ON A THEOREM OF FISHER CONCERNING THE HOMEOMORPHISM GROUP OF A MANIFOLD

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An n-manifold M^n is a connected, separable metric space each point of which has an open neighborhood whose closure is homeomorphic to the n-cell I^n. An *internal cell* of M^n is a subset Q of M^n for which there exists a homeomorphism of Euclidean space E^n into M^n such that Q is the image of the unit n-cell of E^n. Alternatively, Q is a topological n-cell in the interior $\mathring{M}^n$ of M^n whose boundary $\dot{Q}$ is locally flat in M^n [1]. A homeomorphism h of M^n is *supported* on a set $K \subset M^n$ if $h(x) = x$ whenever $x \notin K$. Suppose that $H(M^n)$ denotes the group of all homeomorphisms of M^n onto M^n and $FH(M^n)$ denotes the subgroup generated by homeomorphisms supported on internal cells. Then according to Fisher [2] $FH(M^n)$ *is simple and is the intersection of all nontrivial normal subgroups of* $H(M^n)$.

Suppose $\varepsilon > 0$ and $FH_\varepsilon(M^n)$ denotes the subgroup of $FH(M^n)$ generated by homeomorphisms supported on internal cells of diameter less than ε. The purpose of this note is to prove that

$$FH(M^n) = \bigcap_{\varepsilon > 0} FH_\varepsilon(M^n),$$

that is, a homeomorphism h is in $FH(M^n)$ if and only if for each $\varepsilon > 0$, h is the composition of homeomorphisms supported on subsets of the interior of M^n of diameter less than ε. A similar theorem holds for the piecewise linear case.

The following lemma has a straightforward proof.

LEMMA 1. *Let* $I^n = I^{n-1} \times I^1$ *and suppose* X *is a compact subset of* I^n *such that* $X \cap \dot{I}^n \subset I^{n-1} \times 0$. *Then there is a piecewise linear homeomorphism* h *of* I^n *such that* $h|\dot{I}^n = 1$ *and* $h(X) \subset I^{n-1} \times [0, 1/2)$.

LEMMA 2. *Let* h *be a homeomorphism of* $I^n = I^{n-1} \times I^1$ *onto itself such that* $h|\dot{I}^n = 1$ *and* $h(I^{n-1} \times 1/2) \subset I^{n-1} \times [1/3, 2/3]$. *Then there exists a homeomorphism* h' *of* I^n *such that*

$$h'|(\dot{I}^n \cup I^{n-1} \times [0, 1/4] \cup I^{n-1} \times [3/4, 1]) = 1 \quad \textit{and} \quad h'|I^{n-1} \times 1/2 = h|I^{n-1} \times 1/2.$$

Proof. Let g be a piecewise linear homeomorphism of $I^{n-1} \times [1/4, 3/4]$ onto $I^{n-1} \times [0, 1]$ that is the identity on $I^{n-1} \times [1/2, 2/3]$. Let $h': I^n \to I^n$ be defined by

$$h'(x) = \begin{cases} x, & x \in I^{n-1} \times ([0, 1/4] \cup [3/4, 1]) \\ g^{-1}hg(x), & x \in I^{n-1} \times [1/4, 3/4] \end{cases}$$

Remark. If h is piecewise linear, so is h'.

LEMMA 3. *Let* $h: I^n \to I^n$ *be a homeomorphism such that* $h|\dot{I}^n = 1$. *Then* h *is the composition of five homeomorphisms, each the identity on* $\dot{I}^n$, *and each supported on one of the cells*

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$$I_1 = I^{n-1} \times [0, 1/2],\ I_2 = I^{n-1} \times [1/2, 1],\ I_3 = I^{n-1} \times [1/4, 3/4].$$

Proof. Apply Lemma 1 (with a suitable change of parameter) to get a piecewise linear homeomorphism g_1 of I_1 onto I_1 such that $g_1 \mid \dot{I}_1 = 1$ and

$$g_1(h(I^{n-1} \times 1/2) \cap I_1) \subset I^{n-1} \times (1/4, 1/2].$$

We may think of g_1 as a piecewise linear homeomorphism of I^n by requiring g_1 to be the identity on $I^n - I_1$. Similarly, there exists a piecewise linear homeomorphism g_2 of I^n such that g_2 is the identity on $\dot{I}^n$, g_2 is supported on I_2 and

$$g_2(h(I^{n-1} \times 1/2) \cap I_2) \subset I^{n-1} \times [1/2, 3/4).$$

Hence $g_2 g_1 h(I^{n-1} \times 1/2) \subset I^{n-1} \times (1/4, 3/4)$. Applying Lemma 2 (with a suitable change of parameters), we can get a homeomorphism g_3 of I^n such that g_3 is the identity on $\dot{I}^n$, g_3 is supported on I_3, and

$$g_3 \mid I^{n-1} \times 1/2 = g_2\, g_1\, h \mid I^{n-1} \times 1/2.$$

Then $g_3^{-1} g_2 g_1 h \mid I^{n-1} \times 1/2 = 1$. Let g_4, g_5 be defined by

$$g_4 = \begin{cases} (g_3^{-1} g_2 g_1 h)^{-1} & \text{on } I_1 \\ 1 & \text{on } I^n - I_1 \end{cases}$$

$$g_5 = \begin{cases} (g_3^{-1} g_2 g_1 h)^{-1} & \text{on } I_2 \\ 1 & \text{on } I^n - I_2. \end{cases}$$

Then $h = g_1^{-1} g_2^{-1} g_3 g_4^{-1} g_5^{-1}$.

Remark. If h is piecewise linear so are the g_i.

Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be real numbers with $a_i < b_i$. Then the set Q^n of all points $x = (x_1, \cdots, x_n) \in E^n$ such that $a_i \leq x_i \leq b_i$ $(i = 1, \cdots, n)$ will be called an *n-cube*. Let

$$\triangle(Q^n) \equiv \text{measure of } Q^n = \sum_{i=1}^{n} (b_i - a_i).$$

Suppose I^n is the unit n-cube of E^n, and suppose $H_\varepsilon(I^n)$ $(0 < \varepsilon \leq 1)$ is the subgroup of homeomorphisms of I^n generated by homeomorphisms that are the identity on $\dot{I}^n$ and that are supported on n-cubes of measure less than or equal to ε. (The boundaries of these cubes one allowed to intersect $\dot{I}^n$.) Let

$$H_0(I^n) = \bigcap_{\varepsilon > 0} H_\varepsilon(I^n).$$

Obviously, $\varepsilon > \delta > 0$ implies $H_\varepsilon(I^n) \supset H_\delta(I^n)$. On the other hand, suppose Q^n is an n-cube on I^n of measure $\triangle = \triangle(Q^n)$ and h is a homeomorphism of Q^n which is the identity on $\dot{Q}^n$. Let e be the length of a longest side of Q^n; that is, let e be the maximum value of the various $b_i - a_i$. Then $e \geq \triangle/n$ so

$$\Delta - \frac{e}{2} \leq \Delta - \frac{\Delta}{2n}.$$

Lemma 3 implies that h is the composition of homeomorphisms of Q^n which are supported on cubes of measure no greater than $\Delta - \Delta/2n$. Hence

$$H_1(I^n) = H_{1-\frac{1}{2n}}(I^n) = H_{(1-\frac{1}{2n})^2}(I^n) = \cdots;$$

that is, $H(I^n) = H_0(I^n)$. Thus we have proved the desired result.

THEOREM. *Let* h *be a homeomorphism of* I^n *that is the identity on* $\dot{I}^n$, *and let* $\varepsilon > 0$. *Then* h *is the composition of a finite sequence of homeomorphisms each the identity on* $\dot{I}^n$ *and each supported on an* n*-cube of measure less than* ε. *If* h *is piecewise linear then so are the composing homeomorphisms.*

COROLLARY 1. *Let* M^n *be a manifold, let* h *be a homeomorphism of* M^n *supported on an internal* n*-cell, and let* $\varepsilon > 0$. *Then* h *is the composition of a finite sequence of homeomorphisms of* M, *each supported on a closed subset of* $\mathring{M}$ *of diameter less than* ε.

COROLLARY 2. *Let* M^n *be a combinatorial manifold* ($n \neq 4$), *let* h *be a piecewise linear homeomorphism of* M^n *supported on a (topological) internal cell, and let* $\varepsilon > 0$. *Then* h *is the composition of a finite sequence of piecewise linear homeomorphisms of* M^n, *each supported on a closed subset of diameter less than* ε.

Proof. This corollary follows directly from the Theorem if h is supported on an internal combinatorial cell. Otherwise, we argue as follows. By hypothesis h is supported on the internal cell Q^n. Let U be a neighborhood of Q^n homeomorphic to E^n. Then U inherits a piecewise linear structure from M. By theorems of Stallings [4] and Moise [3], U is piecewise linearly equivalent to the ordinary combinatorial structure E^n. Hence U is the monotone union of combinatorial n-cells C_i^n (each of which are polyhedra in M). One of these, say $C_{i_0}^n$, must contain Q^n in its interior. Then h is supported on $C_{i_0}^n$.

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