AN ANALYTICAL APPROACH TO THE DIFFERENTIAL EQUATIONS OF THE BIRTH-AND-DEATH PROCESS

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1. INTRODUCTION

This paper presents a purely analytical approach to the problems of uniqueness and existence of solutions (separate or simultaneous) for the forward and the backward differential equations of the so-called birth-and-death process. The conditions imposed on the solutions $\{p_{mn}(t)\}$ are of the types

$$p_{mn}(0) = \delta_{mn} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases} \tag{1.1}$$

$$p_{mn}(t) \geq 0, \tag{1.2}$$

$$\sum_n p_{mn}(t) \leq 1; \tag{1.3}$$

no attention is paid to the so-called semigroup property (compare [6] and [9]). The variable t is usually restricted to a fixed finite interval; this is definitely more general than the case $0 \leq t < \infty$ which is considered by most authors, mainly in order to permit use of the Laplace transform of $p_{mn}(t)$.

In our method of proof, neither uniqueness nor existence offers any great difficulty. The major effort is spent in obtaining, for many cases of interest, an explicit representation of all the possible solutions. In particular (see Theorem 9.3) such a representation is obtained for the case where the required solution is to satisfy both the forward and the backward differential equations, in addition to the conditions (1.1) to (1.3) above. An explicit representation for a certain subclass of such solutions in $0 \leq t < \infty$ was already obtained in the interesting paper [7] by Karlin and McGregor.

The conditions involved are highly redundant. In view of this, the problems on hand can be approached from many different directions. Our approach is probably closest to that of Arley and Borchsenius [1] and that of Reuter and Ledermann [10]. An approach involving the analytical theory of continued fractions was announced by Koopman [8]. A great variety of other methods, often for more general situations, can be found in the papers of Feller [2] to [6].

2. STATEMENT OF THE PROBLEM

In this paper, λ_n and μ_n $(n = 0, 1, 2, \cdots)$ denote given nonnegative real numbers. By T we denote a fixed positive number (allowing occasionally that $T = +\infty$).

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We shall study the uniqueness and existence of a system $\{p_{mn}(t)\}$ $(m, n = 0, 1, \cdots)$ of infinitely differentiable functions defined on the interval $0 \le t \le T$, satisfying the initial conditions

$$p_{mn}(0) = \delta_{mn} \tag{2.1}$$

and satisfying, also for $0 \le t \le T$, one or more of the following conditions [each for a specified (possibly empty) set of pairs (m, n) of nonnegative integers]:

First, the so-called forward differential equations

$$(\lambda_n + \mu_n + D)p_{mn} = \lambda_{n-1} p_{m,n-1} + \mu_{n+1} p_{m,n+1}\,. \tag*{(I)$_{mn}$}$$

Here, $D = \frac{d}{dt}$. The case $n = 0$ is to be interpreted as

$$(\lambda_0 + \mu_0 + D)p_{m0} = \mu_1 p_{m1}\,. \tag*{(I)$_{m0}$}$$

Second, the so-called backward differential equations

$$(\lambda_m + \mu_m + D)p_{mn} = \lambda_m p_{m+1,n} + \mu_m p_{m-1,n}\,. \tag*{(II)$_{mn}$}$$

The case $m = 0$ is to be interpreted as

$$(\lambda_0 + \mu_0 + D)p_{0n} = \lambda_0 p_{1n}\,. \tag*{(II)$_{0n}$}$$

Finally, the conditions

$$p_{mn}(t) \ge 0\,, \tag{2.2}$$

$$|p_{mn}(t)| \le 1\,, \tag{2.3}$$

and

$$\left|\sum_{j=0}^{n} p_{mj}(t)\right| \le 1\,. \tag{2.4}$$

As was first shown by Feller [3] (see also Section 6), there always exists a well-defined system $\{p_{mn}(t)\}$ satisfying each of the above conditions for all $0 \le t < \infty$ and all m, n. This system will be denoted by $\{\phi_{mn}(t)\}$. It moreover has the so-called semigroup property

$$\phi_{mn}(t + t') = \sum_{j=0}^{\infty} \phi_{mj}(t)\,\phi_{jn}(t')\,.$$

Assuming that $\lambda_n > 0$ for $n \ge 0$ and that $\mu_n > 0$ for $n \ge 1$, and using known results from the theory of moments and orthogonal polynomials, Karlin and McGregor [7, p. 529] proved that $\{\phi_{mn}(t)\}$ is the only system satisfying all of the above conditions $(I)_{mn}$, $(II)_{mn}$, (2.2), (2.3) and (2.4) for all $0 \le t < \infty$ if and only if

$$\sum_{n=1}^{\infty}\left(\frac{\lambda_0\cdots\lambda_{n-1}}{\mu_1\cdots\mu_{n-1}}\frac{1}{\mu_n}+\frac{\mu_1\cdots\mu_{n-1}}{\lambda_0\cdots\lambda_{n-1}}\frac{\mu_n}{\lambda_n}\right)=\infty. \tag{2.5}$$

In Sections 7 to 9, using totally different methods, we shall obtain a generalization of this result. We shall also study the explicit form of all solutions in case (2.5) does not hold.

The remaining part of this section consists of some more or less obvious remarks indicating the pattern of reasoning to be followed in subsequent sections. In the following, the argument t is restricted to the interval $0 \le t \le T$ with $T > 0$ and fixed.

Let $n \ge 0$ be fixed, and consider the question whether or not the conditions $(2.1)_{mn}$ and $(II)_{mn}$ $(m = 0, 1, \cdots)$ together determine the $p_{mn}(t)$ $(m = 0, 1, \cdots)$ uniquely, in other words, the question whether or not

$$(\lambda_m + \mu_m + D)u_m = \lambda_m u_{m+1} + \mu_m u_{m-1} \qquad (m \ge 0,\ u_{-1} \equiv 0), \tag{2.6}$$

together with

$$u_m(0) = 0 \qquad (m \ge 0),$$

implies that $u_m(t) = 0$ $(m = 0, 1, \cdots;\ 0 \le t \le T)$.

The answer is clearly affirmative if $\lambda_m = 0$ for infinitely many m. For if $\lambda_{M-1} = 0$, then (2.6) for $0 \le m < M$ is a system of the form $Du = Au$, with $u = (u_0, \cdots, u_{M-1})$ and with A as an $M \times M$ matrix of constant coefficients. Thus it suffices to consider the case where $\lambda_m > 0$ for $m \ge p$. Let $p \ge 0$ be minimal; that is, either $p = 0$ or else $p \ge 1$ and $\lambda_{p-1} = 0$. In particular, $u_m(t) = 0$ for $0 \le m \le p - 1$. Therefore, if we let $v_m = u_{m+p}$ $(m = 0, 1, \cdots)$ and

$$\beta_m = \mu_{m+p}/\lambda_{m+p} \ge 0, \qquad \gamma_m = 1/\lambda_{m+p} > 0 \qquad (m = 0, 1, \cdots), \tag{2.7}$$

the question above leads to the following problem, which will be studied in Section 4.

The basic uniqueness problem. Let $\beta_n \ge 0$ and $\gamma_n > 0$ $(n = 0, 1, \cdots)$ be constants. Let the system $\{v_n(t)\}$ $(n = 0, 1, \cdots;\ 0 \le t \le T)$ satisfy

$$(1 + \beta_n + \gamma_n D)v_n = v_{n+1} + \beta_n v_{n-1} \qquad (n \ge 0,\ v_{-1} \equiv 0),$$

$$v_n(0) = 0 \qquad (n \ge 0)$$

and some additional condition (say, $|v_n(t)| \le 1$). Determine whether this implies that $v_n(t) = 0$ for all $0 \le t \le T$ $(n = 0, 1, \cdots)$. If not, find all possible solutions $\{v_n(t)\}$.

Next, let m be a *fixed* integer $(m \ge 0)$, and consider the question whether or not the two conditions $(2.1)_{mn}$ and $(I)_{mn}$ $(n = 0, 1, \cdots)$ determine $p_{mn}(t)$ $(n = 0, 1, \cdots)$ uniquely. Put

$$q_{mn}(t) = \mu_0 \int_0^t p_{m0}(\tau)\,d\tau + \sum_{j=0}^{n-1} p_{mj}(t), \tag{2.8}$$

in particular,

$$q_{m0}(t) = \mu_0 \int_0^t p_{m0}(\tau)\, d\tau .$$

Then the condition $(2.1)_{mn}$ $(n \geq 0)$ is equivalent to

$$\begin{aligned} q_{mn}(0) &= 0 \quad \text{if } m \leq n, \\ &= 1 \quad \text{if } m > n. \end{aligned}$$

Further, $(I)_{mn}$ $(n = 0, 1, \cdots)$ and (2.8) imply that, with $q_n = q_{mn}(t)$ and $p_n = p_{mn}(t)$,

$$\begin{aligned} Dq_n &= \mu_0\, p_0 + \sum_{j=0}^{n-1} (\lambda_{j-1} p_{j-1} + \mu_{j+1} p_{j+1} - \lambda_j p_j - \mu_j p_j) \\ &= \mu_n p_n - \lambda_{n-1} p_{n-1} = \mu_n (q_{n+1} - q_n) - \lambda_{n-1}(q_n - q_{n-1}) \end{aligned}$$

when $n = 0, 1, \cdots$ $(\lambda_{-1} = 0)$. Consequently, the $q_{mn}(t)$ satisfy

(2.9) $$(\lambda_{n-1} + \mu_n + D) q_{mn} = \lambda_{n-1} q_{m,n-1} + \mu_n q_{m,n+1} \qquad (n \geq 0,\ q_{m,-1} \equiv 0);$$

conversely, (2.9) and (2.8) imply $(I)_{mn}$ for $n \geq 0$. In particular, as to the uniqueness problem for the forward differential equations $(I)_{mn}$, it suffices to consider the system

$$(\lambda_{n-1} + \mu_n + D) u_n = \lambda_{n-1} u_{n-1} + \mu_n u_{n+1} \qquad (n \geq 0,\ u_{-1} \equiv 0),$$

with the initial conditions

$$u_n(0) = 0 \qquad (n \geq 0).$$

If $\mu_n = 0$ for infinitely many n, then $u_n(t) \equiv 0$ for all $n \geq 0$. Thus, it suffices to consider the case where $\mu_n > 0$ for $n \geq q$ with $q \geq 0$ minimal. In particular, $u_n(t) \equiv 0$ if $0 \leq n \leq q - 1$. Consequently, letting $v_m(t) = u_{m+q}(t)$ $(m = 0, 1, \cdots)$ and

(2.10) $$\beta_n = \lambda_{n-1+q}/\mu_{n+q} \geq 0, \qquad \gamma_n = 1/\mu_{n+q} > 0, \qquad (n = 0, 1, \cdots),$$

we again have the basic uniqueness problem above (note that $\lambda_{-1} = 0$).

In dealing with the problem whether or not, under certain side conditions, (2.1) combined with both $(I)_{mn}$ and $(II)_{mn}$ (all m, n) determines the system $\{p_{mn}(t)\}$ uniquely, one may clearly assume that $\lambda_n > 0$ for $n \geq p$ and $\mu_n > 0$ for $n \geq q$. In order to avoid certain minor complications, we shall assume in this case that $p = 0$ and that $q = 0$ or $q = 1$; thus

(2.11) $$\lambda_n > 0 \quad \text{for } n \geq 0, \qquad \mu_n > 0 \quad \text{for } n \geq 1.$$

Suppose that (2.11) holds. Let the polynomial $Q_n(s)$ of degree n be defined by

$$Q_{-1} = 0, \qquad Q_0 = 1$$

and

$$(\lambda_n + \mu_n + s)Q_n = \lambda_n Q_{n+1} + \mu_n Q_{n-1} \qquad (n \geq 0).$$

Then the system

$$(2.12) \qquad (\lambda_n + \mu_n + D)u_n = \lambda_n u_{n+1} + \mu_n u_{n-1} \qquad (0 \leq n < N,\ u_{-1} \equiv 0)$$

is equivalent to the system

$$u_n(t) = Q_n(D)\, u_0(t) \qquad (0 \leq n \leq N).$$

Further, if

$$(2.13) \qquad (\lambda_n + \mu_n + D)v_n = \lambda_{n-1} v_{n-1} + \mu_{n+1} v_{n+1} \qquad (0 \leq n < N,\ v_{-1} \equiv 0),$$

then

$$(2.14) \qquad u_n(t) = \frac{\mu_1 \cdots \mu_n}{\lambda_0 \cdots \lambda_{n-1}} v_n(t) \qquad (n \geq 0,\ u_0 = v_0),$$

satisfies (2.12). Hence, the system (2.13) is equivalent to the system

$$v_n(t) = \pi_n Q_n(D)\, v_0(t) \qquad (0 \leq n \leq N),$$

where

$$\pi_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}, \qquad \pi_0 = 1;$$

(here, and throughout the paper, we adopt the standard convention that an empty product shall have the value 1; thus, if $a_{jk} = b_{j+1} \cdots b_k$ for $j < k$, then $a_{kk} = 1$).

In particular, $(\mathrm{I})_{mn}$ for $n = 0, 1, \cdots, N - 1$ and m fixed is equivalent to

$$(2.15) \qquad p_{mn}(t) = \pi_n Q_n(D)\, p_{m0}(t) \qquad (0 \leq n \leq N).$$

Similarly, $(\mathrm{II})_{mn}$ for $m = 0, 1, \cdots, M - 1$ and n fixed is equivalent to

$$(2.16) \qquad p_{mn}(t) = Q_m(D)\, p_{0n}(t) \qquad (0 \leq m \leq M).$$

Thus, the combination of $(\mathrm{I})_{mn}$ and $(\mathrm{II})_{mn}$ for all m and n is equivalent to

$$(2.17) \qquad p_{mn}(t) = \pi_n Q_m(D)\, Q_n(D)\, p_{00}(t) \qquad (m, n \geq 0).$$

This in turn implies the well-known symmetry relation

$$\frac{1}{\pi_n} p_{mn}(t) = \frac{1}{\pi_m} p_{nm}(t).$$

If one adds to (2.17) the condition (2.1), then all derivatives of $p_{00}(t)$ at $t = 0$ are prescribed (in a noncontradictory manner, for $\phi_{00}(t)$ will do). Actually, if (2.17) is given, it would suffice to require (2.1) only for $n = 0$, say.

One also concludes that the uniqueness problem for (2.1), $(\mathrm{I})_{mn}$, and $(\mathrm{II})_{mn}$ combined can be formulated as follows. Let $u_{00}(t)$ be an infinitely differentiable function on $0 \leq t \leq T$, having at $t = 0$ all its derivatives equal to zero. Suppose further that

$$u_{mn}(t) = \pi_n Q_m(D) Q_n(D) u_{00}(t)$$

satisfies certain conditions (say $\Sigma_n |u_{mn}| \leq 2$ for each m). Determine whether this implies $u_{00}(t) = 0$ $(0 \leq t \leq T)$. If not, find all solutions $u_{00}(t)$. In case one only allows functions on $0 \leq t < \infty$ having a representation

$$p_{00}(t) = \int e^{-xt} \nu(dx) \qquad (0 \leq t < \infty),$$

with ν as a finite and regular Borel measure, one is led to the problems considered by Karlin and McGregor [7].

We mentioned that (2.17) implies $(I)_{mn}$ and $(II)_{mn}$ for all m and n. Conversely, (2.17) is implied by each of the following, (compare (2.15) and (2.16)).

(i) $(I)_{mn}$ for all m, n and $(II)_{m0}$ for all m.

(ii) $(II)_{mn}$ for all m, n and $(I)_{0n}$ for all n.

(iii) $\begin{cases} (I)_{mn} \text{ for } m \geq n + 2 \text{ and for } m = 0, \\ (II)_{mn} \text{ for } m \leq n - 1 \text{ and for } n = 0. \end{cases}$

This shows that the simultaneous conditions $(I)_{mn}$ and $(II)_{mn}$ (all m, n) are highly redundant. Given these conditions, each of (2.1) to (2.4) is also quite redundant. For (2.2) and (2.3) this can be seen from the following result.

Let $n \geq 1$ be fixed, and assume only $(II)_{mn}$ for $0 \leq m \leq n - 1$ and $p_{mn}(0) = 0$ for $0 \leq m \leq n - 1$. Then the condition $p_{nn}(t) \geq 0$ $(0 \leq t \leq T)$ implies that $p_{mn}(t) \geq 0$, $(0 \leq t \leq T)$, for all $0 \leq m \leq n$. Further, $|p_{nn}(t)| \leq 1$, $(0 \leq t \leq T)$ implies that $|p_{mn}(t)| \leq 1$ $(0 \leq t \leq T)$ for all $0 \leq m \leq n$.

We shall omit the proof, since it would use formula (3.31) (with $q = 0$, $L_n = Q_n$).

3. AUXILIARY RESULTS

In the next few sections, T and β_n, γ_n $(n = 0, 1, 2, \cdots)$ denote fixed real numbers such that $T > 0$ and

$$\beta_n \geq 0, \quad \gamma_n > 0 \qquad (n = 0, 1, 2, \cdots). \tag{3.1}$$

Consider the infinite system of differential equations

$$(1 + \beta_n + \gamma_n D) u_n = u_{n+1} + \beta_n u_{n-1} \qquad (n = 0, 1, \cdots). \tag{3.2}$$

Equation (3.2) with $n = 0$ is to be interpreted as

$$(1 + \beta_0 + \gamma_0 D) u_0 = u_1 .$$

A system $\{u_n(t)\}$ $(n = 0, 1, \cdots)$ of real- or complex-valued functions defined for $0 \leq t \leq T$ is called a solution of (3.2) in $(0, T)$ if (i) each $u_n(t)$ is continuous for

$0 \leq t \leq T$ and differentiable for $0 < t < T$, and (ii) equations (3.2) hold for $0 < t < T$. Then, in fact, each $u_n(t)$ is infinitely differentiable for $0 \leq t \leq T$, and (3.2) also holds at $t = 0$ and at $t = T$.

We want to study the uniqueness and existence of solutions $\{u_n(t)\}$ of (3.2) in $(0, T)$ with initial values

$$u_n(0) = \delta_{nj}, \tag{3.3}$$

($j \geq 0$ a fixed integer), and such that one or more further conditions (such as uniform boundedness) are satisfied.

Let $L_n(s)$ $(n = 0, 1, 2, \cdots)$ denote the polynomial of degree n defined by $L_0 = 1$ and

$$(1 + \beta_n + \gamma_n s)L_n = L_{n+1} + \beta_n L_{n-1} \qquad (n = 0, 1, \cdots). \tag{3.4}$$

For $n = 0$, (3.4) is to be interpreted as $(1 + \beta_0 + \gamma_0 s)L_0 = L_1$; in view of this, it is convenient to define $L_{-1} = 0$. Observe that (3.2) is equivalent to

$$L_n(D)\, u_0(t) = u_n(t) \qquad (n = 0, 1, \cdots). \tag{3.5}$$

Let us first study the polynomials $L_n(s)$. By (3.4),

$$s\gamma_j L_j = (L_{j+1} - L_j) - \beta_j(L_j - L_{j-1}) \qquad (j = 0, 1, \cdots, k).$$

Multiplying by $\beta_{j+1} \cdots \beta_k$ (= 1 if $j = k$) and adding, one obtains

$$L_{k+1} - L_k = \beta_0\beta_1 \cdots \beta_k + s \sum_{j=0}^{k} (\beta_{j+1} \cdots \beta_k)\gamma_j L_j. \tag{3.6}$$

This in turn implies

$$L_n(s) = 1 + \sum_{k=0}^{n-1} \left(\beta_0 \cdots \beta_k + s \sum_{j=0}^{k} (\beta_{j+1} \cdots \beta_k)\gamma_j L_j(s) \right). \tag{3.7}$$

Let

$$L_n(s) = \sum_{r=0}^{\infty} \lambda_{nr} s^r = \sum_{r=0}^{n} \lambda_{nr} s^r \qquad (\lambda_{nr} = 0 \text{ if } r > n). \tag{3.8}$$

Then, by (3.7),

$$\lambda_{n0} = 1 + \sum_{k=0}^{n-1} \beta_0 \cdots \beta_k \geq 1 \tag{3.9}$$

and

$$\lambda_{nr} = \sum_{k=0}^{n-1} \sum_{j=0}^{k} (\beta_{j+1} \cdots \beta_k)\gamma_j \lambda_{j,r-1} = \sum_{j=0}^{n-1} \gamma_j \lambda_{j,r-1} \sum_{k=j}^{n-1} \beta_{j+1} \cdots \beta_k \tag{3.10}$$

if $r \geq 1$. By induction, $\lambda_{nr} \geq 0$. Taking only the term with $j = n - 1$ and $k = j$, we see by induction that $\lambda_{nr} > 0$ for $n \geq r$. Hence, the first inner sum is at least $\gamma_k \lambda_{k,r-1}$ (and is therefore positive) if $k \geq r - 1$; thus, for $r \geq 1$ fixed, λ_{nr} is strictly increasing for $n \geq r$, with $\lambda_{nr} = 0$ for $n < r$. In particular, by (3.8), $L_n(s)$ is *strictly increasing* in both n and s (except that $L_0(s) \equiv 1$), for s real and positive; (this much is also a direct consequence of (3.7)).

LEMMA 3.1. *If the series*

$$\sum_{j=0}^{\infty} \gamma_j \sum_{k=j}^{\infty} \beta_{j+1} \cdots \beta_k \tag{3.11}$$

diverges, then

$$\lim_{n\to\infty} L_n(s) = \infty \qquad \text{for each } s > 0. \tag{3.12}$$

On the other hand, if the series (3.11) *converges, then the limit*

$$\lim_{n\to\infty} L_n(s) = L_\infty(s)$$

exists for each complex number s, *in fact uniformly in each bounded set.*

It follows from (3.18) below that (3.12) in turn implies that $\{L_n(s)\}$ is unbounded for each $s = \sigma + i\tau$ with $\sigma > 0$. A certain special case of Lemma 3.1 was already used by Karlin and McGregor [7, p. 504]. Their proof does carry over, but in any case the following demonstration seems simpler.

Proof. Let M denote the value of (3.11). If $M = \infty$, then (3.10) with $r = 1$ and $\lambda_{j0} \geq 1$ implies that $\lambda_{n1} \to \infty$ as $n \to \infty$; but $L_n(s) \geq \lambda_{n1} s$ for $s > 0$.

Conversely, suppose that $M < \infty$. Let $\varepsilon_j = \gamma_j \sum_{k=j}^{\infty} \beta_{j+1} \cdots \beta_k$; then $\sum_{j=0}^{\infty} \varepsilon_j = M$. By (3.10),

$$\lambda_{nr} \leq \sum_{j<n} \varepsilon_j \lambda_{j,r-1} \qquad \text{if } r \geq 1.$$

Hence, λ_{nr} is not larger than λ_{n0} times the sum of $\varepsilon_{j_1} \varepsilon_{j_2} \cdots \varepsilon_{j_r}$ over the sets of nonnegative integers $(j_1, \cdots, j_r)$ satisfying $j_1 < j_2 < \cdots < j_r < n$. Consequently,

$$\lambda_{nr} \leq \lambda_{n0} \Big(\sum_{j<n} \varepsilon_j\Big)^r / r! \leq \lambda_{n0} M^r / r!.$$

Finally, by (3.9), with $j = 0$ in (3.11)),

$$\lambda_{n0} \leq 1 + (\beta_0/\gamma_0) M.$$

In view of (3.8), this yields the second assertion. As a by-product we find the inequality

$$|L_\infty(s)| \leq [1 + (\beta_0/\gamma_0)M] e^{M|s|}.$$

LEMMA 3.2. *Let* $n \geq 1$ *be a fixed integer. Let* m *denote the largest integer with* $0 \leq m \leq n - 1$ *and* $\beta_m = 0$, *and let* $m = 0$ *if no such integer exists. Then the polynomials* $L_{n-1}(s)$ *and* $L_n(s)$ *have* $L_m(s)$ *as a common divisor of maximal degree. Let*

$$(3.13) \qquad L_{n-1}(s) = L_m(s)\Lambda^*_{n-1}(s) \quad \text{and} \quad L_n(s) = L_m(s)\Lambda_n(s).$$

Then the $n - m$ *zeros of* $\Lambda_n(s)$ *are distinct negative real numbers. The same is true for the* $n - m - 1$ *zeros of* $\Lambda^*_{n-1}(s)$. *Finally, if* $m < n - 1$, *then the zeros of* $\Lambda_n(s)$ *and* $\Lambda^*_{n-1}(s)$ *separate each other.*

Proof. By (3.4), if $\beta_{n-1} = 0$ or $n = 1$, the assertion holds trivially with $m = n - 1, \Lambda^*_{n-1} = 1$, and $\Lambda_n = 1 + \beta_{n-1} + \gamma_{n-1}s$.

Let $n \geq 1$ be fixed and suppose that the assertions above hold. In proving the assertions for n replaced by $n + 1$, we may assume that $\beta_n > 0$. Let

$$(3.14) \qquad \Lambda_{n+1}(s) = -\beta_n\Lambda^*_{n-1}(s) + (1 + \beta_n + \gamma_n s)\Lambda_n(s)$$

and $\Lambda^*_n(s) = \Lambda_n(s)$; then it follows from (3.4) that (3.13) holds with n replaced by $n + 1$. If we apply (3.14) with s as a zero of $\Lambda_n(s)$, our induction assumption, together with $\Lambda^*_{n-1}(0) > 0$ and $\Lambda_{n+1}(0) > 0$, easily implies that the $n + 1 - m$ zeros of $\Lambda_{n+1}(s)$ are distinct negative real numbers separated by the $n - m$ zeros of $\Lambda_n(s) = \Lambda^*_n(s)$.

LEMMA 3.3. *Let* m, n *be integers* $(m < n)$. *Then there exists a (unique) infinitely differentiable bounded function* $k_{mn}(t)$ *on* $t \geq 0$ *such that*

$$(3.15) \qquad k_{mn}(t) > 0 \quad \text{if } t > 0, \qquad \lim_{t \to \infty} k_{mn}(t) = 0,$$

and

$$(3.16) \qquad \int_0^\infty e^{-st} k_{mn}(t)\,dt = \frac{L_m(s)}{L_n(s)} \quad \text{if } \Re s \geq 0.$$

It follows that

$$\int_0^\infty k_{mn}(t)\,dt = \frac{\lambda_{m0}}{\lambda_{n0}} \leq 1,$$

and for σ *and* y *real* $(\sigma \geq 0)$,

$$(3.17) \qquad \left|\frac{L_m(\sigma + iy)}{L_n(\sigma + iy)}\right| \leq \frac{L_m(\sigma)}{L_n(\sigma)} \leq 1 \qquad (m < n);$$

in particular (with $m = 0$),

$$(3.18) \qquad |L_n(\sigma + iy)| \geq L_n(\sigma) \geq 1.$$

More precisely, let

(3.19) $$\Delta_n(s) = \frac{L_{n-1}(s)}{L_n(s)} = \int_0^\infty e^{-st} d_n(t)\, dt \qquad (n \geq 1).$$

Then $|\Delta_n(\sigma + iy)|$ *is strictly decreasing in* $|y|$ $(\sigma \geq 0)$. *Further,*

$$d_n(t) > 0, \qquad \int_0^\infty d_n(t)\, dt = \Delta_n(0) \leq 1.$$

Finally, for $0 \leq m < n$,

(3.20) $$k_{mn}(t) = d_{m+1}(t) * d_{m+2}(t) * \cdots * d_n(t),$$

where a star denotes convolution.

Proof. In view of the relation

$$L_m(s)/L_n(s) = \Delta_{m+1}(s)\Delta_{m+2}(s)\cdots\Delta_n(s) \qquad (0 \leq m < n),$$

it suffices to prove that $d_n(t) > 0$, together with the fact that $|\Delta_n(\sigma + iy)|$ is decreasing in $|y|$.

Let $n \geq 1$ be *fixed,* and let $m < n$ satisfy the condition in Lemma 3.2. By Lemma 3.2,

(3.21) $$\Delta_n(s) = \frac{\Lambda^*_{n-1}(s)}{\Lambda_n(s)} = c\,\frac{\prod_1^{p-1}(s + \eta_\nu)}{\prod_1^{p}(s + \xi_\nu)}.$$

Here, c is a positive constant, and $p = n - m \geq 1$. Further, $-\xi_1 > \cdots > -\xi_p$ and $-\eta_1 > \cdots > -\eta_{p-1}$ denote the distinct real and negative zeros of $\Lambda_n(s)$ and $\Lambda^*_{n-1}(s)$, respectively. Moreover,

(3.22) $$0 < \xi_\nu < \eta_\nu < \xi_{\nu+1} \qquad (\nu = 1, \cdots, p-1).$$

By (3.19) and (3.21),

$$d_n(t) = \sum_{\nu=1}^{p} c_{n\nu}\, e^{-t\xi_\nu},$$

where

$$c_{n\nu} = \Lambda^*_{n-1}(\xi_\nu)/\Lambda'_n(\xi_\nu) > 0,$$

by (3.22) and $\Delta_n(0) > 0$; this proves that $d_n(t) > 0$.

Moreover, one easily sees from (3.21) that

$$\frac{1}{y}\frac{\partial}{\partial y}\log|\Delta_n(\sigma + iy)| = -|s + \xi_p|^{-2} - \sum_{\nu=1}^{p-1}\{|s + \xi_\nu|^{-2} - |s + \eta_\nu|^{-2}\},$$

where $s = \sigma + iy$. By (3.22), the latter right-hand side is strictly negative when $\sigma \geq 0$, and this proves that $|\Delta_n(\sigma + iy)|$ is strictly decreasing in $|y|$.

The following lemma is essentially known. Lemma 3.5 is an easy consequence of it.

LEMMA 3.4. *Let* μ *be a finite regular Borel measure carried by* $[0, T]$ $(T > 0)$. *Further, let* $p \geq 0$ *be a fixed integer, and let* $k(t)$ *be a function on* $0 \leq t \leq T$ *having there a continuous* p-*th derivative. Suppose that*

$$k^{(\nu)}(0) = 0 \qquad \text{for } 0 \leq \nu < p$$

if no point $t_0 \in [0, T]$ *carries a nonzero* μ-*mass; otherwise, suppose that*

$$k^{(\nu)}(0) = 0 \qquad \text{for } 0 \leq \nu \leq p.$$

Then the function

$$g(t) = \int_{0 \leq \tau < t} k(t - \tau)\, \mu(d\tau) \qquad (0 \leq t \leq T)$$

(defined by a Lebesgue-Stieltjes integral) has a continuous p-*th derivative. Moreover, for* $0 \leq t \leq T$,

$$g^{(\nu)}(t) = \int_{0 \leq \tau < t} k^{(\nu)}(t - \tau)\, \mu(d\tau) \qquad (\nu = 0, \cdots, p);$$

thus,

$$g^{(\nu)}(0) = 0 \qquad \text{for } 0 \leq \nu \leq p.$$

Proof. It suffices to consider the case $p = 1$. We can write

$$(g(t_0 + h) - g(t_0))/h = J_1 + J_2,$$

where

$$J_1 = \int_0^{t_0} k'(t_0 - \tau + \theta h)\, \mu(d\tau) \to \int_0^{t_0} k'(t_0 - \tau)\, \mu(d\tau)$$

by the bounded convergence theorem. Further,

$$J_2 = \frac{1}{h} \int_{t_0}^{t_0+h} k(t_0 + h - \tau)\, \mu(d\tau) \to 0$$

if either $k(t) = O(t)$ (that is, $k(0) = 0$) and t_0 is a point of continuity for μ, or $k(t) = o(t)$ (that is $k(0) = 0$).

LEMMA 3.5. *Let* $k(t)$ *be a function on* $0 \leq t \leq T$ *having there a continuous* p-*th derivative and satisfying*

$$k^{(\nu)}(0) = 0 \qquad \text{for } 0 \leq \nu < p.$$

Let further f(t) *be a function on* $0 \le t \le T$ *having there a continuous* q*-th derivative and satisfying*

$$f^{(\nu)}(0) = 0 \qquad \text{for } 0 \le \nu < q.$$

Then

$$g(t) = \int_0^t k(t - \tau)\, f(\tau)\, d\tau \qquad (0 \le t \le T)$$

has a continuous $(p + q)$*-th derivative and satisfies*

$$g^{(\nu)}(0) = 0 \qquad \text{for } 0 \le \nu < p + q + 1.$$

More precisely

$$g^{(\lambda+\nu)}(t) = \int_0^t k^{(\lambda)}(t - \tau)\, f^{(\nu)}(\tau)\, d\tau \tag{3.23}$$

if $\lambda \le p$ *and* $\nu \le q$. *Hence, for* $\nu \le q$,

$$g^{(p+1+\nu)}(t) = \int_0^t k^{(p+1)}(t - \tau)\, f^{(\nu)}(\tau)\, d\tau + k^{(p)}(0)\, f^{(\nu)}(t), \tag{3.24}$$

provided that, moreover, k(t) *has a bounded* $(p + 1)$-th *derivative.*

Definition. If f(t) is a continuous function on $0 \le t \le T$, then by $L_n(D)^{-1}f$ we denote the unique solution g(t) of the differential equation $L_n(D)\, g(t) = f(t)$ $(0 \le t \le T)$ for which $g^{(\nu)}(0) = 0$ for $0 \le \nu < n$.

We assert that

$$L_n(D)^{-1} f = \int_0^t k_{0n}(t - \tau)\, f(\tau)\, d\tau = k_{0n} * f. \tag{3.25}$$

After all, by (3.16) and $L_0 = 1$,

$$\int_0^\infty e^{-st} k_{0n}(t)\, dt = L_n(s)^{-1} \qquad (\Re s \ge 0). \tag{3.26}$$

Here, $L_n(s)$ is a polynomial of degree n with leading coefficient $\lambda_{nn} > 0$; thus,

$$k_{0n}^{(\nu)}(0) = 0 \quad \text{for } 0 \le \nu \le n - 2, \qquad k_{0n}^{(n-1)}(0) = 1/\lambda_{nn}. \tag{3.27}$$

Integration by parts in (3.26) gives

$$\int_0^\infty e^{-st}\, k_{0n}^{(\nu)}(t)\, dt = \frac{s^\nu}{L_n(s)} - \delta_{\nu n}/\lambda_{nn} \tag{3.28}$$

if $\nu = 0, 1, \cdots, n$. Hence, $L_n(D)\, k_{0n}(t)$ has its Laplace transform equal to zero; thus,

$$L_n(D)\, k_{0n}(t) \equiv 0 .$$

It now follows from (3.23), (3.24) and (3.27) (with $k = k_{0n}$, $p = n - 1$, $\nu = q = 0$) that the right-hand side of (3.25) has all the properties required of $g = L_n(D)^{-1} f$.

Observe that, by (3.15), (3.16) and (3.28),

$$k_{mn}(t) = L_m(D)\, k_{0n}(t) > 0 \qquad (0 \leq m < n, \text{ all } t > 0) . \tag{3.29}$$

Further, for $m < n < n + j$ we have, by (3.20), $k_{m,n+j} = k_{n,n+j} * k_{mn}$, hence, by (3.16),

$$\int_0^a k_{m,n+j}(t)\, e^{-st} dt \leq [L_n(s)/L_{n+j}(s)] \int_0^a k_{mn}(t)\, e^{-st} dt \leq \int_0^a k_{mn}(t)\, e^{-st} dt \tag{3.30}$$

whenever $m < n < n + j$, $s \geq 0$, and $a \geq 0$.

Finally, if $f(t)$ has for $0 \leq t \leq T$ a continuous q-th derivative and $f^{(\nu)}(0) = 0$ for $0 \leq \nu < q$, then, by (3.23), (3.25), (3.27) and (3.29),

$$L_q(D)\, L_m(D)\, L_n(D)^{-1} f = k_{mn}(t) * \{ L_q(D)\, f(t) \} \tag{3.31}$$

whenever $0 \leq m < n$ and $0 \leq t \leq T$; in particular,

$$L_q(D)\, L_n(D)^{-1} f = L_n(D)^{-1} L_q(D) f .$$

4. UNIQUENESS THEOREMS

We are now in a position to treat the uniqueness problem stated at the beginning of Section 3.

Let $T > 0$ be fixed, and let $\{u_n(t)\}$ be a solution of (3.2) in $(0, T)$ satisfying

$$u_n(0) = 0 \qquad (n = 0, 1, \cdots) . \tag{4.1}$$

Then, by (3.5), $u_0(t)$ is an infinitely differentiable function in $0 \leq t \leq T$,

$$u_n(t) = L_n(D)\, u_0(t) \qquad (n \geq 0,\ 0 \leq t \leq T) ,$$

and finally all derivatives of $u_0(t)$ are equal to 0 at $t = 0$.

In other words, by (3.25), $\{u_n(t)\}$ is a sequence of continuous functions in $0 \leq t \leq T$, such that

$$u_0(t) = k_{0n}(t) * u_n(t) \qquad (0 \leq t \leq T) \tag{4.2}$$

for all $n \geq 1$. Letting

$$\alpha_n(t) = L_n(s) \int_0^t k_{0n}(\tau)\, e^{-s\tau} d\tau \tag{4.3}$$

($s \geq 0$ *fixed*), we see that this implies

$$\alpha_n(t) * [u_n(t)\, e^{-st}] = L_n(s) \int_0^t u_0(\tau)\, e^{-s\tau}\, d\tau \qquad (n \geq 1,\ 0 \leq t \leq T). \tag{4.4}$$

But, by (3.15) and (3.16),

$$0 \leq \alpha_n(t) \leq 1 \qquad (n \geq 1,\ t \geq 0), \tag{4.5}$$

and consequently

$$\int_0^t |u_n(\tau)|\, e^{-s\tau} d\tau \geq L_n(s) \left| \int_0^t u_0(\tau)\, e^{-s\tau}\, d\tau \right|.$$

In particular, $u_0(t) \equiv 0$ $(0 \leq t \leq T)$ if at $t = T$ the left-hand side tends to 0. Moreover, from Lemma 3.1 we obtain the following result.

THEOREM 4.1. *Let* $\{u_n(t)\}$ *be a nontrivial solution of* (3.2) *in* (0, T), *satisfying* (4.1). *Suppose moreover that, for infinitely many positive integers* n, *either*

$$\int_0^T |u_n(t)|\, dt \leq 1 \tag{4.6}$$

or

$$|u_n(t)| \leq 1 \qquad (0 \leq t \leq T).$$

Then the series (3.11) *converges.*

Refining the proof of Theorem 4.1, one obtains the following generalization. However, we shall not make any use of it.

THEOREM 4.2. *Let* $\{\varepsilon_n\}$ $(n = 1, 2, \cdots)$ *be a fixed sequence of complex constants. Suppose that there exists a nontrivial complex-valued solution of* (3.2) *in* (0, T), *satisfying* (4.1) *and*

$$\int_0^T \left| \sum_{n=1}^{N} \varepsilon_n u_n(t) \right| dt \leq 1 \tag{4.7}$$

for all $N \geq 1$. *Then, for each real number* $s \geq 0$,

$$\sup_N \left| \sum_{n=1}^{N} \varepsilon_n L_n(s) \right| < \infty. \tag{4.8}$$

That Theorem 4.1 is implied by Theorem 4.2 can be seen as follows. Let $s > 0$ be fixed, and suppose that (3.11) diverges; then, by Lemma 3.1, $L_n(s) \to \infty$ as $n \to \infty$. Letting

$$L_n(s)\, e_n = \int_0^T |u_n(t)|\, dt,$$

we see from (4.6) that $\liminf e_n = 0$. Let $1 \leq n_\nu < n_{\nu+1} < \cdots$ be such that $e_1 \leq 1$ and $e_{n_{\nu+1}} \leq e_{n_\nu}/2$, and define ε_{n_ν} by

$$\varepsilon_{n_\nu} L_{n_\nu}(s)\, e_{n_\nu} = e_{n_\nu} - e_{n_{\nu+1}},$$

$\varepsilon_n = 0$ if $n \neq n_\nu$ ($\nu = 1, 2, \cdots$). Then (4.7) holds for all $N \geq 1$, but (4.8) does not hold—a contradiction.

Proof of Theorem 4.2. Let

$$v_n(t) = \int_0^t e^{-s\tau} u_n(\tau)\, d\tau. \tag{4.9}$$

Then, by (4.4),

$$\alpha_n(t) * v_n(t) = L_n(s) \int_0^t dt' \int_0^{t'} u_0(\tau)\, e^{-s\tau}\, d\tau.$$

Hence, letting

$$g_N(t) = \sum_{n=1}^{N} \varepsilon_n L_n(s) \int_0^t (t - \tau)\, u_0(\tau)\, e^{-s\tau}\, d\tau, \tag{4.10}$$

we find that

$$g_N(t) = \sum_{n=1}^{N} \alpha_n * \varepsilon_n v_n = \sum_{n=1}^{N-1} (\alpha_n - \alpha_{n+1}) * w_n + \alpha_N * w_N, \tag{4.11}$$

where

$$w_n(t) = \sum_{m=1}^{N} \varepsilon_m v_m(t).$$

By (4.7) and (4.9), $|w_n(t)| \leq 1$ $(0 \leq t \leq T)$. Moreover, by the first inequality (3.30) (with $m = 0$ and $j = 1$) and (4.3),

$$\alpha_n(t) - \alpha_{n+1}(t) \geq 0 \qquad (n \geq 1,\ t \geq 0).$$

Hence, together with (4.5), (4.11) implies that $|g_N(t)| \leq t + t = 2t$ for all $N \geq 1$, $0 \leq t \leq T$. By the definition (4.10) of $g_N(t)$, this implies (4.8). For otherwise $u_0(t) \equiv 0$, hence $u_n(t) = L_n(D)\, u_0(t) \equiv 0$ $(0 \leq t \leq T)$.

We shall now investigate the case where (3.11) converges. Here (Lemma 3.1), the limit

$$L_\infty(s) = \lim_{n \to \infty} L_n(s) \tag{4.12}$$

exists for each complex s, *uniformly* on bounded sets. By (3.17), (4.13) $$|L_\infty(s)|^{-1} \leq |L_n(s)|^{-1} \leq |L_m(s)|^{-1} \leq 1$$

if $0 < m < n$, $\Re s \geq 0$ (if $s \neq 0$, then the inequality signs hold in (4.13)). Recall that $L_n(s)$ is a polynomial of degree n ($n = 0, 1, 2, \cdots$).

Now, consider the function $k_{0n}(t)$ defined by (3.26). Putting

(4.14) $$k_{0n}(t) = 0 \qquad \text{if } t < 0,$$

we see from (3.27) that $k_{0n}(t)$ admits a continuous derivative of order $n - 2$ ($-\infty < t < \infty$). In fact, by (3.28),

$$k_{0n}^{(\nu)}(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} s^\nu L_n(s)^{-1} e^{st}\, ds$$

for $\nu = 0, 1, \cdots, n - 2$, $-\infty < t < \infty$. It follows, since (4.12) holds uniformly in bounded sets, and by (4.13), that for each fixed $\nu \geq 0$,

(4.15) $$\lim_{n\to\infty} k_{0n}^{(\nu)}(t) = k_{0\infty}^{(\nu)}(t)$$

uniformly in t ($-\infty < t < \infty$), where

(4.16) $$k_{0\infty}^{(\nu)}(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} s^\nu L_\infty(s)^{-1} e^{st}\, ds.$$

Consequently, $k_{0\infty}^{(\nu)}(t)$ is precisely the ν-th derivative of the function $k_{0\infty}(t) = k_{0\infty}^{(0)}(t)$. Observe that, by (4.14) and (4.15),

$$k_{0\infty}(t) = 0 \qquad \text{if } t < 0,$$

hence,

(4.17) $$k_{0\infty}^{(\nu)}(0) = 0 \qquad (\nu = 0, 1, 2, \cdots).$$

Finally, letting

(4.18) $$k_{m\infty}(t) = L_m(D)\, k_{0\infty}(t),$$

we deduce from (3.15), (3.29) and (4.15) the relations

(4.19) $$k_{m\infty}(t) \geq 0, \qquad \lim_{t\to\infty} k_{m\infty}(t) = 0,$$

while, by (4.16),

(4.20) $$\int_0^\infty k_{m\infty}(t)\, e^{-st}\, dt = L_m(s)\, L_\infty(s)^{-1}$$

if $\Re s \geq 0$ ($m = 0, 1, \cdots$). In particular, by (3.8),

$$0 < \int_0^\infty k_{m\infty}(t)\,dt = \lambda_m/\lambda_{\infty 0} \leq 1, \tag{4.21}$$

where λ_m is given by (3.9) and

$$\lambda_{\infty 0} = 1 + \sum_{k=0}^{\infty} \beta_0 \beta_1 \cdots \beta_k .$$

We conclude from (4.17), (4.18), (4.19) and (4.21) that $\{u_m(t) = k_{m\infty}(t)\}$ defines a nontrivial solution of (3.2) in $(0, \infty)$, satisfying (4.1), and satisfying further (4.6) for all $n \geq 0$ and all $T > 0$. More precise results are contained in Theorems 4.4 and 4.5.

LEMMA 4.3. *Suppose that* (3.11) *converges. Let* $T > 0$ *be fixed, and let* $g(t)$ *be any integrable function on* $0 \leq t \leq T$ *such that*

$$\int_0^t k_{0\infty}(t - \tau)\, g(\tau)\, d\tau = 0 \qquad \text{for } 0 \leq t \leq T. \tag{4.22}$$

Then,

$$g(t) = 0 \quad \textit{for almost all } 0 \leq t \leq T.$$

Proof. By Lemma 3.4, (4.17), and (4.18), it follows from (4.22) that

$$\int_0^t k_{n\infty}(t - \tau)\, g(\tau)\, d\tau = 0 \qquad \text{for } 0 \leq t \leq T,\ n \geq 0. \tag{4.23}$$

Now, consider the nonnegative measure

$$\mu_n(A) = \int_A k_{n\infty}(t)\, dt \tag{4.24}$$

of total variation at most 1 on $(0, \infty)$. By (4.20) and (4.12), its Fourier transform $L_n(iy)/L_\infty(iy)$ tends to 1 as $n \to \infty$. Thus, μ_n converges weakly to the measure of mass 1 at 0; in particular,

$$\lim_{n \to \infty} \int_0^t k_{n\infty}(t)\, dt = 1 \qquad \text{if } t > 0. \tag{4.25}$$

Consider further the bounded and continuous function

$$G(t) = \begin{cases} \int_0^t g(\tau)\, d\tau & \text{if } 0 \leq t \leq T, \\ 0 & \text{if } t < 0. \end{cases}$$

By (4.14), (4.23) implies

$$\int_0^T G(t - \tau)\, \mu_n(d\tau) = 0 \qquad \text{for } 0 \leq t \leq T,\ n \geq 0.$$

It follows from the weak convergence mentioned above that $G(t) = 0$ for $0 \leq t \leq T$; hence, $g(t) = 0$ for almost all $0 \leq t \leq T$.

For later use, we have, (assuming that (3.11) converges),

$$k_{n\infty}(t) > 0 \qquad \text{for all } t > 0,\ n \geq 0. \tag{4.26}$$

In particular, by (4.20),

$$L_\infty(s) = O(e^{\varepsilon|s|}) \qquad \text{for each } \varepsilon > 0.$$

Proof. By (4.20) and (3.16),

$$k_{n\infty}(t) = \int_0^t k_{nN}(t - \tau)\, k_{N\infty}(\tau)\, d\tau \tag{4.27}$$

for $n < N$. Now let n and $t_0 > 0$ be fixed, and suppose that $k_{n\infty}(t_0) = 0$. By (3.29), (4.19), (4.27) and the continuity of $k_{N\infty}(t)$,

$$k_{N\infty}(t) = 0 \qquad \text{for } 0 < t < t_0 \text{ and all } N > n.$$

But this clearly contradicts (4.25).

THEOREM 4.4. (i) *Suppose there exists a nontrivial solution* $\{u_n(t)\}$ *in* $(0, T)$ *of* (3.2), *satisfying* (4.1) *and the condition*

$$\liminf_{n\to\infty} \int_0^T |u_n(t)|\, dt < \infty. \tag{4.28}$$

Then the series (3.11) *converges.*

Moreover, there exists a finite and regular Borel measure μ *on* $[0, T]$ *such that, for* $n \geq 0$ *and* $0 \leq t \leq T$,

$$u_n(t) = \int_0^t k_{n\infty}(t - \tau)\, \mu(d\tau). \tag{4.29}$$

(ii) *Conversely, suppose that the series* (3.11) *converges, and let* μ *be any regular Borel measure on* $[0, \infty)$. *Then* (4.29) *defines a solution of* (3.2) *in* $(0, \infty)$ *that satisfies* (4.1) *and the inequality*

$$\int_0^T |u_n(t)|\, dt \leq \int_0^T |\mu(d\tau)|$$

for all $n \geq 0$ *and all* $T > 0$.

THEOREM 4.5. (i) *Suppose there exists a nontrivial solution of* (3.2) *in* (0, T), *satisfying* (4.1) *and the inequality*

$$|u_n(t)| \leq 1 \qquad (0 \leq t \leq T) \tag{4.30}$$

for infinitely many positive integers n. *Then the series* (3.11) *converges.*

Moreover, there exists a unique measurable function f(t) *on* $0 \leq t \leq T$ *such that*

$$|f(t)| \leq 1 \qquad \textit{for almost all } t \textit{ in } 0 \leq t \leq T, \tag{4.31}$$

and

$$u_n(t) = \int_0^t k_{n\infty}(t - \tau)\, f(\tau)\, d\tau. \tag{4.32}$$

In fact, f(t) *is equal to the weak limit in* $L_1[0, T]$ *of* $\{u_n(t)\}$, *in the sense that for any bounded and measurable function* g(t) *on* $0 \leq t \leq T$,

$$\lim_{n\to\infty} \int_0^T g(t)\, u_n(t)\, dt = \int_0^T g(t)\, f(t)\, dt. \tag{4.33}$$

In particular, if $u_n(t) \geq 0$ $(0 \leq t \leq T)$ *for infinitely many* n, *then* $f(t) \geq 0$ *for almost all* $0 \leq t \leq T$.

(ii) *Conversely, suppose that the series* (3.11) *converges, and let* f(t) *be any measurable function on* $0 \leq t \leq T$ *satisfying* (4.31). *Then* (4.32) *defines a solution of* (3.2) *in* $(0, \infty)$ *that satisfies* (4.1) *and the inequality*

$$|u_n(t)| \leq 1 \qquad \textit{for all } n \geq 0,\ t \geq 0.$$

Note that $f(t) \geq 0$ $(t \geq 0)$ *implies* $u_n(t) \geq 0$ $(n \geq 0,\ t \geq 0)$.

Remark. Assuming that (3.11) converges, one can choose f(t) on $0 \leq t < t_0$ so that (4.32) defines a system $\{u_n(t)\}$ satisfying (4.1), satisfying (3.2) for $0 \leq t < t_0$, and so that $\{u_n(t)\}$ is uniformly bounded in $0 \leq t \leq T$ for each $0 \leq T < t_0$, but not uniformly bounded in $0 \leq t < t_0$. Indeed, it is sufficient to let f(t) be bounded in each interval $0 \leq t \leq T < t_0$ and to tend to ∞ sufficiently fast as $t \to t_0$.

Proof of Theorems 4.4 *and* 4.5. Recall that $\{u_n(t)\}$ is a solution of (3.2) in (0, T), satisfying (4.1), if and only if

(i) $u_0(t)$ is infinitely differentiable in $0 \leq t \leq T$ and has at $t = 0$ all its derivatives equal to 0,

(ii) for $n = 0, 1, \cdots$ and $0 \leq t \leq T$,

$$u_n(t) = L_n(D)\, u_0(t). \tag{4.34}$$

Taking here $T = \infty$, we easily obtain the assertions (ii) of both theorems from Lemma 3.4 and the properties (4.17), (4.18), (4.19), and (4.21) of the $k_{m\infty}(t)$.

Now, let $u_n(t)$ be a nontrivial solution of (3.2) in (0, T) $(T < \infty)$, satisfying (4.1) and either (4.28) or (4.30) for infinitely many n. By Theorem 4.1, the series (3.11) converges. Thus, the functions $k_{n\infty}(t)$ are well-defined and, in fact, positive for $t > 0$ (compare (4.26)). By (4.2), (4.14) and (4.34),

$$u_0(t) = \int_0^T k_{0n}(t - \tau)\, u_n(\tau)\, d\tau = \int_0^T k_{0n}(t - \tau)\, \mu_n(d\tau) \tag{4.35}$$

($n \geq 1$, $0 \leq t \leq T$), where μ_n denotes the finite and regular Borel measure on $[0, T]$ defined by

$$\mu_n(A) = \int_A u_n(t)\, dt.$$

There exists a sequence of integers $1 \leq n_\nu < n_{\nu+1} < \cdots$ such that the μ_{n_ν} are of *uniformly bounded variation*, so that $\{\mu_{n_\nu}\}$ contains a weakly convergent subsequence $\{\mu_{m_\nu}\}$ in the sense that

$$\lim_{n \to \infty} \int g(t)\, \mu_{m_\nu}(dt) = \int g(t)\, \mu(dt)$$

for each continuous function $g(t)$ on $[0, T]$, μ denoting the corresponding weak limit, a finite and regular Borel measure on $[0, T]$. But, for $n \to \infty$, $k_{0n}(t)$ converges uniformly to $k_{0,\infty}$; hence, (4.35) implies that

$$u_0(t) = \int_0^T k_{0\infty}(t - \tau)\, \mu(d\tau) = \int_0^t k_{0\infty}(t - \tau)\, \mu(d\tau).$$

By (4.17), (4.18), (4.34), and Lemma 3.4, this in turn implies (4.29).

In the case of Theorem 4.5, much more can be said. For here, for each sequence $1 \leq n_\nu < n_{\nu+1} < \cdots$ with

$$|u_{n_\nu}(t)| \leq 1 \qquad \text{for } 0 \leq t \leq T,$$

the sequence of elements $\{u_{n_\nu}\}$ in $L_1(0, T)$ is weakly sequentially compact and has a subsequence that is weakly convergent to a function $f(t)$ in $L_1(0, T)$ which thus necessarily satisfies the inequality $|f(t)| \leq 1$. In this case, (4.29) reduces to (4.32). It follows that $|u_n(t)| \leq 1$ for all $n \geq 0$ and all $0 \leq t \leq T$. Therefore, *each* sequence $\{u_{n_\nu}\}$ contains a subsequence converging weakly to a function $f(t)$ such that the representation (4.32) holds. But from Lemma 4.3, the function $f(t)$ in this representation is necessarily unique. This proves that $u_n(t)$ converges weakly to $f(t)$ in the sense of $L_1(0, T)$.

5. EXISTENCE

Let $j \geq 0$ be a fixed integer. We now turn to the existence of a real solution $\{u_n(t)\}$ of (3.2) in $(0, \infty)$ that satisfies the initial conditions (3.3) and for which $0 \leq u_n(t) \leq 1$.

Let $N > j$ be a fixed integer. By Lemma 3.2, the polynomial $L_N(s)$ has all its roots real and negative. Thus, the general solution of $L_N(D)\, u(t) = 0$ is of the form

$$u(t) = \sum_{\nu=1}^{p} \sum_{k=1}^{k_\nu} c_{\nu k} t^{k-1} e^{-\xi_\nu t} \qquad (c_{\nu k} \text{ constant}),$$

where $\xi_1, \cdots, \xi_p$ are the different (positive and real roots) of the equation $L_N(-s) = 0$, and where k_ν is the multiplicity of ξ_ν, $\Sigma k_\nu = N$.

Let ${}_Nf_{0j}$ denote the unique solution of

$$L_N(D)\, {}_Nf_{0j}(t) = 0, \tag{5.1}$$

such that

$$L_n(D)\, {}_Nf_{0j}(t) \equiv {}_Nf_{nj}(t) \qquad \text{(say)} \tag{5.2}$$

satisfies

$${}_Nf_{nj}(0) = \delta_{nj} \qquad (n = 0, 1, \cdots, N-1). \tag{5.3}$$

For brevity, put ${}_Nf_{nj}(t) = u_n(t)$. By (3.4) and (5.2),

$$(1 + \beta_n + \gamma_n D)\, u_n(t) = u_{n+1}(t) + \beta_n u_{n-1}(t)$$

(the last term being zero if $n = 0$). Let

$$\int_0^\infty e^{-st} u_n(t)\, dt = U_n(s) \qquad (\Re s \geq 0).$$

Then, using $u_n(0) = \delta_{nj}$ $(n = 0, 1, \cdots, N-1)$, we obtain for $n = 0, 1, \cdots, N-1$, the relation

$$(1 + \beta_n + \gamma_n s)U_n - \gamma_j \delta_{nj} = U_{n+1} + \beta_n U_{n-1} \tag{5.4}$$

(for $n = 0$ the last term is equal to zero). Now consider the polynomial $B_{nj}(s)$ $(n = 0, 1, \cdots)$ defined by

$$\begin{cases} B_{nj}(s) = 0 \qquad \text{if } n \leq j, \\ (1 + \beta_j + \gamma_j s)B_{jj} + \gamma_j = B_{j+1,j} + \beta_j B_{j-1,j} \end{cases}$$

(thus $B_{j+1,j}(s) = \gamma_j$), together with

$$(1 + \beta_n + \gamma_n s)B_{nj} = B_{n+1,j} + \beta_n B_{n-1,j} \qquad \text{if } n > j. \tag{5.5}$$

$B_{nj}(s)$ is a polynomial of degree $n - j - 1$, for $n \geq j$. Using (3.4) and $L_0 = 1$, we deduce from (5.4) that

$$U_n = U_0 L_n - B_{nj} \qquad \text{if } n \leq N.$$

If we invoke condition (5.1), that is, $U_N(s) = 0$, this yields, for $N > \max(n, j)$,

$$\int_0^\infty e^{-st}\, {}_Nf_{nj}(t)\, dt = L_n(s) \left\{ \frac{B_{Nj}(s)}{L_N(s)} - \frac{B_{nj}(s)}{L_n(s)} \right\}. \tag{5.6}$$

Letting $n > j$, multiplying (5.5) by L_n and subtracting (3.4) multiplied by B_{nj}, we obtain the relation

$$B_{n+1,j}L_n - B_{nj}L_{n+1} = \beta_n(B_{nj}L_{n-1} - B_{n-1,j}L_n).$$

But $B_{jj} = 0$, $B_{j+1,j} = \gamma_j$; thus, for $n \geq j$,

$$\frac{B_{n+1,j}}{L_{n+1}} - \frac{B_{nj}}{L_n} = (\beta_{j+1}\cdots\beta_n)\gamma_j\frac{L_j}{L_n L_{n+1}}.$$

Hence, from $B_{nj} = 0$ for $n \leq j$ we see that

$$\frac{B_{nj}(s)}{L_n(s)} = L_j(s)\sum_{j\leq k<n}(\beta_{j+1}\cdots\beta_k)\gamma_j\frac{1}{L_k(s)L_{k+1}(s)}.$$

Consequently, by (5.6),

$$\int_0^\infty e^{-st}\,{}_N f_{nj}(t)\,dt = L_n L_j \sum_{k=\max(n,j)}^{N-1}\frac{(\beta_{j+1}\cdots\beta_k)\gamma_j}{L_k L_{k+1}}, \tag{5.7}$$

for $N > \max(n, j)$; here $L_m = L_m(s)$.

The relation (5.7) supplies an explicit formula for the solution of (5.1), (5.3). From (3.16) we see, for $k \geq \max(n, j)$, that $L_n L_j/L_k L_{k+1}$ is the Laplace transform of a nonnegative function [namely, $k_{nk}(t) * k_{j,k+1}(t)$]. It follows that

$$0 \leq {}_N f_{nj}(t) \leq {}_{N+1}f_{nj}(t) \tag{5.8}$$

for $N > \max(n, j)$ and for each $t \geq 0$.

Next, let us consider (5.7) when N is large. Let j and $\sigma \geq 0$ be fixed and such that

$$C_j(\sigma) = C = \sigma L_j(\sigma) + \beta_0\cdots\beta_j/\gamma_j > 0.$$

Then either $\sigma > 0$ or $\sigma = 0$ and $\beta_0 > 0, \cdots, \beta_j > 0$. By (3.7),

$$L_k(\sigma) \geq 1 + \sum_{\nu=j}^{k-1}(\beta_0\cdots\beta_\nu + \sigma(\beta_{j+1}\cdots\beta_\nu)\gamma_j L_j(\sigma))$$

$$= 1 + C\sum_{\nu=j}^{k-1}(\beta_{j+1}\cdots\beta_\nu)\gamma_j = \rho_k,$$

say, for $k \geq j$. Hence, $(\beta_{j+1}\cdots\beta_k)\gamma_j = (\rho_{k+1} - \rho_k)/C$; thus,

$$\sum_{k=j}^{\infty}\frac{(\beta_{j+1}\cdots\beta_k)\gamma_j}{L_k(\sigma)L_{k+1}(\sigma)} \leq \frac{1}{C_j(\sigma)} < \infty.$$

Next, let n and j be fixed, let σ be as above, and let $p = 1 + \max(n, j)$. From (5.7) we obtain, for $p \leq M < N$, (5.9) $$ {}_N f_{nj}(t) - {}_M f_{nj}(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} G_{MN}(s)\, e^{st} ds, $$

where

$$ G_{MN}(s) = \frac{L_n(s)\, L_j(s)}{[L_p(s)]^2} \sum_{k=M}^{N-1} (\beta_{j+1}\cdots\beta_k)\gamma_j \frac{[L_p(s)]^2}{L_k(s)\, L_{k+1}(s)}. $$

By (3.17), the absolute value of the k-th term in the latter sum at $x = \sigma + iy$ (y real) is at most equal to its value at $s = \sigma$. Consequently, for $p \le M < N$,

$$ |G_{MN}(\sigma + iy)| \le \left| \frac{L_n(s)\, L_j(s)}{[L_p(s)]^2} \right| \varepsilon_M, $$

where

$$ \varepsilon_M = [L_p(\sigma)]^2 \sum_{k=M}^{\infty} \frac{(\beta_{j+1}\cdots\beta_k)\gamma_j}{L_k(\sigma)\, L_{k+1}(\sigma)} $$

can be made arbitrarily small by choosing M sufficiently large. Hence, by (5.9), the limit

(5.10) $$ \lim_{N\to\infty} {}_N f_{nj}(t) = f_{nj}(t) \quad \text{(say)} $$

exists *uniformly in each finite interval*, and even uniformly for all $t \ge 0$ if σ can be chosen as zero, that is, if $\beta_0 > 0, \cdots, \beta_j > 0$. In view of this uniformity, we see from (5.2) that $f_{0j}(t)$ is infinitely differentiable, while

$$ L_n(D)\, f_{0j}(t) = f_{nj}(t). $$

Further, by (5.3),

$$ f_{nj}(0) = \delta_{nj}, $$

and by (5.8),

(5.11) $$ f_{nj}(t) \ge 0. $$

Finally, by (5.7),

(5.12) $$ F_{nj}(s) = \int_0^\infty e^{-st} f_{nj}(t)\, dt = L_n L_j \sum_{k=\max(n,j)}^{\infty} \frac{(\beta_{j+1}\cdots\beta_k)\gamma_j}{L_k L_{k+1}} $$

(where $L_k = L_k(s)$), for $\Re s > 0$, (for $\Re s \ge 0$ if $\beta_0 > 0, \cdots, \beta_k > 0$).

We further assert that

(5.13) $$ f_{nj}(t) \le 1. $$

In fact, much more can be said. Let s be real and positive. Then, by (5.12),

$$(5.14)\qquad \begin{aligned} \frac{\beta_0}{\gamma_0} F_{n0} + s \sum_{j=0}^{\infty} F_{nj}(s) &= \sum_{k=n}^{\infty} \frac{L_n}{L_k L_{k+1}} \left(\beta_0 \cdots \beta_k + s \sum_{j=0}^{k} (\beta_{j+1} \cdots \beta_k) \gamma_j L_j \right) \\ &= L_n \sum_{k=n}^{\infty} \left(\frac{1}{L_k} - \frac{1}{L_{k+1}} \right) = 1 - \frac{L_n}{L_\infty}. \end{aligned}$$

Here we have used (3.6) and the fact that $L_n(s)$ is increasing in n. We have $L_\infty(s) < \infty$ or $L_\infty(s) = \infty$ according to whether (3.11) converges or diverges; (compare Lemma 3.1). Let

$$(5.15)\qquad k_{n\infty}(t) = 0$$

if (3.11) diverges; otherwise, let $k_{n\infty}(t)$ be defined by (4.20). Dividing (5.14) by s and using (5.12), we see that

$$(5.16)\qquad \frac{\beta_0}{\gamma_0} \int_0^t f_{n0}(\tau)\, d\tau + \int_0^t k_{n\infty}(\tau)\, d\tau + \sum_{j=0}^{\infty} f_{nj}(t) = 1 .$$

By (5.11) and (4.19), each of the terms on the left-hand side is nonnegative; this implies (5.13).

THEOREM 5.1. *Let* $j \geq 0$ *be an integer. Then* (5.12) *defines a solution* $\{f_{nj}(t)\}$ $(n = 0, 1, 2, \cdots)$ *of* (3.2) *in* $(0, \infty)$ *that satisfies*

$$f_{nj}(0) = \delta_{nj}, \qquad \textit{and} \qquad 0 \leq f_{nj}(t) \leq 1 .$$

In fact, even (5.16) *holds.*

Moreover, if $\{u_n(t)\}$ *is any real solution of* (3.2) *in* (0, T) *such that* $u_n(0) = \delta_{nj}$ $(n = 0, 1, \cdots)$, *and*

$$(5.17)\qquad u_N(t) \geq 0 \qquad \textit{for } 0 \leq t \leq T ,$$

for infinitely many positive integers N, *then*

$$(5.18)\qquad u_n(t) \geq f_{nj}(t) \qquad \textit{for all } n \geq 0,\ 0 \leq t \leq T .$$

Proof. Only the last statement remains to be proved. It is closely related to results of Feller [3], [5, p. 536] and of Ledermann and Reuter [10, p. 254].

Let there be given an infinitely differentiable real function $u_0(t)$ on $0 \leq t \leq T$ such that $u_n(t) = L_n(D)\, u_0(t)$ satisfies the initial condition $u_n(0) = \delta_{nj}$. Now consider $\Delta_N(t) = u_0(t) - {}_N f_{0j}(t)$; by (5.2),

$$(5.19)\qquad L_n(D)\, \Delta_N(t) = u_n(t) - {}_N f_{nj}(t) .$$

By (5.3), this reduces to zero when $t = 0$ and $0 < n < N - 1$; thus, $\Delta_N^{(\nu)}(0) = 0$ for $\nu = 0, 1, \cdots, N - 1$. Further, by (5.1), $L_N(D)\, \Delta_N(t) = u_N(t)$. Consequently,

$$\Delta_N(t) = L_N(D)^{-1} u_N(t) ,$$

and hence, by (3.31) (with $q = 0$) and (5.19),

$$u_n(t) - {}_N f_{nj}(t) = k_{nN}(t) * u_N(t)$$

if $0 \leq n \leq N - 1$. From $k_{nN}(t) \geq 0$, we see that if (5.17) holds for infinitely many N then $u_n(t) \geq {}_N f_{nj}(t)$ $(0 \leq t \leq T)$ for infinitely many N, and (5.10) thus implies (5.18).

THEOREM 5.2. *Let* $j \geq 0$ *be a fixed integer,* $T > 0$ *a fixed number. Then, by taking* $h_j(t)$ *as any measurable function on* $0 \leq t \leq T$ *satisfying*

$$0 \leq h_j(t) \leq 1 \qquad (0 \leq t \leq T), \tag{5.20}$$

and letting

$$u_{nj}(t) = f_{nj}(t) + \int_0^t k_{n\infty}(t - \tau)\, h_j(\tau)\, d\tau \tag{5.21}$$

$(n \geq 0,\ 0 \leq t \leq T)$, *we obtain precisely all the solutions* $\{u_{nj}(t)\}$ $(n \geq 0)$ *of* (3.2) *in* (0, T) *such that*

$$u_{nj}(0) = \delta_{nj} \qquad (n = 0, 1, \cdots) \tag{5.22}$$

and

$$0 \leq u_{nj}(t) \leq 1 \qquad \text{for } n \geq 0,\ 0 \leq t \leq T. \tag{5.23}$$

Note that $k_{n\infty}(t) \equiv 0$ *if* (3.11) *diverges. If* (3.11) *converges, there is a one-to-one correspondence between such systems* $\{u_{nj}\}$ *and the functions* $h_j(t)$ *satisfying* (5.20); *it is determined by* (5.21) *and the fact that* $h_j(t)$ *is precisely the weak limit in the sense of* $L_1(0, T)$ *of the corresponding sequence* $\{u_{nj}\}$.

Proof. If (3.11) diverges, then by the convention (5.15), the formula (5.21) merely states that $u_{nj}(t) = f_{nj}(t)$. That this is sufficient follows from Theorem 5.1. That it is also necessary follows from Theorem 4.1 applied to the system

$$v_{nj}(t) = u_{nj}(t) - f_{nj}(t) \qquad (0 \leq t \leq T). \tag{5.24}$$

Now assume that (3.11) converges. That (5.20) and (5.21) are sufficient is a consequence of Theorem 5.1, the second assertion of Theorem 4.5, and the inequality

$$f_{nj}(t) + \int_0^t k_{n\infty}(\tau)\, d\tau \leq 1,$$

the latter following from (5.16).

Conversely, let (3.11) be convergent, and let $\{u_{nj}(t)\}$ be a solution of (3.2) in (0, T) that satisfies (5.22) and (5.23). By the condition $f_{nj}(t) \geq 0$ and the last assertion in Theorem 5.1, the function $v_{nj}(t)$ defined by (5.24) satisfies the relation $0 \leq v_{nj}(t) \leq 1$ for all $n \geq 0$, $0 \leq t \leq T$. Now apply the first assertion of Theorem 4.5 to the system $\{v_{nj}(t)\}$.

THEOREM 5.3. *Let* $h_j(t)$ $(j = 0, 1, \cdots)$ *be a sequence of real and measurable functions on* $0 \leq t \leq T$ $(T > 0,$ *fixed*), *satisfying* (5.20), *and let* $u_{nj}(t)$ *be defined by* (5.21) $(n, j \geq 0,\ 0 \leq t \leq T)$. *Then*

$$\sum_{j=0}^{\infty} u_{nj}(t) = 1 - \frac{\beta_0}{\gamma_0} \int_0^t f_{n0}(\tau)\, d\tau + \int_0^t (h(\tau) - 1)\, k_{n\infty}(t - \tau)\, d\tau \tag{5.25}$$

for $n \geq 0$, $0 \leq t \leq T$, *where*

$$h(t) = \sum_{j=0}^{\infty} h_j(t) \leq \infty \qquad (0 \leq t \leq T). \tag{5.26}$$

If

$$\sum_{j=0}^{\infty} u_{nj}(t) \leq 1 \qquad (0 \leq t \leq T) \tag{5.27}$$

for infinitely many n, *then either* (i) (3.11) *diverges or* (ii) (3.11) *converges and*

$$h(t) \leq 1 \qquad \textit{for almost all } t \ (0 \leq t \leq T).$$

In either case, (5.27) *holds for all* $n \geq 0$.

Finally, suppose that (5.27) *holds for all* $n \geq 0$. *If*

$$\sum_{j=0}^{\infty} u_{nj}(t) = 1 \tag{5.28}$$

for one pair of numbers $n = n_0 \geq 0$, $t = t_0 > 0$, *then* (i) $\beta_p = 0$ *for some* p $(0 \leq p \leq n_0)$, (ii) *the equation* (5.28) *holds for all* n $(n \geq p)$ *and all* t $(0 \leq t \leq t_0)$, (*even for* $n \geq p$ *and* $t \geq 0$ *if* (3.11) *diverges*); *also,* $h(t) = 1$ *for almost all* t $(0 \leq t \leq t_0)$, *if* (3.11) *converges.*

Proof of Theorem 5.3. Note that (5.25) is an immediate consequence of (5.16), (5.21) and (5.26).

Suppose, for the moment, that either (3.11) diverges (thus $k_{n\infty}(t) = 0$) or (3.11) converges and $0 \leq h(t) \leq 1$ for almost all t $(0 \leq t \leq T)$. Then (5.25) implies (5.27). Moreover, by (5.25), at $n = n_0$, $t = t_0 > 0$ the equality sign in (5.27) can hold only if

$$(\beta_0/\gamma_0) \int_0^{t_0} f_{n_0,0}(t)\, dt = 0. \tag{5.29}$$

If (3.11) converges, one moreover needs, by (4.26), the relation $h(t) = 1$ for almost all t $(0 \leq t \leq t_0)$. By (5.12) with $j = 0$, (3.15), and (3.16), it follows from (5.29) that $\beta_0\beta_1 \cdots \beta_k = 0$ for all $k \geq n_0$, in other words, that $\beta_p = 0$ for some p $(0 \leq p \leq n_0)$. The latter in turn implies, by (5.12), that $\beta_0 f_{n0}(t) = 0$ for all $n \geq p$ and all t. This yields the last conclusion of Theorem 5.2.

Finally, consider the case where (3.11) converges and (5.27) holds for infinitely many n. Put

$$v_{nj}(t) = \int_0^t k_{n\infty}(t - \tau)\, h_j(\tau)\, d\tau \qquad (0 \leq t \leq T),$$

$$u_n(t) = \sum_{j=0}^{\infty} v_{nj}(t) \qquad (0 \le t \le T).$$

By (5.20) and (5.26), $v_{nj}(t) \ge 0$ and

$$u_n(t) = \int_0^t k_{n\infty}(t - \tau)\, h(\tau)\, d\tau \qquad (0 \le t \le T). \tag{5.30}$$

Moreover, by (5.21), the relations $f_{nj}(t) \ge 0$, and (5.27), the condition

$$0 \le u_n(t) \le 1 \qquad (0 \le t \le T) \tag{5.31}$$

holds for infinitely many n.

We must prove that the nonnegative function $h(t)$ satisfies $h(t) \le 1$ for almost all $0 \le t \le T$. Let us first assume that

$$\int_0^T h(t)\, dt < \infty . \tag{5.32}$$

Applying Lemma 3.4 to (5.30) with $n = 0$ (and using (4.17), (4.18)), we see that (5.30) defines a solution of (3.2) in $(0, T)$, satisfying (4.1). Using (5.31), we deduce from assertion (i) of Theorem 4.5 that this solution has a representation (4.32) with $|f(t)| \le 1$. In particular, by (5.30) with $n = 0$,

$$\int_0^{\infty} k_{0\infty}(t - \tau)\, (h(\tau) - f(\tau))\, d\tau = 0 \qquad (0 \le t \le T).$$

Hence, from (5.32) and Lemma 4.3, we see that $h(t) - f(t) = 0$ for almost all t $(0 \le t \le T)$; thus, $|h(t)| \le 1$ for almost all t $(0 \le t \le T)$.

It remains to prove (5.32). Let $h_1(t)$ denote any *integrable* functions on $0 \le t \le T$ satisfying $0 \le h_1(t) \le h(t)$. From (5.30) and (5.31), we see that

$$\int_0^t k_{n\infty}(t - \tau)\, h_1(\tau)\, d\tau \le 1 \qquad (0 \le t \le T),$$

for infinitely many n. Hence, letting

$$H_1(t) = \begin{cases} \int_0^t h_1(\tau)\, d\tau & \text{if } 0 \le t \le T, \\ 0 & \text{if } t < 0, \end{cases}$$

we see that

$$\int_0^T H_1(t - \tau)\, \mu_n(d\tau) \le t \qquad (0 \le t \le T)$$

for infinitely many n. Here, the measure μ_n is defined by (4.24). For $n \to \infty$ it converges weakly to the measure of mass 1 at 0. But $H_1(t)$ is continuous, consequently, $H_1(t) \leq t$ $(0 \leq t \leq T)$. Taking the supremum over all possible functions $h_1(t)$, we obtain (5.32).

6. THE SOLUTION $\{\phi_{nj}(t)\}$

Let us now return to the problems mentioned in Section 2. We are given the constants

$$\lambda_n \geq 0, \quad \mu_n \geq 0 \qquad (n = 0, 1, \cdots).$$

Then a solution in $(0, \infty)$ of $(I)_{mn}$, $(II)_{mn}$, and (2.1) to (2.4) is given by

$$\int_0^\infty e^{-st}\phi_{nj}(t)\,dt = \sum_{k=\max(n,j)}^{\infty} (\lambda_n \cdots \lambda_{k-1})(\mu_{j+1} \cdots \mu_k)\frac{R_n R_j}{R_k R_{k+1}},$$

where $R_m = R_m(s)$ denotes the polynomial of degree m defined by $R_{-1} = 0$, $R_0 = 1$, and

$$R_{m+1} = (\lambda_m + \mu_m + s)R_m - \lambda_{m-1}\mu_m R_{m-1} \qquad (m \geq 0).$$

In order to avoid some tedious proofs, we shall show this only for the case where

$$\lambda_m > 0 \quad \text{for } m \geq 0. \tag{6.1}$$

Assume (6.1), and let the polynomial $Q_n(s)$ of degree n be defined by $Q_{-1} = 0$, $Q_0 = 1$, and

$$(\lambda_n + \mu_n + s)Q_n = \lambda_n Q_{n+1} + \mu_n Q_{n-1} \qquad (n \geq 0) \tag{6.2}$$

(thus, $R_n = \lambda_0 \cdots \lambda_{n-1} Q_n$). If we let

$$L_n = Q_n, \quad \beta_n = \mu_n/\lambda_n, \quad \gamma_n = 1/\lambda_n, \tag{6.3}$$

then (3.1) and (3.4) hold, and we can use all the results of the previous sections.

In particular, by (5.12), (see also [6, p. 315]),

$$\int_0^\infty e^{-st}\phi_{nj}(t)\,dt = \sum_{k=\max(n,j)}^{\infty} \frac{\mu_{j+1} \cdots \mu_k}{\lambda_j \cdots \lambda_k}\,\frac{Q_n Q_j}{Q_k Q_{k+1}} \tag{6.4}$$

defines an infinitely differentiable function such that

$$\phi_{nj}(0) = \delta_{nj}, \tag{6.5}$$

$$0 \leq \phi_{nj}(t) \leq 1 \qquad (0 \leq t < \infty),$$

$$Q_n(D)\,\phi_{0j}(t) = \phi_{nj}(t) \qquad (0 \leq t < \infty). \tag{6.6}$$

Finally, by (5.16), (see also [6, p. 315]), (6.7) $$\mu_0 \int_0^t \phi_{n0}(\tau)\, d\tau + \int_0^t k_{n\infty}(\tau)\, d\tau + \sum_{j=0}^{\infty} \phi_{nj}(t) = 1,$$

where

(6.8) $$\int_0^\infty e^{-st} k_{n\infty}(t)\, dt = \frac{Q_n(s)}{Q_\infty(s)} \qquad (\Re s \geq 0)$$

if the series

(6.9) $$\sum_{j=0}^{\infty} \sum_{k=j}^{\infty} \frac{\mu_{j+1} \cdots \mu_k}{\lambda_j \cdots \lambda_k}$$

converges (compare (3.11)), and $k_{n\infty}(t) = 0$, otherwise. In particular, by (4.26),

$$k_{n\infty}(t) > 0 \qquad \text{for } t > 0$$

if (6.9) converges.

By (6.2) and (6.6), $\{\phi_{mj}(t)\}$ satisfies $(II)_{mn}$ for all m, n, that is,

$$(\lambda_m + \mu_m + D)\phi_{mn} = \lambda_m \phi_{m+1,n} + \mu_m \phi_{m-1,n}$$

$(\phi_{-1,n} = 0)$. Further, $\{\phi_{mj}(t)\}$ satisfies $(I)_{mn}$ for all m, n; that is,

(6.10) $$(\lambda_n + \mu_n + D)\phi_{mn} = \lambda_{n-1} \phi_{m,n-1} + \mu_{n+1} \phi_{m,n+1}$$

$(\phi_{m,-1} = 0)$. This is particularly easy to show if $\mu_m > 0$ for all m; for then ϕ_{mn} and ϕ_{nm} differ only by a constant factor, and one can use the transformation (2.14). Otherwise, the result follows on taking the Laplace transform of both members of (6.10) and using (6.2), (6.4) and (6.5); note that by (6.6), it suffices to consider the case $m = 0$ of (6.10).

As was shown in the proof of Theorem 5.3, the condition

(6.11) $$\sum_{j=0}^{\infty} \phi_{nj}(t) = 1$$

for some fixed n and some fixed $t > 0$ can only be satisfied if

(i) the series (6.9) diverges,

(ii) $\mu_m = 0$ for some m $(0 \leq m \leq n)$.

Conversely, if this is true, then (6.11) holds for all $t > 0$ (n fixed). In some sense, it would have been more reasonable to ask for

(6.12) $$\mu_0 \int_0^t \phi_{n0}(\tau)\, d\tau + \sum_{j=0}^{\infty} \phi_{nj}(t) = 1,$$

(the first term being equal to the "probability of absorption in the -1 state during (0, t)"). Clearly by (6.7), (6.12) holds for some fixed n and some fixed $t > 0$ if and only if (6.9) diverges.

Finally, from the second part of Theorem 5.1 (with β_n, γ_n as in (6.3)), we see that, for $j \geq 0$ fixed, the solution $\{\phi_{nj}(t)\}$ $(n \geq 0)$ of $u_n(0) = \delta_{nj}$ and

$$(6.13) \qquad (\lambda_n + \mu_n + D)u_n = \lambda_n u_{n+1} + \mu_n u_{n-1} \qquad (n \geq 0,\ u_{-1} = 0)$$

has the property that, for each *nonnegative* solution of (6.13) in (0, T) with $u_n(0) = \delta_{nj}$,

$$(6.14) \qquad u_n(t) \geq \phi_{nj}(t) \qquad (n \geq 0,\ 0 \leq t \leq T).$$

For convenience, suppose that

$$\lambda_n > 0 \text{ for } n \geq 0, \qquad \mu_n > 0 \text{ for } n \geq 1.$$

Let $\{v_n\}$ be a *nonnegative* solution of $v_n(0) = \delta_{mn}$ and let

$$(\lambda_n + \mu_n + D)v_n = \lambda_{n-1} v_{n-1} + \mu_{n+1} v_{n+1} \qquad (n \geq 0,\ v_{-1} = 0,\ 0 \leq t \leq T).$$

Then the system defined by (2.14) satisfies (6.13), hence, by (6.4) and (6.14),

$$(6.15) \qquad v_n(t) \geq \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} \phi_{nm}(t) = \phi_{mn}(t) \qquad (n \geq 0,\ 0 \leq t \leq T).$$

7. THE BACKWARD DIFFERENTIAL EQUATIONS

Consider the uniqueness problem for the backward equations $(\mathrm{II})_{mn}$ (n fixed) together with the initial conditions (2.1). As was mentioned in Section 2, these equations have a unique solution, namely $\{\phi_{mn}(t)\}$, if $\lambda_m = 0$ for infinitely many m. Otherwise, the problem can be reduced to that of the system (3.2) with β_n and γ_n as in (2.7). Applying Theorem 4.1, we obtain our next result.

THEOREM 7.1. *Suppose that either* $\lambda_m = 0$ *for infinitely many* m *or* $\lambda_m > 0$ *for* $m \geq p$ *and*

$$(7.1) \qquad \sum_{j=p}^{\infty} \left(\frac{1}{\lambda_j} + \sum_{k=j+1}^{\infty} \frac{\mu_{j+1} \cdots \mu_k}{\lambda_j \cdots \lambda_k} \right) = \infty .$$

Let $n \geq 0$ *and* $T > 0$ *be fixed. Then* $\{\phi_{mn}(t)\}$ $(m \geq 0)$ *is the only solution* $\{p_{mn}(t)\}$ $(m \geq 0)$ *in* $0 \leq t \leq T$ *of* $(2.1)_{mn}$ *and* $(\mathrm{II})_{mn}$ $(m = 0, 1, \cdots)$ *such that*

$$(7.2) \qquad \inf_m \int_0^T |p_{mn}(t)|\, dt < \infty .$$

On the other hand, if $\lambda_m > 0$ for $m \geq p$ and (7.1) does not hold, then there are many solutions satisfying (7.2). For convenience, in order to avoid certain tedious complications, let us assume that

(7.3) $$\lambda_m > 0 \quad \text{for } m \geq 0.$$

Let $Q_n(s)$ again be defined by (6.2), and let

$$L_n = Q_n, \quad \beta_n = \mu_n/\lambda_n, \quad \gamma_n = 1/\lambda_n.$$

Then (3.1) and (3.4) hold and, thus, the Theorems 5.2 and 5.3 where $f_{nj}(t) = \phi_{nj}(t)$ (compare Section 6). A consequence is the following.

THEOREM 7.2. *Suppose that* (7.3) *holds and, moreover,*

(7.4) $$\sum_{j=0}^{\infty} \left(\frac{1}{\lambda_j} + \sum_{k=j+1}^{\infty} \frac{\mu_{j+1} \cdots \mu_k}{\lambda_j \cdots \lambda_k} \right) < \infty .$$

Let $k_{n\infty}(t)$ *denote the positive function on* $t > 0$ *defined by*

$$\int_0^\infty e^{-st} k_{n\infty}(t)\, dt = \frac{Q_n(s)}{Q_\infty(s)} \leq 1 \quad (s \geq 0).$$

Let $T > 0$ *be a fixed constant, and let* $h_j(t)$ $(j = 0, 1, \cdots)$ *be real-valued, measurable functions on* $0 \leq t \leq T$, *satisfying the inequalities*

(7.5) $$h_j(t) \geq 0, \quad \sum_{j=0}^{\infty} h_j(t) \leq 1 \quad (0 \leq t \leq T).$$

Then

(7.6) $$p_{nj}(t) = \phi_{nj}(t) + \int_0^t k_{n\infty}(t - \tau) h_j(\tau)\, d\tau \quad (n \geq 0,\ j \geq 0,\ 0 \leq t \leq T),$$

defines a solution $\{p_{nj}\}$ *of* (2.1) *and* $(\mathrm{II})_{mn}$ *(all* m, n) *in* $0 \leq t \leq T$ *satisfying the inequalities*

$$p_{nj}(t) \geq 0, \quad \sum_{j=0}^{\infty} p_{nj}(t) \leq 1 \quad (0 \leq t \leq T).$$

Conversely, if $\{p_{mn}(t)\}$ $(m, n \geq 0)$ *is a system of real functions satisfying these conditions, then*

(i) *for* $n \to \infty$ *(and* j *fixed),* $p_{nj}(t)$ *tends weakly (in the sense of* $L_1[0, T]$) *to a function* $h_j(t)$ $(0 \leq t \leq T)$;

(ii) *these functions satisfy* (7.5) *for almost all* t $(0 \leq t \leq T)$;

(iii) *the system* $\{p_{mn}\}$ *admits the representation* (7.6), *for all* n, j, *and all* t $(0 \leq t \leq T)$.

Finally, for $n \geq 0$ *and* $t_0 > 0$ *fixed,*

$$\sum_{j=0}^{\infty} p_{nj}(t_0) = 1$$

holds if and only if

$$\sum_{j=0}^{\infty} h_j(t) = 1 \quad (0 \leq t \leq t_0), \qquad \mu_0 \mu_1 \cdots \mu_n = 0.$$

8. THE FORWARD DIFFERENTIAL EQUATIONS

The forward equations $(I)_{mn}$ (m fixed) together with the initial conditions (2.1) have a unique solution if $\mu_n = 0$ for infinitely many n. Otherwise, the corresponding uniqueness problem can be reduced to one for the system (3.2) with β_n, γ_n as in (2.10), namely, by means of the transformation (2.8). Applying Theorem 4.1, we obtain the following result.

THEOREM 8.1. *Suppose that either* $\mu_n = 0$ *for infinitely many* n *or* $\mu_n > 0$ *for* $n \geq q$, *and that*

$$\sum_{j=q}^{\infty} \left(\frac{1}{\mu_j} + \sum_{k=j+1}^{\infty} \frac{\lambda_j \cdots \lambda_{k-1}}{\mu_j \cdots \mu_k} \right) = \infty. \tag{8.1}$$

Let $m \geq 0$ *and* $T > 0$ *be fixed. Then* $\{\phi_{mn}(t)\}$ $(n \geq 0)$ *is the only solution* $\{p_{mn}(t)\}$ $(n \geq 0)$ *in* $0 \leq t \leq T$ *of* $(2.1)_{mn}$ *and* $(I)_{mn}$ $(n = 0, 1, \cdots)$ *such that*

$$\inf_n \int_0^T \Big| \sum_{j=0}^{n} p_{mj}(t) \Big| \, dt < \infty. \tag{8.2}$$

On the other hand, if $\mu_n > 0$ for $n \geq q$ and (8.1) does not hold, there exist many solutions satisfying (8.2). To show this, it is necessary and sufficient to find a *nontrivial* solution $\{u_n(t)\}$ $(n \geq 0)$ in (0, T) of the system

$$(\lambda_n + \mu_n + D)u_n = \lambda_{n-1} u_{n-1} + \mu_{n+1} u_{n+1} \qquad (n \geq 0,\ u_{-1} = 0)$$

such that $u_n(0) = 0$ and

$$\inf_n \int_0^T \Big| \sum_{j=0}^{n} u_j(t) \Big| \, dt < \infty;$$

(to see this, consider $p_{mn} - \phi_{mn} = c_m u_n$, where c_m is a constant). Or, equivalently (compare (2.8)), if we let

$$v_n(t) = \mu_0 \int_0^t u_0(\tau)\, d\tau + \sum_{j=0}^{n-1} u_j(t) \qquad (n \geq 0,\ 0 \leq t \leq T),$$

we must have a nontrivial solution $\{v_n(t)\}$ $(n \geq 0)$ in (0, T) of the system

(8.3) $$(\lambda_{n-1} + \mu_n + D)v_n = \lambda_{n-1} v_{n-1} + \mu_n v_{n+1} \qquad (n \geq 0,\ \lambda_{-1} = 0),$$

such that $v_n(0) = 0$ and

$$\inf_n \int_0^T |v_n(t)|\, dt < \infty .$$

We may assume that $q \geq 0$ is minimal with respect to the condition that $\mu_n > 0$, for $n \geq q$; thus, if $q \geq 1$, then $\mu_{q-1} = 0$. Hence, by (8.3) and the condition $v_n(0) = 0$, we necessarily obtain the conclusion

$$v_n(t) = 0 \qquad \text{for } n < q .$$

If we let $w_n = v_{n+q}$ and

(8.4) $$\beta_n = \lambda_{n+q-1}/\mu_{n+q}, \qquad \gamma_n = 1/\mu_{n+q} \qquad (n = 0, 1, \cdots),$$

we thus must have a nontrivial solution in $(0, T)$ of the system

$$(1 + \beta_n + \gamma_n D)w_n = w_{n+1} + \beta_n w_{n-1} \qquad (n \geq 0,\ w_{-1} = 0),$$

such that $w_n(0) = 0$ and

$$\inf_n \int_0^T |w_n(t)|\, dt < \infty .$$

By (8.4), the assumption that (8.1) does not hold is equivalent to the convergence of (3.11). Thus, Theorem 4.4 applies and yields many (in fact, all) such solutions $\{w_n\}$.

More precisely, let L_n $(n = 0, 1, \cdots)$ be defined by (3.4) (β_n, γ_n as in (8.4)) and let

(8.5) $$H_n^{(q)}(s) = \begin{cases} L_{n-q}(s) & \text{if } n \geq q, \\ 0 & \text{if } n < q; \end{cases}$$

in particular, $H_q^{(q)} = 1$ (the upper q is an ordinary index; it does not refer to differentiation). We note three consequences of the results of Sections 3 and 4: First, we observe that, by (3.4),

$$(\lambda_{n-1} + \mu_n + s)\, H_n^{(q)} = \lambda_{n-1} H_{n-1}^{(q)} + \mu_n H_{n+1}^{(q)} \qquad (n \geq 0).$$

In particular, for $n \geq q$, $H_n^{(q)}$ is a polynomial of degree $n - q$. By Lemma 3.2, all its zeros are real and negative. Further, by (3.6) and the condition $\mu_{q-1} = 0$, we see that

(8.6) $$H_n^{(q)} - H_{n-1}^{(q)} = \delta_{nq} + s \sum_{j=q}^{n-1} \frac{\lambda_j \cdots \lambda_{n-2}}{\mu_j \cdots \mu_{n-1}} H_j^{(q)}$$

if $n \geq q$. Second,

$$H_\infty^{(q)}(s) = \lim_{n\to\infty} H_n^{(q)}(s)$$

exists uniformly on each bounded set (compare Lemma 3.1). Further (see (4.20)), the equation

$$(8.7) \qquad H_m^{(q)}(s)/H_\infty^{(q)}(s) = \int_0^\infty h_{m\infty}^{(q)}(t)\, e^{-st} dt \qquad (\Re s \geq 0)$$

defines a nonnegative, infinitely differentiable function $h_{m\infty}^{(q)}(t)$ (equal to 0 if $m < q$) all of whose derivatives vanish at $t = 0$. Moreover, for $m \geq q$ (see also (4.26)),

$$(8.8) \qquad h_{m\infty}^{(q)}(t) > 0 \quad \text{if } t > 0, \qquad \int_0^\infty h_{m\infty}^{(q)}(t)\, dt \leq 1.$$

Also (compare (4.18),

$$h_{m\infty}^{(q)}(t) = H_m^{(q)}(D)\, h_{q\infty}^{(q)}(t).$$

Finally, from Theorems 4.4 and 4.5 we obtain the next proposition.

THEOREM 8.2. *Suppose that* $\mu_n > 0$ *for* $n \geq q$, *with* $q \geq 0$ *minimal, and that further*

$$\sum_{j=q}^{\infty} \left(\frac{1}{\mu_j} + \sum_{k=j+1}^{\infty} \frac{\lambda_j \cdots \lambda_{k-1}}{\mu_j \cdots \mu_k} \right) < \infty.$$

Let $m \geq 0$ *and* $T > 0$ *be fixed. Then the most general solution* $\{p_{mn}(t)\}$ $(n \geq 0)$ *in* $0 \leq t \leq T$ *of* $(2.1)_{mn}$ *and* $(I)_{mn}$ $(n \geq 0)$, *that satisfies* (8.2) *is of the form*

$$p_{mn}(t) - \phi_{mn}(t) = \int_0^t [h_{n+1,\infty}^{(q)}(t-\tau) - h_{n,\infty}^{(q)}(t-\tau)]\, \mu(d\tau) \qquad (n \geq 0,\ 0 \leq t \leq T),$$

where μ *is an arbitrary finite regular Borel measure on* $[0, T]$.

Moreover, if condition (8.2) *is replaced by the stronger condition*

$$(8.9) \qquad \inf_n \sup_{0 \leq t \leq T} \left| \sum_{j=0}^{n} p_{mj}(t) \right| < \infty,$$

then the most general solution is of the form

$$(8.10) \quad p_{mn}(t) - \phi_{mn}(t) = \int_0^t \left(h_{n+1,\infty}^{(q)}(\tau) - h_{n,\infty}^{(q)}(\tau) \right) f(t-\tau)\, d\tau \qquad (n \geq 0,\ 0 \leq t \leq T),$$

where $f(t)$ *denotes a bounded, measurable function on* $0 \leq t \leq T$. *In fact, if* $\{p_{mn}\}$ *is a system on* $0 \leq t \leq T$ *satisfying* (2.1), $(I)_{mn}$ $(n \geq 0)$, *and* (8.9), *then the corresponding function* $f(t)$ *is precisely equal to the weak limit, in the sense of* $L_1(0, T)$, *of the sequence* $\{v_n\}$ *defined by the equation*

(8.11) $$v_n(t) = \sum_{j=0}^{n-1} \Delta_{mj}(t) + \mu_0 \int_0^t \Delta_{m0}(\tau)\, d\tau,$$

where

$$\Delta_{mj}(t) = p_{mn}(t) - \phi_{mn}(t).$$

For convenience, suppose that

$$\lambda_n > 0 \text{ for } n \geq 0, \qquad \mu_n > 0 \text{ for } n \geq 1.$$

Let us apply the above theorem with

$$f(t) = \int_0^t g(\tau)\, d\tau, \qquad \text{where}$$

(8.12) $$0 \leq g(t) \leq \mu_0 \phi_{m0}(t) + k_{m\infty}(t).$$

We assert that the resulting solution $\{p_{mn}(t)\}$ $(n \geq 0)$ of (2.1) and $(I)_{mn}$ $(n \geq 0)$, (m fixed), also satisfies

(8.13) $$p_{mn}(t) \geq 0, \qquad \sum_{n=0}^{\infty} p_{mn}(t) \leq 1.$$

The first inequality follows from the fact that the right-hand side of (8.10) can be written as the convolution of $g(t)$ and the function

$$\int_0^t \left(h^{(q)}_{n+1,\infty}(\tau) - h^{(q)}_{n\infty}(\tau)\right) d\tau,$$

which is nonnegative by (8.6), and (8.8). Further, by (8.10) and (6.7),

$$1 - \sum_{n=0}^{N-1} p_{mn}(t) \geq \mu_0 \int_0^t \phi_{m0}(\tau)\, d\tau + \int_0^t k_{m\infty}(\tau)\, d\tau$$

$$- \int_0^t g(t-\tau)\, d\tau \int_0^\tau \left(h^{(q)}_{N\infty}(x) - h^{(q)}_{0\infty}(x)\right) dx.$$

Here, by (8.8), the inner integral is not greater than 1, hence, by (8.12), we obtain the second inequality (8.13).

The resulting solutions $\{p_{mn}\}$ can be chosen to be distinct from $\{\phi_{mn}\}$ if and only if (8.1) does not hold and, moreover, the right-hand side of (8.12) is not zero almost everywhere (in other words, is positive for $t > 0$), which happens if and only if either (7.4) holds or $\mu_0 > 0$.

On the other hand, if either (8.1) is true or (7.1) is true and $\mu_0 = 0$, then there does not exist *any* solution $\{p_{mn}(t)\}$ $(n \geq 0,\ 0 < t < T,\ m > 0$, and $T > 0$, fixed) distinct from $\{\phi_{mn}(t)\}$ of the forward equations $(I)_{mn}$ and the initial conditions (2.1) such that (8.13) holds for $0 \leq t \leq T$. After all, by (6.15), $p_{mn}(t) \geq \phi_{mn}(t)$. If (8.1) holds, the assertion follows from Theorem (8.1). Otherwise, we see by (6.7) that

$$\sum_{n=0}^{\infty} \phi_{mn}(t) = 1 \qquad (\text{all } t \geq 0).$$

This type of proof has already been used by Reuter and Ledermann [9, p. 255].

9. THE FORWARD AND BACKWARD DIFFERENTIAL EQUATIONS

We now come to the most interesting problem concerning the existence of systems $\{p_{mn}(t)\}$ $(m, n \geq 0)$ on $0 \leq t \leq T$ distinct from the system $\{\phi_{mn}\}$ and satisfying all of the conditions $(I)_{mn}$, $(II)_{mn}$, (2.1), (2.2), (2.3), and (2.4) for all m, n. By Theorem 7.1 and Theorem 8.1, such existence is possible only if $\lambda_m > 0$ for $m \geq p$,

$$\sum_{j=p}^{\infty} \left(\frac{1}{\lambda_j} + \sum_{k=j+1}^{\infty} \frac{\mu_{j+1} \cdots \mu_k}{\lambda_j \cdots \lambda_k} \right) < \infty, \tag{9.1}$$

$\mu_m > 0$ for $m \geq q$, and finally

$$\sum_{j=q}^{\infty} \left(\frac{1}{\mu_j} + \sum_{k=j+1}^{\infty} \frac{\lambda_j \cdots \lambda_{k-1}}{\mu_j \cdots \mu_k} \right) < \infty. \tag{9.2}$$

For convenience, we shall assume that

$$\lambda_n > 0 \text{ for } n \geq 0, \qquad \mu_n > 0 \text{ for } n \geq 1;$$

thus, $p = 1$ and $q = 0$ or $q = 1$ (depending on whether $\mu_0 > 0$ or $\mu_0 = 0$). Letting

$$\pi_n = \left(\frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_{n-1}} \right) \frac{1}{\mu_n}, \qquad \pi_0 = 1,$$

and

$$\rho_n = \left(\frac{\mu_1 \cdots \mu_{n-1}}{\lambda_0 \cdots \lambda_{n-1}} \right) \frac{\mu_n}{\lambda_n} = \frac{1}{\lambda_n \pi_n},$$

we see that (9.1) and (9.2) are equivalent to the conditions

$$\sum_{0 \leq j \leq k} \pi_j \rho_k < \infty \quad \text{and} \quad \sum_{j > k \geq 0} \pi_j \rho_k < \infty,$$

respectively. Together they are equivalent to the conditions

$$\sum_{0}^{\infty} \pi_n < \infty, \qquad \sum_{0}^{\infty} \rho_n < \infty. \tag{9.3}$$

From now on we assume (9.3).

Let us first collect some of the formulae that will be needed or are helpful for better understanding. First, there is the polynomial $Q_n(s)$ defined by the recurrence relation

$$(\lambda_n + \mu_n + s)Q_n = \lambda_n Q_{n+1} + \mu_n Q_{n-1} \qquad (n \geq 0)$$

and the conditions that $Q_0(s) = 1$, $Q_n(s) = 0$ for $n < 0$. Further, (compare (6.8), (4.17) to (4.21), (4.25), and (4.26)),

$$\frac{Q_m(s)}{Q_\infty(s)} = \int_0^\infty k_{m\infty}(t)\, e^{-st}\, dt \qquad (\Re s \geq 0),$$

where

$$k_{n\infty}(t) = Q_n(D)\, k_{0\infty}(t) \begin{cases} > 0 & \text{for } t > 0, \\ = 0 & \text{for } t \leq 0; \end{cases}$$

(9.4) $$\int_0^\infty k_{m\infty}(t)\, dt \leq 1; \qquad \text{and}$$

$$\lim_{m\to\infty} \int_0^t k_{m\infty}(t)\, dt = 1 \qquad \text{for } t > 0.$$

Also, by (6.7),

(9.5) $$\mu_0 \int_0^t \phi_{m0}(\tau)\, d\tau + \int_0^t k_{m\infty}(\tau)\, d\tau + \sum_{n=0}^{\infty} \phi_{mn}(t) = 1.$$

Let $q = 0$ if $\mu_0 > 0$, $q = 1$ if $\mu_0 = 0$. Further, let $H_n^{(q)}(s)$ be defined by the relations

(9.6) $$(\lambda_{n-1} + \mu_n + s)H_n^{(q)} = \lambda_{n-1} H_{n-1}^{(q)} + \mu_n H_{n+1}^{(q)} \qquad (n \geq 0,\ \lambda_{-1} = 0),$$

(9.7) $$H_q^{(q)} = 1, \qquad H_n^{(q)} = 0 \text{ if } \quad n < q.$$

Completely similar to the above formulae, there are the relations (see (8.5)),

(9.8) $$\frac{H_n^{(q)}(s)}{H_\infty^{(q)}(s)} = \int_0^\infty h_{n\infty}^{(q)}(t)\, e^{-st}\, dt \qquad (\Re s \geq 0);$$

(9.9) $$h_{n\infty}^{(q)} = H_n^{(q)}(D) h_{q\infty}^{(q)} \begin{cases} > 0 & \text{for } t > 0,\ n \geq q, \\ = 0 & \text{for } t \leq 0; \end{cases}$$

(9.10) $$\int_0^\infty h_{n\infty}^{(q)}\, dt \leq 1;$$

$$\lim_{n\to\infty} \int_0^t h_{n\infty}^{(q)}\, dt = 1 \qquad \text{if } t > 0. \tag{9.11}$$

Further (see (8.6)),

$$H_n^{(q)} - H_{n-1}^{(q)} = \delta_{nq} + s \sum_{q \le j \le n-1} \frac{\lambda_j \cdots \lambda_{n-2}}{\mu_j \cdots \mu_{n-1}} H_j^{(q)},$$

hence, by (9.8) and (9.9),

$$\int_0^t \left(h_{n+1,\infty}^{(q)}(\tau) - h_{n,\infty}^{(q)}(\tau) \right) d\tau > 0 \qquad \text{if } n \ge 0,\ t > 0. \tag{9.12}$$

Also notice that, by (9.7) and (9.8),

$$h_{0\infty}^{(q)}(t) = 0 \qquad \text{if } q = 1 \text{ (that is, if } \mu_0 = 0).$$

DEFINITION 9.1. *With* $T > 0$ *fixed, we shall denote by* Ω *the collection of systems* $\{p_{mn}(t)\}$ $(m, n \ge 0)$ *of infinitely differentiable functions on* $0 \le t \le T$ *such that* $(I)_{mn}$, $(II)_{mn}$, (2.1), (2.2), (2.4) *hold for all* t $(0 \le t \le T)$ *and all* $m, n \ge 0$.

DEFINITION 9.2. *We denote by* Ω_1 *the collection of real, bounded, and measurable functions* ψ *on* $0 \le t \le T$ *such that*

$$\int_0^t \left(h_{n+1,\infty}^{(q)}(\tau) - h_{n,\infty}^{(q)}(\tau) \right) \psi(t - \tau)\, d\tau \ge 0 \tag{9.13}$$

for all $n \ge 0$, $0 \le t \le T$, *and such that further*

$$0 \le \psi(t) \le 1 + \int_0^t h_{0\infty}^{(q)}(\tau)\, \psi(t - \tau)\, d\tau \tag{9.14}$$

for almost all t $(0 \le t \le T)$. *In particular, any nonnegative and nondecreasing function* $\psi(t)$ *on* $0 \le t \le T$ *such that*

$$\int_{0+}^t \left(1 - \int_0^{t-\tau} h_{0\infty}^{(q)}(x)\, dx \right) d\psi(\tau) \le 1 - \psi(0+)$$

for $0 < t \le T$ *necessarily belongs to* Ω_1 *(note that the expression in square brackets is less than* 1 *if* $\mu_0 > 0$, *equal to* 1 *if* $\mu_0 = 0$). *Functions* ψ *in* Ω_1 *differing only on a set of measure* 0 *will be identified.*

THEOREM 9.3. *Corresponding to each system* $\{p_{mn}\}$ *in* Ω *there exists a unique* $\psi \in \Omega_1$ *such that*

$$p_{mn}(t) = \phi_{mn}(t) + k_{m\infty}(t) * \left(h_{n+1,\infty}^{(q)}(t) - h_{n\infty}^{(q)}(t) \right) * \psi(t) \tag{9.15}$$

for all $0 \le t \le T$ *and all* $m, n \ge 0$ *(the star denotes convolution). Conversely, for each* $\psi \in \Omega_1$, (9.15) *defines a system* $\{p_{mn}\}$ *belonging to* Ω. *Moreover,* (9.16) $$\sum_{n=0}^{\infty} p_{mn}(t) = 1 - \mu_0 \int_0^t \phi_{m0}(\tau)\, d\tau - \int_0^t k_{m\infty}(t-\tau)\psi_1(\tau)\, d\tau$$

($0 \leq t \leq T$, $m \geq 0$), *where* $\psi_1(t)$ *denotes the (nonnegative) difference of the third and second members of* (9.14). *It follows that, for* $m \geq 0$ *and* $t_0 > 0$ *fixed,*

$$\sum_{n=0}^{\infty} p_{mn}(t_0) = 1$$

can hold only when $\mu_0 = 0$ *and* $\psi(t) = 1$ *for* $0 \leq t \leq t_0$.

Proof. Note that the last assertion in Definition 9.2 follows from (9.12) and an integration by parts.

First, let $\psi \in \Omega_1$, and let p_{mn} be defined by (9.15). Further, let

(9.17) $$\Delta_{mn}(t) = p_{mn}(t) - \phi_{mn}(t) = k_{m\infty} * f_n,$$

where

(9.18) $$f_n(t) = \left(h^{(q)}_{n+1,\infty} - h^{(q)}_{n,\infty}\right) * \psi \geq 0$$

by (9.13); thus (2.1) and (2.2) hold. Moreover, $(\mathrm{II})_{mn}$ holds, (say) by (9.4) and Lemma 3.4. That $(\mathrm{I})_{mn}$ holds follows, for instance, from Theorem 8.2, (see (9.15) and (8.10)). It remains to prove (2.4). By the definition of ψ_1,

$$\sum_{n=0}^{N-1} \Delta_{mn}(t) = \sum_{n=0}^{N-1} k_{m\infty} * f_n = k_{m\infty} * \left(h^{(q)}_{N\infty} - h^{(q)}_{0\infty}\right) * \psi$$

$$= h^{(q)}_{N\infty} * (k_{m\infty} * \psi) - k_{m\infty} * (\psi + \psi_1 - 1).$$

But by (9.10) and (9.11), the first term tends to the continuous function $k_{m\infty} * \psi$, as $N \to \infty$. Hence, (9.5) and (9.17) imply (9.16), which in turn implies (2.4). The last assertion of Theorem 9.3 follows from the properties $\phi_{m0}(t) > 0$ and $k_{m\infty}(t) > 0$ for $t > 0$.

As to the uniqueness of ψ (given $\{p_{mn}\}$), it follows from Lemma 4.3, (9.16) and the definition of ψ_1 that it suffices to prove that $\chi(t) = 0$ for $0 \leq t \leq T$ if $\chi(t)$ is a bounded and measurable function on $0 \leq t \leq T$ such that $\chi(t) = h^{(q)}_{0\infty}(t) * \chi(t)$ for $0 \leq t \leq T$. But this readily follows on interation of this equation, with the help of the conditions $h^{(q)}_{0\infty}(t) \geq 0$ and (9.10).

Conversely, let $\{p_{mn}\}$ be a system belonging to Ω. Further, let

(9.19) $$\Delta_{mn}(t) = p_{mn}(t) - \phi_{mn}(t).$$

By the second part of Theorem 7.2, there exist measurable functions $f_n(t)$ on $0 \leq t \leq T$ such that

(9.20) $$f_n(t) \geqq 0, \quad f(t) \leq 1,$$

where

$$f(t) = \sum_{n=0}^{\infty} f_n(t)$$

and such that, moreover,

$$\Delta_{mn}(t) = \int_0^t k_{m\infty}(t - \tau) f_n(\tau) d\tau \tag{9.21}$$

for all t $(0 \leq t \leq T)$ and all $m, n \geq 0$. Indeed, by the second part of Theorem 8.2,

$$\Delta_{mn}(t) = \int_0^t \left(h^{(q)}_{n+1,\infty}(\tau) - h^{(q)}_{n,\infty}(\tau)\right) g_m(t - \tau) d\tau \tag{9.22}$$

for all t $(0 \leq t \leq T)$ and all $m, n \geq 0$. Here, $g_m(t)$ is equal to the weak limit (in the sense of $L_1(0, T)$) of the sequence defined by (8.11). Consequently, by (9.20) and (9.21),

$$g_m = k_{m\infty} * \left(f + \mu_0 \int_0^t f_0(\tau) d\tau\right). \tag{9.23}$$

Now put

$$\psi(t) = f(t) + \mu_0 \int_0^t f_0(\tau) d\tau . \tag{9.24}$$

Then, in the first place, (9.22) and (9.23) imply the representation (9.15). Clearly, by (9.24), $\psi(t)$ is a nonnegative and bounded function on $0 \leq t \leq T$.

In the second place, comparing (9.15) and (9.21), we see from Lemma 4.3 that

$$\left(h^{(q)}_{n+1,\infty} - h^{(q)}_{n\infty}\right) * \psi = f_n \qquad (0 \leq t \leq T). \tag{9.25}$$

Hence, (9.20) implies (9.13). It remains to prove (9.14). By (9.20) and (9.24), this inequality is obvious when $\mu_0 = 0$, for then $h^{(q)}_{0\infty} = 0$. Thus suppose that $\mu_0 > 0$. Then, by (9.6) and (9.7),

$$\mu_0 \left(H^{(q)}_1(s) - H^{(q)}_0(s)\right) = s;$$

thus, by (9.9) and (9.25),

$$\mu_0 \int_0^t f_0(\tau) d\tau = \int_0^t h^{(q)}_{0\infty}(\tau) \psi(t - \tau) d\tau .$$

Hence, (9.20) and (9.24) imply (9.14).

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