SUFFICIENT CONDITIONS FOR SEMICONTINUOUS SURFACE INTEGRALS

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1. INTRODUCTION

Semicontinuous parametric and nonparametric line integrals were introduced by L. Tonelli [14] to aid in establishing existence theorems in the calculus of variations. Following this procedure, McShane [9, 10] proved the first theorems on semicontinuous parametric surface integrals for surfaces having Lipschitzian representations. Subsequently, T. Rado [12] and L. Cesari [3] proved successively stronger theorems (slightly later, but independently, P. V. Reichelderfer [13] published an intermediate result; see also G. M. Ewing [7]). Cesari's theorems deal with arbitrary surfaces of finite area defined on the unit square.

A surface (T, Q) defined on the unit square Q is a mapping T of

$$Q = \{(u, v): 0 \leq u,\ v \leq 1\}$$

into Euclidean three-space $E^3 = \{(x, y, z)\}$. All mappings will be supposed continuous in this paper. We shall say that (T, Q) is of bounded variation or has BV if it has finite area.

We shall call a function of six variables $f(x, y, z, J_1, J_2, J_3) = f(p, J)$ a parametric integrand if it is continuous and positively homogeneous in J. If a BV surface (T, Q) is absolutely continuous, then generalized Jacobians (see [5]) exist almost everywhere in Q, and if f(p, J) is a parametric integrand bounded on T(Q), the Lebesgue-Tonelli integral

$$I(T, Q) = (Q) \int f(T(w), J(w))\,du\,dv \tag{1}$$

exists, where $J(w) = (J_1, J_2, J_3)$ are the generalized Jacobians. This is the integral used by Cesari. The other authors also used this integral; but they restricted themselves to mappings (T, Q) that have sufficiently well behaved Jacobians in the usual sense.

Bouligand [1] showed that in the study of semicontinuous parametric line integrals $\int f(x, \dot{x})dt$, the usual conditions on the Weierstrass function can be replaced by weaker conditions involving the convexity of $f(x, \dot{x})$ in $\dot{x}$ for every x. Later Aronszain (as reported by Pauc [11]) did the same for nonparametric line integrals.

Here, we shall continue the work of McShane, Rado, Reichelderfer, and Cesari on surface integrals, generalizing their results in the following respects:

(1) The domain A of the surface (T, A) will be an arbitrary admissible set, as defined by Cesari [5] and given below;

(2) Absolutely continuous representations of the surfaces will not be used. We shall use the integral defined by Cesari [2], in the extended form given by Cesari and Turner [6], for which no special representation is necessary;

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(3) All conditions that were imposed by previous authors on the Weierstrass function and that presuppose the existence of the partial derivatives $\partial f(x, J)/\partial J_i$ will be replaced by weaker hypotheses involving the convexity of $f(x, J)$ with respect to J.

2. NOTATION

Unless otherwise stated, all results of this section may be found in [5]. We shall denote the interior of a set A by A^o. Let $(T, A)\colon A \to E^3$ be a continuous surface (mapping) of finite area, where A is an admissible set in the sense of Cesari; that is, where A is either an open set in the plane, or A is a pairwise disjoint union of sets of the form $A = J_0 - (J_1 + J_2 + J_3 + \cdots + J_s)^o$ where $J_0, \cdots, J_s$ are Jordan regions, $J_1, \cdots, J_s$ are pairwise disjoint and each $J_1, \cdots, J_s$ is a subset of J_0^o (if the boundary curves J_n^* are polygons, we will call such a domain a figure), or A is an open subset of such a set. Let t_1, t_2, t_3 be the functions that orthogonally project E^3 onto the yz-, zx-, and xy-planes, respectively, (we shall call these planes E_1, E_2, E_3) and let $T_r = t_r T$. Let $\pi \subset A$ be a simple polygonal region, π^* the oriented boundary of π, and $C_{\pi r} = T_r(\pi^*)$ the closed oriented projection of π^* in E_r. Let $O^+(p, C_{\pi r}) = (|O| + O)/2$, $O^-(p; C_{\pi r}) = (|O| - O)/2$. We shall always let S represent a finite set of simple nonoverlapping polygonal regions $\pi \subset A$. For $p \in E_r$, let

$$N(p, T_r) = \sup_{(S)} \sum_{\pi \in S} |O(p; C_{\pi r})|,$$

$$N^+(p, T_r) = \sup_{(S)} \sum_{\pi \in S} O^+(p; C_{\pi r}),$$

$$N^-(p; T_r) = \sup_{(S)} \sum_{\pi \in S} O^-(p; C_{\pi r}),$$

$$u_r(\pi) = (E_r) \int O(p; C_{\pi r}), \quad v_r(\pi) = (E_r) \int |O(p; C_{\pi r})|,$$

$$u(\pi) = (u_1^2 + u_2^2 + u_3^2)^{1/2}, \quad v(\pi) = (v_1^2 + v_2^2 + v_3^2)^{1/2},$$

where the integrations are taken with respect to two dimensional Lebesgue measure. Now $N = N^+ + N^-$ everywhere in the plane except at a countable number of points, and N^+ and N^- are both finite almost everywhere. Let $n(p: T) = N^+ - N^-$, where N^+ and N^- are not simultaneously infinite, let $n = 0$ where $N^+ = N^- = \infty$. The area or total variation of the plane mapping (T_r, A) is $W(T_r, A) = (E_r) \int N(p; T_r)$. The positive, negative, and relative variations are defined by

$$W^+(T_r, A) = (E_r) \int N^+(p; T_r);$$

$$W^-(T_r, A) = (E_r) \int N^-(p; T_r);$$

$$V(T_r, A) = (E_r) \int n(p; T_r);$$

and $V = W^+ - W^-$. If $W(T_r, A) < +\infty$ ($r = 1, 2, 3$), we say that (T, A) is a BV surface. The Geocze area of (T, A) is

$$U(T, A) = \sup_{(S)} \sum_{\pi \in S} u(\pi) = \sup_{(S)} \sum_{\pi \in S} v(\pi).$$

We defined the oriented curves $C_{\pi r}$ above. Let $[C_{\pi r}]$ be the set covered by $C_{\pi r}$. Let absolute value signs denote Lebesgue measure. For every system S associated with (T, A) we define three indices d, m, μ as

$$d(S) = \max\{\operatorname{diam} T(\pi): \pi \in S\};$$

$$m(S) = \max\{|\sum_{\pi}[C_{\pi r}]|; r = 1, 2, 3\};$$

$$\mu(S) = \max\{U(T, A) - \sum_{\pi} u(\pi), U(T_r, A) - \sum_{\pi} |u_r(\pi)|: r = 1, 2, 3\}.$$

For every surface (T, A) and each $\varepsilon > 0$ there exist systems S with indices $d, m, \mu < \varepsilon$.

Let $f(p, J)$ be a parametric integrand bounded on $T(A)$. Cesari defined the following integral by the limit, which he proved exists,

$$H(T, A; f) = \lim \sum_{\pi \in S} f(p_\pi, u_1(\pi), u_2(\pi), u_3(\pi)), \tag{2}$$

where p_π is any point of $T(\pi)$ and the limit is taken as d, m, μ tend to zero. Let $V(\pi) = (V(T_1, \pi), V(T_2, \pi), V(T_3, \pi))$. In [15], it was shown that $H(T, A; f)$ could be defined by

$$H(T, A; f) = \lim \sum_{\pi \in S} f(p_\pi, v(\pi))$$

with p_π and the limit taken as before. Also, if $\alpha = (\alpha_{ij})$ is an orthogonal matrix and $\tilde{T} = \alpha T$, then

$$H(\tilde{T}, A; g) = H(T, A; f),$$

where $g(p, J) = f(\alpha^{-1} p, \alpha^{-1} J)$.

Let (T, A) be a BV surface. Let $\Gamma(A)$ be the set of all components $g \subset A$ of $T^{-1}(p)$ as p varies over all $T(A)$. Let $\mathscr{G}, \mathscr{F}, \mathscr{B}_0$ be the class of all subsets of A that, respectively, are unions of the $g \in \Gamma(A)$ and open in A, are compact, and are Borel sets.

Let K be any element of $\mathscr{B}_0$, and define

$$
(3) \qquad
\begin{aligned}
\phi(K) &= \inf\{U(T, G): G \supset K,\ G \in \mathscr{G}\},\\
\phi_r^+(K) &= \inf\{W^+(T_r, G): G \supset K,\ G \in \mathscr{G}\},\\
\phi_r^-(K) &= \inf\{W^-(T_r, G): G \supset K,\ G \in \mathscr{G}\},\\
V_r(K) &= \phi_r^+(K) - \phi_r^-(K).
\end{aligned}
$$

In [4, 5] Cesari showed that these are measures if A is compact. This result was extended to any admissible set A in [16]. It was also shown in [16] that ϕ_r^+ and ϕ_r^- are mutually singular and therefore form a Jordan decomposition of V_r. Moreover, for any $K \in \mathscr{B}_0$,

$$\phi(K) = \sup\{\textstyle\sum[V_1^2(B) + V_2^2(B) + V_3^2(B)]^{1/2}\},$$

where the sum is taken over all members B of a decomposition of K into disjoint sets of $\mathscr{B}_0$ and the supremum is taken over all such partitions of K. This makes V_r absolutely continuous with respect to ϕ, so we may take a Radon-Nikodym derivative $\theta_r(w) = dV_r/d\phi$. It is shown in [6] that

$$\|\theta\| = (\theta_1^2 + \theta_2^2 + \theta_3^2)^{1/2} = 1 \quad \text{a.e. } (\phi).$$

We may define a surface integral over (T, A) with respect to a bounded parametric integrand f as

$$(4) \qquad I(T, A; f) = (A)\int f(T(w), \theta(w))\, d\phi.$$

It is shown in [6] that this integral coincides with (1).

We will need more results proved in [6]. Let $B \subset A$ be an admissible set so that (T, B) is also a BV surface. Let $\Gamma(B)$ be the collection of maximal components of constancy for this mapping. Let $\Gamma^*(B)$ be the set of all $g \in \Gamma(B)$ that are continua and contained in B°, and let $\hat{B}$ be the set covered by $\Gamma^*(B)$. Then $\Gamma^*(B) \subset \Gamma(B)$; B is open in the plane and $V(T, \hat{B}) = V(T, B)$. Furthermore there are σ-algebras of Borel sets corresponding to the mappings (T, B) and $(T, \hat{B})$ and also measures defined for these mappings which are analogous to (3). But the σ-algebra $\mathscr{B}_1$ for $(T, \hat{B})$ is a subalgebra of $\mathscr{B}_0$ and also a subalgebra of that for (T, B), and the similarly defined measures for $(T, \hat{B})$, (T, B) and (T, A) are identical on $\mathscr{B}_1$; hence we will not introduce new notation for these measures or the Radon-Nikodym derivatives for $(T, \hat{B})$. Thus for the bounded parametric integrand f,

$$I(T, B; f) = (\hat{B})\int f(T(w), \theta(w))\, d\phi.$$

An elementary fact which we will need is that if (T, A) is a BV mapping, then for any $\varepsilon > 0$ there is a compact admissible set $B \subset A$ such that

$$V(T, A) - V(T, B) < \varepsilon.$$

In fact B may be chosen to be a figure.

3. SOME LEMMAS

The concepts of normal integrand and regular integrand are old. The terms semi-regular and semi-normal have also been used. We shall employ the following definitions of these concepts already used by W. Fleming and L. C. Young [8].

Definition 1. The integrand $f(x, J)$ is called positive semi-regular at x_0 (on a set $D \subset E^3$) if $f(x_0, J)$ is convex in J (for all $x_0 \in D$).

Definition 2. The integrand $f(x, J)$ is called positive semi-normal (henceforth PSN) at x_0 (on a set $D \subset E^3$) if $f(x, J)$ is positive semi-regular at x_0 and $f(x_0, J) + f(x_0, -J) > 0$ for all $J \neq 0$ (and all $x_0 \in D$).

LEMMA 1. *Let* $f(x, J)$ *be* PSN *at* x_0. *Then the set* H *of all* $d = (d_1, d_2, d_3)$ *such that*

$$f(x_0, J) \geq (d, J) = d_1 J_1 + d_2 J_2 + d_3 J_3$$

for all J *is convex, closed and has a nonvoid interior. Moreover, if* $d \in H$ *has distance* τ *from the boundary of* H, *then* $f(x_0, J) \geq (d, J) + \tau \| J \|$ *for all* J.

Proof. Suppose first that $f(x_0, J) \geq 0$ for all J. Then $d = 0 \in H$, so H is non-void. Moreover, if $d^1, d^2 \in H$ and $0 \leq \alpha \leq 1$, then

$$f(x_0, J) - (\alpha d^1 + (1 - \alpha) d^2, J) = \alpha [f(x_0, J) - (d^1, J)] + (1 - \alpha)[f(x_0, J) - (d^2, J)] \geq 0$$

for all J. Hence H is convex. If H had no interior, then H would be contained in a 2-dimensional manifold of E^3 since $(0, 0, 0) \in H$. Then there would be a vector $c \in H$ such that $(c, d) = 0$ for all $d \in H$.

Now $f(x_0, c) + f(x_0, -c) > 0$ so we may suppose that $f(x_0, c) > 0$. But, $f(x_0, J)$ being convex in J, there exists a linear function of J, say

$$(d, J) = d_1 J_1 + d_2 J_2 + d_3 J_3,$$

such that $f(x_0, J) \geq (d, J)$ for all J and $f(x_0, c) = (d, c)$. Then $d \in H$ so $(d, c) = 0$, but with $J = c$ it must also be that $f(x_0, c) = (d, c) > 0$, which is a contradiction. Thus H has an interior. It is obvious that H is closed.

Now suppose $d \in H$ has distance τ from the boundary of H. Then for any J, let $d^* = d + \tau J / \| J \| \in H$. Then

$$f(x_0, J) - (d, J) = f(x_0, J) - (d^*, J) + (d^* - d, J) \geq \tau \| J \|$$

as desired.

For an arbitrary PSN function $f(x_0, J)$, let d' be such that $f(x_0, J) \geq (d', J)$ for all J, and let $f'(x_0, J) = f(x_0, J) - (d', J)$, so that f' is PSN at x_0 and $f'(x_0, J) \geq 0$ for all J. Let H' and H be the sets defined above for f' and f, respectively. Then

$$f(x_0, J) - (d + d', J) = f'(x_0, J) - (d, J)$$

implies that $H = d' + H'$, and the desired properties of H follow immediately from those of H'.

LEMMA 2. *Let* $f(x, J)$ *be* PSN *at* x_0. *Let* $\| J_0 \| = 1$. *Then for every* $\varepsilon > 0$ *there exist a* $\delta > 0$ *and a* $d \in E^3$ *such that*

(a) $f(x, J) > (d, J)$ *for all* J *if* $\|x - x_0\| < \delta$; *and*

(b) $f(x, J) < (d, J) + \varepsilon$ *if* $\|J - J_0\| < \delta$, $\|J\| = 1$, *and* $\|x - x_0\| < \delta$.

Proof. Let $z(J) = (d^*, J)$ be a supporting function for $f(x_0, J)$ at $J = J_0$. Thus $f(x_0, J) \geq (d^*, J)$ for all J and $f(x_0, J_0) = (d^*, J_0)$. Let H be the convex body defined in Lemma 1. Then $d^* \epsilon H$ is a boundary point of H. Let d be an interior point of H with $\|d - d^*\| < \varepsilon/3$. Let $\tau > 0$ be the distance from d to the boundary of H, so $\tau \leq \varepsilon/3$. Now by the continuity of $f(x, J)$, there is a $\delta > 0$ with $\delta < \varepsilon/3\|d\|$ such that

$$|f(x, J) - f(x_0, J_0)| < \varepsilon/3 \quad \text{if} \quad \|x - x_0\| < \delta, \ \|J - J_0\| < \delta$$

and

$$|f(x, J) - f(x_0, J)| < \tau \quad \text{for all } \|J\| = 1 \text{ if } \|x - x_0\| < \delta .$$

Thus if $\|x - x_0\| < \delta$ and $\|J - J_0\| < \delta$,

$$\begin{aligned} f(x, J) - (d, J) &= [f(x, J) - f(x_0, J_0)] + [f(x_0, J_0) - (d^*, J_0)] \\ &\quad + (d^* - d, J_0) + (d, J_0 - J) \\ &< \varepsilon/3 + 0 + \varepsilon/3 + \varepsilon/3 = \varepsilon ; \end{aligned}$$

and if $\|J\| = 1$ and $\|x - x_0\| < \delta$, then

$$f(x, J) - (d, J) = [f(x, J) - f(x_0, J)] + [f(x_0, J) - (d, J)] > -\tau + \tau\|J\| = 0 .$$

LEMMA 3. *Let* (T_0, A) *be a* BV *plane mapping,* $T_0(u, v) = (x(u, v), y(u, v))$. *Then for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *with the following property:*

If (T, A) *is another* BV *plane mapping into the* $x - y$ *plane with* $\|T(w) - T_0(w)\| < \delta$ *for all* $w \epsilon A$, *then there exists an* $\eta > 0$ *such that for all systems* S *with indices* $d, m, \mu < \eta$ *with respect to* (T, A), *there exists a subsystem* S' *with*

$$|V(T_0, A) - \Sigma' V(T, q)| < \varepsilon ,$$

where Σ' *denotes a sum over all* $q \epsilon S'$.

Proof. Let $\tau (\tau > 0, \ \tau < \varepsilon/8)$ be such that

$$(G) \int N(p; T_0, A) < \varepsilon/8$$

for every measurable set $G \subset E_2$ with $|G| < \tau$. Let S^* be a system of polygonal regions $t \subset A$ with indices $d, m, \mu < \tau$. Then, if Σ^* denotes a sum over all $t \epsilon S^*$ and C_{0t} is the closed curve $T_0(t^*)$ which is oriented as t^* is,

$$0 \leq U(T_0, A) - \Sigma^* v(t, T_0) \leq U(T_0, A) - \Sigma^* |u(t, T_0)| < \tau .$$

Hence

$$0 \leq (E_2) \int N(p; T_0, A) - \Sigma^* (E_2) \int |O(p; C_{0t})| < \tau ,$$

and the set $H_0^* \subset E_2$ where $N(p; T_0, A) > \Sigma^* |O(p; C_{0t})|$ has measure $|H_0^*| < \tau$. Thus

$$n(p; T_0, A) = \sum^* O(p; C_{0t}) \quad \text{for all } p \in E_2 - H_0^* - D_0 = E_2 - H_0 ,$$

where $H_0 = H_0^* + D_0$ and D_0 is the set of measure zero where

$$n(p; T_0, A) \neq N^+(p; T_0, A) - N^-(p; T_0, A) .$$

Moreover, $n(p; T_0, t) = O(p; C_{0t})$ except in $[C_{0t}]$. Thus if $B = \Sigma^* [C_{0t}]$, then $|B| < \tau$, and

$$|V(T_0, A) - \sum^* V(T_0, t)| = |(E_2) \int n(p; T_0, A) - \sum^* (E_2) \int n(p; T_0, t)|$$

$$= |(B + H_0) \int [n(p; T_0, A) - \sum^* n(p; T_0, t)]|$$

$$\leq (B + H_0) \int [N^+(p; T_0, A) - \sum^* N^+(p; T_0, t)]$$

$$+ (B + H_0) \int [N^-(p; T_0, A) - \sum^* N^-(p; T_0, t)]$$

$$= (B + H_0) \int [N(p; T_0, A) - \sum^* N(p; T_0, t)] < \frac{2\varepsilon}{8} .$$

Let B_ρ be the closed ρ neighborhood of B. Then $\lim |B_\rho| = |B| < \tau$ as $\rho \to 0$. Therefore for some $\delta > 0$, $|B_\rho| < \tau$ for all ρ satisfying the condition $0 < \rho < 2\delta$.

Let (T, A) be any BV plane mapping into E_2 such that $|T(w) - T_0(w)| < \delta$ for all $w \in A$. Let $\lambda > 0$ be such that $(G) \int N(p; T, A) < \varepsilon/8$ for $G \subset E_2$, $|G| < \lambda$. Since $\lim |B_\gamma - B_\delta| = 0$ as $\gamma \to \delta^+$, there is a γ satisfying the conditions $0 < \delta < \gamma < 2\delta$ and $|B_\gamma - B_\delta| < \lambda$.

Let $\eta = \min(\lambda, \gamma - \delta, \tau)$, and let S be any system of polygonal regions $\pi \subset A$ with indices d, m, $\mu < \eta$ with respect to (T, A). Let S_t denote the set of those $\pi \in S$ such that $\pi t \neq 0$, and let $t' = t^o + \Sigma_t \pi^o$, where Σ_t denotes a sum over all $\pi \in S_t$. Then $t' \supset t^o$, and

$$N(p; T, t') \geq N(p; T, t^o) = N(p; T, t) \quad \text{for all } p \in E_2 .$$

Let Σ_t^* denote a sum over all $\pi \notin S_t$ and Σ denote a sum over all $\pi \in S$. Now

$$0 \leq W(T, A) - \sum |v(\pi, T)| = (E_2) \int [N(p, T, A) - \sum |O(p; C_\pi)|] < \eta \leq \tau < \frac{\varepsilon}{8} .$$

Thus the set H^* where $N(p; T, A) - \Sigma |O(p; C_\pi)| > 0$ has measure $|H^*| < \varepsilon/8$. But

$$N(p; T, A) - \sum |O(p; C_\pi)| \geq [N(p; T, t') - \sum_t |O(p, C_\pi)|]$$
$$+ \sum_t^* [N(p; T, \pi) - |O(p; C_\pi)|] \geq 0$$

everywhere in E_2. (Note that $N(p; T, A)$ is super-additive as a function of A.) Thus if H_t^* is the set where $N(p; T, t') > \Sigma_t |O(p; C_\pi)|$, then $H_t^* \subset H^*$. Let $S_t' \subset S_t$ be the set of all $\pi \in S_t$ such that $T(\pi)B_\delta = 0$. If $\pi t^* \neq 0$, then $T(\pi)B_\delta \neq 0$ since, for some point $w \in \pi t^*$, $\|T(w) - T_0(w)\| < \delta$ and $T(w) \in B_\delta$ since $T_0(w) \in B$. Thus $\pi \subset t^o$ for every $\pi \in S_t'$. Also if $\pi \in S_t - S_t'$, then $T(\pi)B_\delta \neq 0$ and $T(\pi) \subset B_\gamma$ since $\operatorname{diam}[T(\pi)] < \eta \leq \gamma - \delta$; thus $O(p; C_\pi) = 0$ for all $p \in E_2 - B_\gamma$. Therefore

$$N(p; T, t') = \Sigma_t' |O(p; C_\pi)| \quad \text{for all } p \in E_2 - B_\gamma - H_t^*,$$

where Σ_t' denotes a sum over all $\pi \in S_t'$. But

$$N(p; T, t') \geq N(p, T, t) \geq \Sigma_t' |O(p; C_\pi)|$$

everywhere in E_2, so $N(p; T, t) = \Sigma_t' |O(p; C_\pi)|$ for all $p \in E_2 - B_\gamma - H_t^*$. Hence $n(p; T, t) = \Sigma_t' O(p; C_\pi)$ for all $p \in E_2' - B_\gamma - H_t$; here $H_t = H_t^* + D_t$, and D_t is the set of measure zero where

$$n(p; T, t) \neq N^+(p; T, t) - N^-(p; T, t).$$

A fundamental theorem concerning the topological index states that if C_0, C are two closed curves with Frechet distance $\|C_0, C_1\| < \delta$ and $p \in E_2$ has distance at least δ from C_0, then $O(p; C_0) = O(p; C_1)$. Thus, since $\|C_t, C_{0t}\| < \delta$,

$$n(p; T, t) = O(p; C_t) = O(p; C_{0t}) = n(p; T_0, t) \quad \text{for all } p \in E_2 - B_\delta.$$

Therefore $n(p; T_0, t) = \Sigma_t' O(p; C_\pi)$ for all $p \in E_2 - B_\gamma - H_t$.

Let $K = \Sigma^* H_t$. Then $\Sigma^* N(p; T, t) = \Sigma^* \Sigma_t' |O(p; C_\pi)|$ for all $p \in E_2 - B_\gamma - K$. Moreover, $|K| < \eta \leq \tau$ and $|K| < \eta \leq \lambda$, since

$$K = \Sigma^* (D_t + H_t^*) \subset \Sigma^* D_t + H^* \quad \text{and } |K| \leq |\Sigma^* D_t| + |H^*| = |H^*| < \eta.$$

Therefore

$$\Sigma^* n(p; T_0, t) = \Sigma^* \Sigma_t' n(p; T, \pi)$$

for all $p \in E_2 - B_\gamma - F - K$, where $F = \Sigma[C_\pi]$ and $|F| < \eta \leq \tau$, $|F| < \eta \leq \lambda$. Let $S' = \Sigma^* S_t'$, and let Σ' denote a sum over S'. Then

$$|V(T_0, A) - \Sigma' V(T, \pi)| \leq |V(T_0, A) - \Sigma^* V(T_0, t)| + |\Sigma^* V(T_0, t) - \Sigma'(T, \pi)|$$

$$< \frac{2\varepsilon}{8} + |(E_2) \int \left[\Sigma^* n(p; T_0, t) - \Sigma' n(p; T, \pi) \right] |$$

$$= \frac{\varepsilon}{4} + |(B_\gamma + K + F) \int \left[\Sigma^* n(p; T_0, t) - \Sigma' n(p; T, \pi) \right] |$$

$$\leq \frac{\varepsilon}{4} + (B_\gamma + K + F) \int N(p; T_0, A) + (B_\gamma + K + F) \int \Sigma' N(p; T, \pi)$$

$$\leq \frac{5\varepsilon}{8} + (K + F) \int N(p; T, A) + (B_\gamma - B_\delta) \int N(p; T, A)$$

$$\leq \frac{5\varepsilon}{8} + 2\frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \varepsilon.$$

4. SEMICONTINUITY THEOREMS

Let (T_0, A_0) and (T, A) be two surfaces. Let us recall the definition of (Fréchet) distance between these surfaces. This is defined if and only if there is a sense preserving homeomorphism σ from A_0 to A. In this case the distance is defined to be

$$\inf_{\{\sigma\}} \sup_{w \in A_0} \|T_0(w) - T(\sigma(w))\|.$$

The infimum is taken over all such homeomorphisms σ. If this distance is zero, we regard (T_0, A_0) and (T, A) as representing the same Fréchet surface. This is reasonable since they have the same area (see [5]), and the integral (4) over each is the same (see [6]). Then with σ, (T_0, A_0), and (T, A) as above, the second surface has a representation $(T\sigma, A_0)$ with A_0 as domain. Thus any surface whose distance from (T_0, A_0) is defined may be regarded as having domain A_0. Moreover, it is obviously true that the class of all surfaces whose distance from (T_0, A_0) is less than δ consists of all surfaces with a representation (T, A_0) such that $\|T_0(w) - T(w)\| < \delta$ for all $w \in A_0$.

Thus we may, in discussions of semicontinuous integrals, regard all surfaces as having the same domain.

THEOREM 1. *Suppose* $f(p, J)$ *is a parametric integrand defined on* E^6. *Suppose* (T_0, A) *is a* BV *surface. Suppose that for some* $\rho > 0$, *the set*

$$U = \{p: \text{dist } (T_0(A), p) < \rho\}$$

is such that $f(p, J) \geq 0$ *for all* J *if* $p \in U$. *Suppose* $f(p, J)$ *is uniformly continuous and uniformly bounded on* $Z = \{(p, J): p \in U, \|J\| = 1\}$. *Suppose that almost every* $(\phi_0)w_0 \in A$ *has the property that, for each* $\varepsilon > 0$, *there exist a* $\sigma > 0$ *and a linear function* $\psi_0(J) = (b_0, J)$ *such that if* $\|p - T_0(w_0)\| < \rho_0$,

(a) $f(p, J) \geq \psi_0(J)$ *for all* J, *and*

(b) $f(p, J) \leq \psi_0(J) + \varepsilon \|J\|$ *if* $\left\| \frac{J}{\|J\|} - \theta_0(w_0) \right\| < \sigma$,

where ϕ_0 *is the measure induced on* A *by* (T_0, A) *and* $\theta_0 = (\theta_{01}, \theta_{02}, \theta_{03})$ *are the associated Radon-Nikodym derivatives. Then* $I(T, A) = (A) \int f(p, J)\, d\phi$ *is lower semi-continuous at* (T_0, A) *in the class of all surfaces* (T, A) *comparable to* (T_0, A).

Proof. Let M be such that $M \geq \sup\{f(p, J): (p, J) \in Z\}$ and $M \geq U(T_0, A)$. Let $\varepsilon > 0$ be given. By Lusin's theorem, there is a compact set $F \subset \hat{A}$ with $F \in \mathscr{B}_0$, where $\mathscr{B}_0$ is the class of Borel subsets corresponding to (T_0, A), such that $\phi_0(A - F) < \varepsilon/(6M)$, $\theta_0(w)$ is continuous on F, $\|\theta(w)\| = 1$ on F, and every continuum $g \subset F$ with $g \in \Gamma$ has the property that there exist a $\rho' > 0$ and a function $\psi(J) = (b, J)$ such that if $\|p - T_0(g)\| < \rho'$,

$$(5) \quad \begin{aligned} &f(p, J) \geq \psi(J) \quad \text{for all J and} \\ &f(p, J) \leq \psi(J) + \frac{\varepsilon}{6M} \|J\| \quad \text{if } \left\| \frac{J}{\|J\|} - \theta_0(g) \right\| < \rho'. \end{aligned}$$

Given $g \subset F$ with $g \in \Gamma$, let ρ' and $\psi(J) = (b, J)$ be the constant and function defined relative to g above. For every $w' \in g$, there is a $\delta' > 0$ such that

$$\|T_0(w) = T_0(w')\| = \|T_0(w) - T_0(g)\| < \rho'/2 \quad \text{if } \|w - w'\| < \delta', \quad \text{and}$$

$$\|\theta_0(w) - \theta_0(w')\| = \|\theta_0(w) - \theta_0(g)\| < \rho' \quad \text{if } \|w - w'\| < \delta', \quad w \in F.$$

The circles $\{w: \|w - w'\| < \delta'\}$ cover g as w' varies in g. Since g is compact, a finite number cover g. Let the union of the members of such a cover be $G(g)$, and let $H(g)$ be the open set consisting of all continua of constancy contained in $G(g)$. Thus $H(g) \in \mathscr{G}$. Therefore, if $\operatorname{dist}(p, T_0 H(g)) < \rho'/_2$, there is a $w' \in H(g)$ such that $\|T_0(w') - p\| < \rho'/_2$. Thus

$$\|p - T_0(g)\| \leq \|p - T_0(w')\| + \|T_0(w') - T_0(g)\| < \rho'/_2 + \rho'/_2 = \rho';$$

hence, from (5),

$$\text{(6)} \qquad \begin{aligned} &f(p, J) \geq \psi(J) \quad \text{for all } J, \text{ and} \\ &f(p, \theta_0(w)) \leq \psi(\theta_0(w)) + \frac{\varepsilon}{6M} \quad \text{if } w \in H(g)F. \end{aligned}$$

The open sets $H(g)$ cover F. Therefore, there are a finite number that cover F since F is compact. Let the members of such a cover be $H_j = H(g_j)$ $(j = 1, \cdots, \nu)$, and let the associated constants and functions be ρ_j and $\psi_j(J) = (b_j, J)$. Then the inequality $\|b_j\| \leq M$ follows from the inequalities

$$M \geq f\left(T_0(g_j), \frac{b_j}{\|b_j\|}\right) \geq \left(b_j, \frac{b_j}{\|b_j\|}\right) = \|b_j\|.$$

The sets H_j are open and therefore admissible. Let K_1 be a figure, $K_1 \subset H_1$, such that $\phi_0(H_1) - U(T_0, K_1) < \varepsilon/(6M\nu)$. Let $\tilde{K}_1$ be the compact set that is the union of all continua $g \in \Gamma$ which intersect K_1. Thus $\tilde{K}_1 \in \mathscr{B}_0$ and $K_1 \subset \tilde{K}_1 \subset H_1$. Then $H_2 - \tilde{K}_1$ is open in E_2; hence it is admissible. Similarly, for $i = 2, 3, \cdots, \nu$, let K_i be a figure $K_i \subset H_i - (\tilde{K}_1 + \cdots + \tilde{K}_{i-1})$ such that

$$\phi_0[H_i - (\tilde{K}_1 + \cdots + \tilde{K}_{i-1})] - U(T_0, K_i) < \varepsilon/(6M\nu) \qquad (i = 2, \cdots, \nu),$$

and let $\tilde{K}_i$ be the union of all continua $g \in \Gamma$ that intersect K_i. Then the figures K_j $(j = 1, \cdots, \nu)$ are disjoint, and

$$\text{(7)} \qquad \begin{aligned} 0 &\leq U(T_0, A) - \sum_{j=1}^{\nu} U(T_0, K_j) \\ &\leq \phi_0(F) + \frac{\varepsilon}{6M} - \sum_{j=1}^{\nu} U(T_0, K_j) \\ &\leq \phi_0(H_1 + \cdots + H_\nu) - \sum_{j=1}^{\nu} U(T_0, K_j) + \frac{\varepsilon}{6M} \end{aligned}$$

$$\leq \phi_0(H_1) + \phi_0(H_2 - H_1) + \cdots + \phi_0[H_\nu - (H_1 + H_2 + \cdots + H_{\nu-1})]$$

$$- \sum_{j=1}^{\nu} U(T_0, K_j) + \frac{\varepsilon}{6M} < \frac{2\varepsilon}{6M}.$$

Let $P_j = F\hat{K}_j$, so $P_j \in \mathscr{B}_0$. Let $D = \Sigma_{j=1}^{\nu} (\hat{K}_j - P_j) \subset A - F$, so $\phi_0(D) < \varepsilon/(6M)$. Also

$$\phi_0\left(A - \sum_{j=1}^{\nu} \hat{K}_j\right) = \phi_0(A) - \sum_{j=1}^{\nu} \phi_0(\hat{K}_j) = U(T_0, A) - \sum_{j=1}^{\nu} U(T_0, \hat{K}_j) < \frac{2\varepsilon}{6M}$$

from (7). Therefore, using (6), we see that

$$I(T_0, A) = \sum_{j=1}^{\nu} (\hat{K}_j) \int f(T_0(w), \theta_0(w))\, d\phi_0 + \left(A - \sum_{j=1}^{\nu} K_j\right) \int f(T_0(w), \theta_0(w))\, d\phi_0$$

$$\leq \sum_{j=1}^{\nu} (P_j) \int f(T_0(w), \theta_0(w))\, d\phi_0 + (D) \int f(T_0(w), \theta_0(w))\, d\phi_0 + \varepsilon/3$$

$$\leq \sum_{j=1}^{\nu} (P_j) \int \left[b_j \cdot \theta_0(w) + \frac{\varepsilon}{6M}\right] d\phi_0 + \frac{\varepsilon}{6} + \frac{\varepsilon}{3} \tag{8}$$

$$= \sum_{j=1}^{\nu} (\hat{K}_j) \int [b_j \cdot \theta_0(w)]\, d\phi_0 + (D) \int [b_j \cdot \theta_0(w)]\, d\phi_0 + \frac{2\varepsilon}{3}$$

$$\leq \sum_{j=1}^{\nu} (b_j; V_0(\hat{K}_j)) + \frac{5\varepsilon}{6},$$

where $V_0(\hat{K}_j) = [V_{01}(\hat{K}_j), V_{02}(\hat{K}_j), V_{03}(\hat{K}_j)]$.

If $b_j \neq 0$, let λ_j be the normal plane to b_j, and let it be the plane $z = 0$ if $b_j = 0$. Let α_j be any linear orthogonal transformation of E_3 into itself:

$$\alpha_j(x, y, z) = (\xi, \eta, \zeta),$$

$$\xi = \alpha_{11} x + \alpha_{12} y + \alpha_{13} z,$$

$$\eta = \alpha_{21} x + \alpha_{22} y + \alpha_{23} z,$$

$$\zeta = \alpha_{31} x + \alpha_{32} y + \alpha_{33} z,$$

with $\zeta = 0$ the plane λ_j. Then $(\alpha_{31}, \alpha_{32}, \alpha_{33})$ is a positive multiple of b_j, and $\alpha_j(b_j) = (0, 0, \|b_j\|)$. Let $(T_0', K_j) = (\alpha_j T_0, K_j)$, and consider the plane mapping (T_{03}', K_j) of the K_j into the planes λ_j.

Use $\varepsilon/(6M\nu)$ in Lemma 3 with these mappings, and obtain constants δ_j. Let

$$\sigma = \min\left\{\frac{\varepsilon}{5M}, \rho, \frac{\rho_j}{2} : j = 1, \cdots, \nu\right\} .$$

Let (T, A) be any surface defined on A such that $\|T(w) - T_0(w)\| < \sigma$ for all $w \in A$. Let $(T', K_j) = (\alpha_j T, K_j)$, and let (T'_3, K_j) be the corresponding mappings of K_j into E_3 and λ_j, respectively. Then by Lemma 3, there exist constants η_j such that to every finite system S_j of $q \subset H_j$ with indices d, m, $\mu < \eta_j$, there corresponds a subsystem S'_j such that

$$|V(T'_{03}, K_j) - \sum_j{}' V(T'_3, q)| < \frac{\varepsilon}{6M\nu}, \tag{9}$$

where Σ'_j denotes the sum over all $q \in S'_j$.

Notice that $T(K_j)$ is contained in the $\rho_j/2$-neighborhood of the set $T_0(H_j)$, so that inequality (5) holds for all $p \in T(K_j)$. Let ϕ denote the measure corresponding to (T, A) and let $\theta(w)$ be the associated Radon-Nikodym derivative. Then, since $f(b, J) \geq 0$,

$$\begin{aligned} I(T, A) &\geq \sum_{j=1}^{\nu} \sum_j{}' I(T, q) \\ &= \sum_{j=1}^{\nu} \sum_j{}' (\hat{q}) \int f(T(w), \theta(w))\, d\phi \\ &\geq \sum_{j=1}^{\nu} \sum_j{}' (\hat{q}) \int (b_j, \theta(w))\, d\phi \\ &= \sum_{j=1}^{\nu} \left[b_j \cdot \sum_j{}' V(\hat{q})\right], \end{aligned}$$

where $V(\hat{q}) = (V_1(\hat{q}), V_2(\hat{q}), V_3(\hat{q}))$.

From (8),

$$I(T, A) - I(T_0, A) \geq \sum_{j=1}^{\nu} b_j \cdot \left(\sum_j{}' V(\hat{q}) - V_0(\hat{K}_j)\right) - \frac{5\varepsilon}{6}.$$

The inner products $b_j \cdot [\Sigma'_j V(\hat{q}) - V_0(\hat{K}_j)]$ can be evaluated in the (ξ, η, ζ) coordinate systems. Hence

$$\left|b_j \cdot \left(\sum_j{}' V(\hat{q}) - V_0(\hat{K}_j)\right)\right| = \|b_j\| \cdot \left\|\sum_j{}' V(T'_3, \hat{q}) - V(T'_{03}, K_j)\right\| < \frac{\varepsilon}{6\nu}$$

from (9). Therefore

$$I(T, A) - I(T_0, A) > -\frac{5\varepsilon}{6} - \frac{\varepsilon}{6} = -\varepsilon ,$$

and I(T, A) is lower semicontinuous at (T_0, A) as desired.

The statement of Theorem 1 can be simplified if slightly stronger hypotheses and the definition of a PSN integrand are used. Thus the following theorem is an immediate consequence of Theorem 1 and Lemma 2.

THEOREM 2. *Let* $f(p, J)$ *be a parametric integrand that is* PSN *for all* $p \in E_3$ *and* $f(p, J) \geq 0$ *for all* $(p, J) \in E^6$. *Let* (T_0, A) *be a* BV *surface, and let* $f(p, J)$ *for* $\|J\| = 1$ *be bounded in a neighborhood of* $T_0(A)$. *Then the integral* $I(T_0, A)$ *is lower semicontinuous at* (T_0, A).

Weaker lower semicontinuity theorems can be proved if $f(p, J)$ is only semiregular.

THEOREM 3. *Let* $f(p, J)$ *be a positive semiregular integrand. Let* (T_0, A) *be a* BV *surface such that for* $\|J\| = 1$, $f(p, J)$ *is bounded in some neighborhood of* $T_0(A)$. *Then* $I(T_0, A)$ *is lower semicontinuous in every class of surfaces with uniformly bounded area.*

Proof. Let $\mathscr{K}$ be a class of surfaces with areas less than L for some fixed constant L. Given $\varepsilon > 0$, let $f^*(p, J) = f(p, J) + \varepsilon \|J\|/(2L)$. Then f^* satisfies the hypotheses of Theorem 2. Thus there is a δ such that if (T, A) is a surface whose distance from (T_0, A) is less than δ,

$$I(T, A; f^*) - I(T_0, A; f^*) > -\varepsilon/2.$$

Thus

$$I(T, A; f) - I(T_0, A; f) = I(T, A; f^*) - I(T_0, A; f^*) - \varepsilon U(T, A)/(2L) + \varepsilon U(T_0, A)/(2L)$$

$$> -\varepsilon/2 - \varepsilon/2 = -\varepsilon, \quad \text{if } (T, A) \in \mathscr{K}.$$

REFERENCES

1. G. Bouligand, *Essai sur l'unitê des methodes directes,* Mém. Soc. Roy. Sci. Liège, (3) 19 (1943), 88 pp.

2. L. Cesari, *La nozione di integrale sopra una superficie in forma parametrica,* Ann. Scuola Norm. Super. Pisa (2) 13 (1944) 78-117 (1948).

3. ———, *Condizioni sufficienti per la semicontinuità degli integrali sopra una superficie in forma parametrica,* Ann. Scuola Norm. Super. Pisa (2) 14 (1945), 47-79 (1948).

4. ———, *L'area di Lebesgue come una misura,* Rend. Mat. e Appl. (5) 14 (1955) 655-673.

5. ———, *Surface Area,* Annals of Math. Studies, no. 35, Princeton University Press, Princeton, 1956.

6. L. Cesari and L. Turner, *Surface integral and Radon-Nikodym derivatives,* Rend. Circ. Mat. Palermo, (2)7 (1958), 143-154.

7. G. M. Ewing, *Surface integrals of the Weierstrass type,* Duke Math. J. 18 (1951), 275-286.

8. W. Fleming and L. C. Young, *A generalized notion of boundary,* Trans. Amer. Math. Soc. 76 (1954), 457-484.

9. E. J. McShane, *On the semicontinuity of double integrals of the calculus of variations,* Ann. of Math. (2) 33 (1932), 460-484.

10. E. J. McShane, *Integrals over surfaces in parametric form*, Ann. of Math. (2) 34 (1933), 815-838.

11. Ch. Y. Pauc, *Les méthodes directes en calcul des variations et en géométrie differentielle*, Thèse, Université de Paris, 1941. ix + 139 pp.

12. T. Radó, *On the semi-continuity of double integrals in parametric form*, Trans. Amer. Math. Soc. 51 (1942), 336-361.

13. P. V. Reichelderfer, *On the semi-continuity of double integrals*, Trans. Amer. Math. Soc. 67 (1949), 275-285.

14. L. Tonelli, *Fondamenti di calcolo delle variazioni*, vols. 1 and 2, N. Zanichelli, Bologna, 1921-23.

15. L. Turner, *An invariant property of Cesari's surface integral*, Proc. Amer. Math. Soc. 9 (1958), 920-925.

16. ———, *Measures induced on a σ-algebra by a surface*, Duke Math. J. 26 (1959), 501-509.

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