SECOND MAIN THEOREM WITHOUT EXCEPTIONAL INTERVALS ON ARBITRARY RIEMANN SURFACES

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1. In the present paper an integrated form of the second main theorem is established for meromorphic functions on an arbitrary Riemann surface W. The theorem is valid for all compact bordered subregions of W.

An analogue of Nevanlinna's classical reasoning is first used to derive the first main theorem and a preliminary form of the second main theorem. A method introduced in [41] for analytic mappings is then applied to reach the final form of the second main theorem in its full generality. The proximity function is here expressed in terms of $\overset{+}{\log}$.

Literature on meromorphic functions on Riemann surfaces and on analytic mappings is listed in the References.

2. The basic idea of our approach is as follows. In the classical theory no estimate valid for all values of the variable r is known for the remainder term in the second main theorem. It is the integral of the integral of the remainder that can be given a dominating function. The remainder itself might have an arbitrarily wild behavior in certain intervals that can be estimated but which must be omitted in stating the second main theorem. These exceptional intervals, together with the related change of the coordinate system for a varying subregion of a Riemann surface, prohibit the use of directed limits in deriving the defect relation. However, on an arbitrary Riemann surface ordinary limits cannot be employed, for there is no one single parameter that gives an exhaustion of the entire surface. Thus the exceptional intervals block any attempt at transferring the classical theory to arbitrary Riemann surfaces.

This difficulty can be overcome by the following simple device. The conventional proximity function gives the mean proximity to a point a of an image *curve*. We replace this by the mean proximity to a of the corresponding image *region*, and then integrate that. Analytically this means that, in some sense, we bring all quantities involved to the same level of integration. Then the remainder in the second main theorem can be estimated for *every* subregion, directed limits may be used, and the theory can be developed on arbitrary Riemann surfaces.

For the classical cases of the plane and the disk one obtains as a by-product an elementary proof of the defect relation, and a second main theorem without exceptional intervals.

1. GENERALIZATION OF JENSEN'S FORMULA

3. Let $\overline{\Omega}$ be a compact bordered Riemann surface with border β_Ω, and let p denote the capacity function in Ω with pole at a given $\zeta \in \Omega$. By definition,

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$$p(z) - \log|z - \zeta| \to 0 \quad \text{as } z \to \zeta,$$

and $p(z) = k = \text{const.}$ on β_Ω.

Given a continuous real-valued function f on β_Ω, the solution $v(z)$ of the Dirichlet problem can be expressed in the form

$$v(\zeta) = \frac{1}{2\pi} \int_{\beta_\Omega} f\, dp^*.$$

To see this let α be a level line $p = c$ near ζ, oriented to leave ζ to its left. Then by Green's formula,

$$\int_{\beta - \alpha} v\, dp^* - p\, dv^* = 0,$$

and the statement follows on letting $c \to -\infty$.

There is a trivial relation between p and the Green's function: $g(\zeta, t) = k - p(t)$. For the proof let δ be a level line $g(\zeta, t) = c_1$ near t, encircling t counterclockwise. Then

$$\int_{\beta_\Omega - \alpha - \delta} g\, dp^* - p\, dg^* = 0,$$

and the statement follows on letting $c \to -\infty$, $c_1 \to \infty$. It is also a direct consequence of the well-known symmetry $g(\zeta, t) = g(t, \zeta)$.

4. Let W be an arbitrary open Riemann surface. We consider the class L of functions u on W, harmonic except for logarithmic singularities $\lambda_i \log|z - z_i|$ at z_i $(i = 1, 2, \cdots)$ with integral coefficients λ_i. By definition, the class M_e consists of locally meromorphic functions

(1) $$w = e^{u + iu^*} \quad (u \in L).$$

The conjugate function u^* has periods around z_i and along some cycles in W. Every branch of w is meromorphic, the branches differing by multiplicative constants c with $|c| = 1$. The modulus $|w|$ is single-valued throughout W. The class M_e contains the class M of (single-valued) meromorphic functions on W.

5. Given $\zeta \in W$, let Ω be a relatively compact subregion containing ζ and bounded by a finite set β_Ω of analytic Jordan curves. Denote by a_μ, b_ν the zeros and poles of a given $w \in M_e$ on W. We first assume that $w(\zeta) \neq 0, \infty$, and that no a_μ, b_ν is on β_Ω. Consider on Ω the function

(2) $$v(z) = \log|w(z)| + \sum_{a_\mu \in \Omega} g(z, a_\mu) - \sum_{b_\nu \in \Omega} g(z, b_\nu),$$

where each a_μ, b_ν is taken as many times as indicated by its multiplicity. Clearly $v(z)$ is harmonic on $\overline{\Omega}$, and

(3) $$\log|w(\zeta)| = \frac{1}{2\pi} \int_{\beta_\Omega} \log|w(z)|\, dp^* - \sum_{a_\mu \in \Omega} (k - p(a_\mu)) + \sum_{b_\nu \in \Omega} (k - p(b_\nu)).$$

If an a_μ or b_ν is on β_Ω, we first apply (3) to a slightly smaller region $\Omega_{k-\varepsilon} \subset \Omega$ bounded by the level lines $p = k - \varepsilon$, and then let $\varepsilon \to 0$. Since all terms in the equation are continuous in ε, (3) remains valid for Ω.

6. Suppose now $w(\zeta) = 0$ or ∞. A branch of w near ζ then has the Laurent expansion

$$w(z) = c_\lambda (z - \zeta)^\lambda + c_{\lambda+1} (z - \zeta)^{\lambda+1} + \cdots, \tag{4}$$

and the other branches are obtained through a multiplication by constants $e^{i\alpha}$. The same is true of the branches of the function

$$\phi(z) = e^{p(z)+ip^*(z)} \in M_e$$

and of

$$\psi(z) = w(z) \cdot \phi(z)^{-\lambda} = c_\lambda + \varepsilon(z - \zeta) \in M_e, \tag{5}$$

where $\varepsilon(z - \zeta) \to 0$ as $z \to \zeta$. On applying (3) to $\psi(z)$, one obtains

$$\log|c_\lambda| = \frac{1}{2\pi} \int_{\beta_\Omega} \log|w|\,dp^* - \lambda k - \sum{}' (k - p(a_\mu)) + \sum{}' (k - p(b_\nu)), \tag{6}$$

the sums Σ' being extended over points in $\overline{\Omega} - \zeta$.

7. For $-\infty < h \leq k$ consider the region $\Omega_h \subset \Omega$ bounded by the level lines $p = h$. Let $n(h, a)$ be the number of a-points, $a = 0$ or ∞, of w in $\overline{\Omega}_h$, counted with their multiplicities. It is understood that $n(-\infty, a)$ is the multiplicity (≥ 0) of the a-point at ζ. Then

$$\sum{}' (k - p(w^{-1}(a))) = \int_{-\infty}^{k} (k - h)d(n(h, a) - n(-\infty, a)) = \int_{-\infty}^{k} (n(h, a) - n(-\infty, a))dh,$$

and $\lambda = n(-\infty, 0) - n(-\infty, \infty)$. We set

$$N(h, a) = \int_{-\infty}^{h} (n(h, a) - n(-\infty, a))dh + n(-\infty, a)h \tag{7}$$

and also use the notations $N(\Omega, a) = N(k, a)$, $N(\Omega, w) = N(\Omega, \infty)$. We have obtained the following result.

JENSEN'S FORMULA ON RIEMANN SURFACES. *For a locally meromorphic function* $w \in M_e$ *with single-valued modulus on an arbitrary open Riemann surface* W,

$$\log|c_\lambda| = \frac{1}{2\pi} \int_{\beta_\Omega} \log|w|\,dp^* + N(\Omega, w) - N\left(\Omega, \frac{1}{w}\right). \tag{8}$$

8. We set

$$m(\Omega, w) = \frac{1}{2\pi} \int_{\beta_\Omega} \overset{+}{\log} |w| \, dp^*,$$

and we see that

$$\frac{1}{2\pi} \int_{\beta_\Omega} \log |w| \, dp^* = m(\Omega, w) - m\left(\Omega, \frac{1}{w}\right).$$

The counterpart of Nevanlinna's characteristic function is

(9) $$T(\Omega) = T(\Omega, w) = m(\Omega, w) + N(\Omega, w),$$

and Jensen's formula (8) takes the form

(10) $$\log |c_\lambda| = T(\Omega, w) - T\left(\Omega, \frac{1}{w}\right)$$

for all $w \in M_e$.

2. FIRST MAIN THEOREM ON W

9. In the present section we consider differences $w - a$, and we therefore restrict our attention to the class $M \subset M_e$ of single-valued meromorphic functions w on W.

For $a \neq \infty$ we define the counterpart of Nevanlinna's *proximity function* as

(11) $$m(\Omega, a) = m\left(\Omega, \frac{1}{w - a}\right) = \frac{1}{2\pi} \int_{\beta_\Omega} \overset{+}{\log} \frac{1}{|w - a|} \, dp^*,$$

and the counterpart of the *counting function* as

(12) $$N(\Omega, a) = N\left(\Omega, \frac{1}{w - a}\right) = \int_{-\infty}^{k} (n(h, a) - n(-\infty, a))dh + n(-\infty, a)k.$$

For $a = \infty$ the definitions were given in Section 1.

We apply Jensen's formula (10) to $w - a$. Clearly $N(\Omega, w - a) = N(\Omega, w)$, while

$$m(\Omega, w - a) \leq m(\Omega, w) + \overset{+}{\log} |a| + \log 2.$$

We can now state the following (see [16, 39]):

FIRST MAIN THEOREM ON RIEMANN SURFACES. *For a meromorphic function on an arbitrary Riemann surface* W,

(13) $$m(\Omega, a) + N(\Omega, a) = T(\Omega) + \phi(a),$$

where $|\phi(a)| \leq \overset{+}{\log} |a| + \log 2 + |\log |c||$, c *being the first Laurent coefficient of* $w - a$ *at* ζ.

The theorem expresses a universal property of meromorphic functions w on compact bordered subsurfaces Ω of arbitrary open Riemann surfaces W, regardless of topological or conformal properties of Ω or W. The theorem simply states that in the Poisson-Jensen decomposition of $\log|w(z) - a|$ into positive and negative harmonic functions on Ω their values at ζ must add up to $\log|w(\zeta) - a|$. The value of the positive component of $\log|w(z)|$ at ζ is the characteristic $T(\Omega)$.

10. The Ahlfors-Shimizu interpretation of the characteristic continues to be valid. The chordal distance between the sterographic images of w and a is denoted by $[w, a]$, and the proximity function for β_h: $p = h \leq k$ is defined as

$$m(h, a) = \frac{1}{2\pi} \int_{\beta_h} \log \frac{1}{[w, a]} \, dp^* .$$

The counting function is, by definition,

$$N(h, a) = \int_{h_0(a)}^{h} n(h, a) dh ,$$

where $h_0(a)$ is so chosen that $\lim_{h \to -\infty}(m(h, a) + N(h, a)) = 0$.

For any two values a, b one obtains at once the equalities

$$\frac{dm(h, b)}{dh} - \frac{dm(h, a)}{dh} = \frac{1}{2\pi} \int_{\beta_h} \frac{d}{dh} \log\left|\frac{w - a}{w - b}\right| dp^* = \frac{1}{2\pi} \int_{\beta_h} d \arg \frac{w - a}{w - b}$$

$$= n(h, a) - n(h, b) ,$$

and integration from $-\infty$ to h gives the result $m(h, a) + N(h, a) = m(h, b) + N(h, b)$. The common value of the sum is taken as the characteristic function $T(h)$. For $h = k$, $T(k) = T(\Omega)$, the first main theorem takes the familiar Shimizu-Ahlfors form

$$m(\Omega, a) + N(\Omega, a) = T(\Omega) . \tag{14}$$

One integrates (14) over the area elements $d\omega(a)$ of the sphere A above the w-plane. Since $\iint_A \log (1/[w, a]) d\omega(a)$ is independent of w,

$$T(\Omega) = \frac{1}{\pi} \iint_A N(\Omega, a) \, d\omega(a) + \text{const.} \tag{15}$$

For more general distributions (15) is replaced by an inequality, obtained by integrating (14) over an arbitrary positive mass distribution $d\mu$ on the w-plane with total mass unity:

$$\int_{(a)} N(\Omega, a) d\mu(a) \leq T(\Omega) . \tag{16}$$

It is clear that, as in the classical theory, the two forms (9) and (14) of $T(\Omega)$ differ by a bounded quantity and can be used interchangeably in the sequel.

3. SECOND MAIN THEOREM ON W

11. Denote by W_h the relatively compact region with boundary β_h. Let $w \in M$ be a meromorphic function on W, and set

(17) $$w_P = \frac{dw}{dz} \Big/ \frac{dP}{dz} = \lambda c_\lambda (z - \zeta)^\lambda + \cdots ,$$

where dP is the differential $dp + i\,dp^*$. Returning to the $\overset{+}{\log}$-form, we write

$$m(h, a) = \frac{1}{2\pi} \int_{\beta_h} \overset{+}{\log} \frac{1}{|w - a|} dp^* \quad \text{for } a \neq \infty ,$$

and

$$m(h, w) = m(h, \infty) = \frac{1}{2\pi} \int_{\beta_h} \overset{+}{\log} |w| dp^* .$$

The number of a-points, with their multiplicities, of w in $\overline{W}_h - W_0$ is denoted by n(h, a). For the counting function we now choose

(18) $$N(h, a) = \int_0^h n(h, a) dh ,$$

and we set $N(h, w) = N(h, \infty)$, $T(h) = m(h, w) + N(h, w)$.

12. One is able to show that $\Sigma_1^q m(h, a_\nu)$ for $a_\nu \neq \infty$ is asymptotically $m(h, w_P^{-1})$ and that m(h, w) is essentially $m(h, w_P)$.

First consider the function

$$f = \sum_1^q \frac{1}{w - a_\nu} = \frac{1}{w - a_\mu} \left(1 + \sum_{\nu \neq \mu} \frac{w - a_\mu}{w - a_\nu} \right).$$

It is known (see [26, p. 242]) that

$$\sum_{\mu=1}^q m(h, a_\mu) \leq m(h, f) + O(1) ,$$

where

(19) $$m(h, f) = m(h, f w_P w_P^{-1}) \leq m(h, w_P^{-1}) + m\left(h, \sum_{\mu=1}^q \frac{w_P}{w - a_\mu} \right).$$

The modification of the counting function to (18) only contributes a term O(h) in the Jensen formula (10) applied to w_P:

(20) $$m(h, w_P^{-1}) = m(h, w_P) + N(h, w_P) - N(h, w_P^{-1}) + O(h) .$$

Furthermore,

(21) $$m(h, w_P) = m(h, ww_P w^{-1}) \leq m(h, w) + m(h, w_P w^{-1}).$$

Set $a_{q+1} = \infty$, and add $m(h, w)$ to both sides of (20). On the right the first main theorem (13) can be applied, β_0 again only contributing a term $O(1)$. It gives rise to $2T(h)$, while the counting functions add up to

$$-2N(h, w) + N(h, w_P) - N(h, w_P^{-1}) = -2N(h, w) + N(h, w_z) - N(h, w_z^{-1}) + N(h, P_z^{-1}),$$

there being no poles of P_z in $W - W_0$. We can also choose the parametric disc $|z - \zeta| < 1$ with $p(z) - \log|z - \zeta| \to 0$ for $z \to \zeta$, so small that W_0 contains no zeros of P_z. The number $n(h, P_z^{-1})$ of the zeros of P_z is the sum of the indices at the singularities of the vector field grad p. By Lefschetz' fixed point theorem, this sum, for any differentiable vector field, is the Euler characteristic $e(h)$ of $W_h - W_0$. We conclude (compare with [39]) that

$$N(h, P_z^{-1}) = E(h) = \int_0^h e(h)dh.$$

The remaining terms form the counting function

$$N_1(h) = \int_0^h n_1(h)dh$$

for multiple points of w, each k-tuple point counted $k - 1$ times:

$$N_1(h) = N(h, w_z^{-1}) + (2N(h, w) - N(h, w_z)).$$

We have arrived at the following result.

PRELIMINARY FORM OF THE SECOND MAIN THEOREM ON RIEMANN SURFACES.

(22) $$\sum_1^{q+1} m(h, a_\mu) < 2T(h) - N_1(h) + E(h) + S(h),$$

where

(23) $$S(h) = m\left(h, \frac{w_P}{w}\right) + m\left(h, \sum_1^q \frac{w_P}{w - a_\mu}\right) + O(h).$$

13. To estimate $S(h)$, consider a unit mass distribution $d\mu(a)$ of density $\rho(a)$ over the a-plane. The mass on the image surface of $W_h - W_0$ under w is

(24) $$M(h) = \int_0^h \int_{\beta_h} |w_P|^2 \rho(w)dp^* dh = \int_{(a)} n(h, a)d\mu(a),$$

its h-derivative is

$$M'(h) = \int_{\beta_h} |w_P|^2 \rho(w) dp^*,$$

and its integral is

(25) $$Q(h) = \int_0^h M(h)dh = \int_{(a)} N(h, a)d\mu(a) < T(h) + O(h).$$

In the special case $\rho(w) = 1/[2\pi^2|w|^2(1 + (\log|w|)^2)]$,

(26) $$m\left(h, \frac{w_P}{w}\right) < \frac{1}{4\pi}\int_{\beta_h} \overset{+}{\log}(|w_P|^2 \rho(w))dp^* + \frac{1}{4\pi}\int_{\beta_h} \log(1 + (\log|w|)^2)dp^* + O(1),$$

where the first term is estimated by the convexity of the logarithm,

$$\frac{1}{2\pi}\int_{\beta_h} \overset{+}{\log}(|w_P|^2 \rho(w))dp^* < \overset{+}{\log} M'(h) + O(1),$$

and the second term has the standard estimate

$$\frac{1}{2\pi}\int_{\beta_h} \log\sqrt{1 + (\log|w|)^2}\, dp^* \leq \log \frac{1}{2\pi}\int_{\beta_h} (1 + |\log|w||)dp^*$$

$$= \log\left(m(h, w) + m\left(h, \frac{1}{w}\right) + 1\right)$$

$$\leq \overset{+}{\log} T(h) + O(h).$$

14. Changing our notations slightly, we take two disjoint relatively compact bordered subregions: W_0 with border β_0 and Ω with border $\beta_0 \cup \beta_\Omega$. To economize symbols, denote by p the harmonic function on $\overline{\Omega}$ with $p = 0$ on β_0, $p = k = k(\Omega)$, a constant, on β_Ω, and $\int_{\beta_0} dp^* = 2\pi$.

For $0 \leq h \leq k$ consider the level line β_h in Ω. Formulas (22) through (26) continue to hold without modification. For any continuous function ϕ in $[0, k]$ set

$$\phi_2(h) = \int_0^h \int_0^y \phi(x)dx\,dy.$$

Then the preliminary form of the second main theorem gives

(27) $$\sum_1^{q+1} m_2(h, a_\mu) < 2T_2(h) - N_{1(2)}(h) + E_2(h) + S_2(h),$$

where

$$S_2(h) = m_2\left(h, \frac{w_P}{w}\right) + m_2\left(h, \sum_1^q \frac{w_P}{w - a_\mu}\right) + O(h^3). \tag{28}$$

To estimate m_2 we have

$$m\left(h, \frac{w_P}{w}\right) = \frac{1}{4\pi}\int_{\beta_h} \overset{+}{\log}(|w_P|^2 \rho(w))dp^* + \overset{+}{\log} T(h) + O(h), \tag{29}$$

where the inequality

$$\int_{\beta_h} \overset{+}{\log}(|w_P|^2 \rho(w))dp^* \leq 2\pi \overset{+}{\log} M'(h) + O(h)$$

gives the result

$$\int_0^h \int_{\beta_y} \overset{+}{\log}(|w_P|^2 \rho(w))dp^*\, dy \leq 2\pi h \overset{+}{\log}\left(\frac{1}{h} M(h)\right) + O(h^2)$$

$$\leq 2\pi h \overset{+}{\log} M(h) + O(h^2).$$

In virtue of the inequalities

$$\int_0^h x \overset{+}{\log} M(x)dx \leq h \int_0^h \overset{+}{\log} M(x)dx$$

$$\leq h^2 \overset{+}{\log}\left(\frac{1}{h} Q(h)\right)$$

$$\leq h^2 \overset{+}{\log} T(h) + O(h^3),$$

we obtain the estimate

$$\int_0^h \int_0^y \int_{\beta_x} \overset{+}{\log}(|w_P|^2 \rho(w))dp^*\, dx\, dy \leq 2\pi h^2 \overset{+}{\log} T(h) + O(h^3).$$

Similarly,

$$\int_0^h \int_0^y \overset{+}{\log} T(x)dx\, dy \leq h^2 \overset{+}{\log} T_2(h) + O(h^3),$$

and we conclude that

$$m_2\left(h, \frac{w_P}{w}\right) \leq O(h^2 \log T(h) + h^3) + O(h^2 \log T_2(h)). \tag{30}$$

On substituting this into (28) and taking $h = k$, we obtain our main result.

SECOND MAIN THEOREM ON ARBITRARY RIEMANN SURFACES. *For every* $\Omega \subset W$,

$$(31) \qquad \sum_{1}^{q} m_2(k, a_\mu) < 2T_2(k) - N_{1(2)}(k) + E_2(k) + S_2(k)$$

with

$$(32) \qquad S_2(k) = O(k^3 + k^2 \log (T(k)\, T_2(k))) .$$

Observe that the variable here is the region Ω which determines $k = k(\Omega)$.

15. Using directed limits, we define for canonical regions Ω,

$$(33) \qquad \delta(a) = 1 - \lim_{\Omega \to W} \sup \frac{N_2(k, a)}{T_2(k)},$$

$$(34) \qquad \theta = \lim_{\Omega \to W} \inf \frac{N_{1(2)}(k)}{T_2(k)},$$

$$(35) \qquad \eta = \lim_{\Omega \to W} \inf \frac{E_2(k)}{T_2(k)},$$

and obtain our last result.

DEFECT AND RAMIFICATION RELATIONS ON ARBITRARY RIEMANN SURFACES. *For functions with* $\lim_{\Omega \to W} (S_2(k)/T_2(k)) = 0$

$$(36) \qquad \sum \delta(a_\mu) + \theta \leq 2 + \eta .$$

In particular, there can be at most $2 + \eta$ Picard values. The classical bound 2 is met by functions for which T_2 grows more rapidly than E_2.

The condition $S_2(k)/T_2(k) \to 0$ or, equivalently, $(k^3 + k^2 \log T(k))/T_2(k) \to 0$, is natural and is obviously satisfied even by such functions as $T(k) = k^\alpha$ with $\alpha > 1$ and $T(k) = e^{\alpha k}$ with $\alpha > 0$.

16. In case W is a W_p surface, that is, W has a capacity function p with compact level lines, then β_0, β_Ω can be chosen as level lines, and the two meanings of p coincide. In (31), (32), k instead of Ω can be taken as the variable and the directed limits in formulas (33) through (35) replaced by ordinary limits as $k \to \sup_W p$. By L'Hospital's rule the subindex 2 can here be dropped and δ, θ can then be used in the traditional sense. In the special cases of the plane and the disk, we have in (31) a new second main theorem without exceptional intervals and an elementary proof of the defect and ramification relations.

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