \in W$; since $x \in C(R)$, $x^{n+1} = x^n x \in WC(R) = (0)$. We have shown that C(R) is a nil ideal. Since R has no nil ideals, we conclude that C(R) = (0), which was our objective. The theorem now has been proved.

Application 1.

In [2] we studied rings R in which $(xy)^n = x^n y^n$ for some integer $n > 1$, and also rings R in which $(x + y)^n = x^n + y^n$. For such rings we "proved" that C(R) is a nil ideal, however there was a gap in the proof in that we applied Amitsur's theorem which holds for algebras in the wider context of a ring. However, what we did does prove the results for algebras over a field. It is easy to verify that the conditions used carry over to ideals and homomorphic images and that if R is of characteristic 0 they extend from R to R*. Thus all the requirements of the theorem of the present note are fulfilled, allowing us to conclude that C(R) is a nil ideal. In consequence, *all the theorems proved in* [2] *are true as stated in that paper.*

Application 2.

If R is any ring, we define the sequence of higher commutators of x and y as follows:

$$[x, y]_1 = xy - yx, \cdots, [x, x, \cdots, x, y]_n = x[x, \cdots, x, y]_{n-1} - [x, \cdots, x, y]_{n-1}x.$$

The ring R is said to satisfy the nth Engel condition if $[x, \cdots, x, y]_n = 0$ for all x, y in R. In [3] we proved that in a ring R satisfying the nth Engel condition, C(R) must be a nil ideal.

We give an alternate proof of this result here. Exactly as in [3] we prove that if R is a semi-simple ring satisfying the nth Engel condition, then it is commutative. (The proof is easy and does not even require that n be fixed.) Consequently, if J(R) denotes the Jacobson radical of R, then $C(R) \subset J(R)$. To verify that conditions (1) and (2) of the theorem of this note hold if P(R) means the nth Engel condition on R is a triviality. We now verify condition (3).

Let A be an algebra over a field F satisfying the nth Engel condition $[x, \cdots, x, y]_n = 0$ for all x, y in A. Suppose that $c \in C(A)$; thus

$$c = \sum_{i=1}^{m} u_i(r_i s_i - s_i r_i)v_i .$$

Let A_0 be the subalgebra of A generated by the u_i, r_i, s_i, v_i; A_0 is a finitely generated algebra and satisfies the polynomial identity $[x, x, \cdots, x, y]_n = 0$. Thus by Amitsur's theorem, $J(A_0)$ is nil. Since $A_0/J(A_0)$ is semi-simple and satisfies the nth Engel condition, it must be commutative. Therefore, $C(A_0) \subset J(A_0)$ and so must also be nil. Since $c \in C(A_0)$, c must be nilpotent. However, c was an arbitrary element of C(A), which leads us to conclude that C(A) is a nil ideal.

We have seen that the nth Engel condition is a condition on a ring R satisfying the hypothesis of our theorem; hence we can conclude that, in any ring R which satisfies the nth Engel condition, the commutator ideal C(R) must be a nil ideal (and so, since it satisfies a polynomial identity, a locally nilpotent ideal).

The conditions imposed on P(R) are not difficult to check for a concrete P, especially conditions (1) and (2). If P is a condition given by polynomial identities, it is generally true that these conditions hold.

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