THE SIGNS OF SOME CONSTANTS ASSOCIATED WITH THE RIEMANN ZETA-FUNCTION

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1. In a paper by Chowla and Briggs [1] the following proposition is proved:

THEOREM 1. *If*

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} a_n (s-1)^n, \tag{1}$$

then

$$a_n = \frac{(-1)^n \gamma_n}{n!}, \tag{2}$$

where

$$\gamma_n = \lim_{N\to\infty} \left[\sum_{k=1}^{N} \frac{\log^n k}{k} - \frac{\log^{n+1} N}{n+1} \right].$$

In connection with his investigations on the constants γ_n, using the functional equation for the zeta-function, Briggs [2] has established the result below.

THEOREM 2. *Infinitely many* γ_n *are positive, and infinitely many are negative.*

The purpose of this paper is to extend Theorem 2.

2. Let A be the set of all positive integers n such that $a_n \neq 0$: $A = \{n \mid a_n \neq 0\}$. Further define

$$A_1 = \{n \mid a_n \neq 0 \text{ and } (-1)^n = 1\}, \quad A_1^+ = \{n \mid a_n > 0 \text{ and } (-1)^n = 1\}, \text{ and}$$

$$A_1^- = \{n \mid a_n < 0 \text{ and } (-1)^n = 1\}.$$

By analogy, let

$$A_2 = \{n \mid a_n \neq 0 \text{ and } (-1)^n = -1\},$$

$$A_2^+ = \{n \mid a_n > 0 \text{ and } (-1)^n = -1\}, \quad A_2^- = \{n \mid a_n < 0 \text{ and } (-1)^n = -1\};$$

$$B = \{n \mid \gamma_n \neq 0\}, \quad B^+ = \{n \mid \gamma_n > 0\}, \quad B^- = \{n \mid \gamma_n < 0\}.$$

We denote the cardinal number of a set E by kE; $\aleph_0$ is the cardinal number of the set of all positive integers.

Received May 18, 1962.

3. Since

$$\zeta(s) - \frac{1}{s-1}$$

is an entire transcendental function, it is evident that $kA = \aleph_0$. We can write

$$\zeta(s) - \frac{1}{s-1} = \sum_{n\in A_1^-} a_n(s-1)^n + \sum_{n\in A_1^+} a_n(s-1)^n + \sum_{n\in A_2^+} a_n(s-1)^n$$

$$+ \sum_{n\in A_2^-} a_n(s-1)^n.$$

By setting $s = t + 1$ and then $s = -t + 1$ in the last identity and adding the results, we conclude that

(3) $$\zeta(t+1) + \zeta(-t+1) = 2\left(\sum_{n\in A_1^-} a_n t^n + \sum_{n\in A_1^+} a_n t^n\right);$$

and if we subtract the results, we obtain the formula

(4) $$\zeta(t+1) - \zeta(-t+1) - \frac{2}{t} = 2\left(\sum_{n\in A_2^+} a_n t^n + \sum_{n\in A_2^-} a_n t^n\right).$$

Observe that if $t = 2m + 1$, then the left-hand sides of both equation (3) and equation (4) approach 1 as $m \to +\infty$ through integral values. The right-hand sides of these equations, therefore, cannot be polynomials. Thus $kA_1 = kA_2 = \aleph_0$. Also, with this same substitution, if $kA_1^- < \aleph_0$, then the right-side of equation (3) approaches $+\infty$ as $m \to +\infty$; if $kA_1^+ < \aleph_0$, then the right-side of equation (3) approaches $-\infty$ as $m \to +\infty$. In either case there is a contradiction. Similarly, by using equation (4), assuming that $kA_2^+ < \aleph_0$ or that $kA_2^- < \aleph_0$ leads to a contradiction.

We thus have the following result (announced without proof in [3]).

THEOREM 3. *For the coefficients in the expansion* (1), *each of the inequalities*

$$a_{2n} > 0, \; a_{2n} < 0, \; a_{2n-1} < 0, \; a_{2n-1} > 0$$

holds for infinitely many n.

COROLLARY. *Infinitely many* a_n *are positive and infinitely many are negative.*

By combining Theorem 3 and equation (2) we obtain an extension of Theorem 2.

THEOREM 4. *Each of the inequalities*

$$\gamma_{2n} < 0, \; \gamma_{2n} > 0, \; \gamma_{2n-1} < 0, \; \gamma_{2n-1} > 0$$

holds for infinitely many n.

Consequently, Theorem 2 is a corollary of Theorem 4.

Remark. Theorem 2 can also be obtained directly by considering the equality

$$\zeta(-t) + \frac{1}{t+1} = \sum_{n \in B^-} \frac{\gamma_n}{n!}(t+1)^n + \sum_{n \in B^+} \frac{\gamma_n}{n!}(t+1)^n .$$

Under each of the assumptions $kB^- < \aleph_0$, $kB^+ < \aleph_0$, a contradiction is reached by taking the limit on both sides as $t = 2m \to \infty$ $(m = 1, 2, \cdots)$.

REFERENCES

1. W. E. Briggs and S. Chowla, *The power series coefficients of* $\zeta(s)$, Amer. Math. Monthly, 62 (1955), 323-325.

2. W. E. Briggs, *Some constants associated with the Riemann zeta-function,* Michigan Math. J. 3, (1955-56), 117-121.

3. D. Mitrović, *Sur la fonction* ζ *de Riemann,* C. R. Acad. Sci. Paris, 245 (1957), 885-886.

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