CELLULARITY OF SETS IN PRODUCTS

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1. INTRODUCTION

There is no known factorization $R^n = X \times Y$ of euclidean n-space R^n in which neither factor is locally euclidean, although factorizations are known in which one factor fails to be locally euclidean (see [2] and [1]). There is a class of nonlocally euclidean spaces, which we call "pinched spaces" (see Section 5), and it seems likely that if X and Y are pinched spaces, then $X \times Y$ is euclidean space. We cannot show this, but, as a corollary to our main theorem, we have the conclusion that $X \times Y$ is a homotopy manifold.

The crucial question turns out to be whether certain sets are cellular (as defined by M. Brown [3]), and our main result is the following.

THEOREM 1. *Let* M^m *and* N^n *be combinatorial manifolds, and let* A *and* B *be absolute retracts in* Int M *and* Int N, *respectively. If* $\sup\{m\text{-dim } A, n\text{-dim } B\} \geq 2$, *then* $A \times B$ *is cellular in* $M \times N$. *In fact, if* $M \times N$ *is triangulated as a combinatorial manifold, then* $A \times B$ *is the intersection of combinatorial* $(m + n)$*-cells in* $M \times N$.

In the above context, $A \times B$ will be said to be *combinatorially cellular* in $M \times N$.

2. NESTED SEQUENCES OF MANIFOLDS

We collect here some results needed in proving Theorem 1.

(i) *Let* A *be an absolute retract in* Int M, *and let* U *be an open neighborhood of* A. *Then there exists a finite combinatorial manifold* H, *with nonempty boundary, such that*

$$A \subset \text{Int } H \subset H \subset U.$$

Such an H may be obtained as a small regular neighborhood of the closed simplicial neighborhood of A in a sufficiently fine subdivision of M.

(ii) *Let* $A \subset \text{Int } H$ *as in* (i). *Then there exists a neighborhood* V *of* A *such that* $V \subset \text{Int } H$ *and the inclusion* $i: V \to H$ *is null-homotopic.*

Since H is an absolute neighborhood retract, there exists an $\varepsilon > 0$ with the property that if f and g are maps of a space K into H such that $\rho(f(k), g(k)) < \varepsilon$ for each $k \in K$, then f and g are homotopic in H. Let r be a retraction of H onto A, and choose V to be an open set such that $A \subset V \subset \text{Int } H$ and $\rho(x, r(x)) < \varepsilon$ for each x in V. Since A is contractible, V is the required neighborhood of A.

(iii) *There exists a sequence* $\{H_i\}$ *of finite combinatorial* m*-manifolds, with nonempty boundaries, such that* $H_{i+1} \subset \text{Int } H_i$, $A = \bigcap_i H_i$, *and each inclusion* $H_{i+1} \to H_i$ *is homotopically trivial.* This follows immediately from (i) and (ii).

The following result is proved in [8].

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LEMMA 1. *Suppose* $M_1, M_2, \cdots, M_{k-r+1}$ *is a sequence of finite combinatorial* k-*manifolds such that each* M_i *is a combinatorial subspace of* M_{i+1} *and each inclusion* $M_i \to M_{i+1}$ *is homotopically trivial. If* Y *is a subcomplex of* M_1 *such that* $\dim Y \leq k - r - 1$ *and* $r \geq 2$, *then* Y *lies in a combinatorial* k-*cell in* M_{k-r+1}.

Returning to an absolute retract A in the interior of a combinatorial manifold M^m, we define a sequence $\{H_i\}$ as in (iii) to be a *special sequence for* A (*relative to* M) if it satisfies the following additional condition: if Y is a subcomplex of H_{i+1} and $\dim Y \leq m - 3$, then Y lies in a combinatorial m-cell in H_i. Using this terminology, we have established the following result.

LEMMA 2. *If* A *is an absolute retract in the interior of a combinatorial manifold* M^m, *then special sequences for* A *exist. Indeed, each nested sequence of* m-*manifolds closing down on* A *contains a special subsequence.*

For example, let $\{H_i\}$ be chosen as in (iii), and let $H_i' = H_{1+i(m-2)}$. By Lemma 1, $\{H_i'\}$ is a special sequence for A.

3. SPINES OF MANIFOLDS

By a *spine* of a combinatorial manifold M with boundary we mean a subcomplex K of M such that $M \downarrow K$, that is, such that M may be changed into K by a finite sequence of Whitehead elementary collapsings [9]. Thus M is a regular neighborhood of any spine of M. It is easy to see that an n-manifold with nonempty boundary has an (n - 1)-dimensional spine. Our next lemma concerns n-manifolds that have (n - 2)-dimensional spines.

LEMMA 3. *Let* A *be an absolute retract of dimension at most* n - 2 *in the interior of a combinatorial* n-*manifold* Q^n, *with* $\text{Bd}\, Q^n \neq \emptyset$. *Then there exists a combinatorial* n-*manifold* N *such that* $A \subset \text{Int}\, N \subset N \subset Q^n$ *and* N *has a spine of dimension* n - 2.

Proof. The result is obvious for $n \leq 2$. Suppose $n \geq 3$. Now Q^n has a spine that lies in its (n - 1)-skeleton and contains its (n - 2)-skeleton. A small regular neighborhood T of this spine will consist of a regular neighborhood of the (n - 2)-skeleton with an n-cell attached for each (n - 1)-simplex. The n-cell can be considered to be the (n - 1)-simplex, slightly thickened. It follows that, in some subdivision of Q^n, T contains a disjoint collection of arcs $\alpha_1, \cdots, \alpha_k$ such that each α_i has its end-points in Bd T and $\text{Int}\, \alpha_i \subset \text{Int}\, T$, and such that if the interior of a small regular neighborhood of each α_i is removed, then the resulting manifold has an (n - 2)-dimensional spine. By Whitehead's theorem [9] on uniqueness of regular neighborhoods, it follows that Q^n contains such a collection of arcs (here, and later, we permit ourselves to use the same notation for Q^n after subdivision).

The proof will be completed by showing that for i = 1, ⋯, k, there exists a piecewise linear homeomorphism h_i of Q^n onto Q^n which is the identity on Bd Q^n and outside an arbitrarily small neighborhood of α_i, and which has the property that $A \cap h_i(\alpha_i) = \emptyset$. For then the h_i can be pieced together in the obvious manner to obtain a piecewise linear homeomorphism h of Q^n onto Q^n which is fixed on Bd Q^n and has the property that

$$A \cap \bigcup_{i=1}^{k} h(\alpha_i) = \emptyset .$$

We then choose N to be Q^n minus the interior of a small regular neighborhood of each $h(\alpha_i)$.

Recall a special case of a definition in [10]. If β_1, β_2 are 1-cells contained as subcomplexes in Int M^n, we say that β_1 and β_2 *differ by a cellular move* across the 2-cell D if (Int D) $\cap (\beta_1 \cup \beta_2) = \emptyset$ and Bd D has $\overline{\beta_1 - \beta_2}, \overline{\beta_2 - \beta_1}$ as an equatorial decomposition. The proof of Lemma 3 of [10] reveals that if such a D exists, then there exists a piecewise linear homeomorphism of M^n onto M^n that throws β_1 onto β_2 and is fixed on $\overline{\beta_1 - D}$ and outside an arbitrarily small neighborhood of D. We now use this result to obtain h_i.

Subdivide Q^n so that a 1-cell β_i in Int Q^n contains $\alpha_i \cap A$ and $\beta_i \subset$ Int α_i. Let C_i be the closed star of α_i in the second barycentric subdivision of Q^n. Then C_i is an n-cell. Subdivide Q^n twice more barycentrically, and let C_i' be the closed star of β_i. Then C_i' is an n-cell, and

$$\alpha_i \cap A \subset \text{Int } C_i' \subset C_i' \subset \text{Int } C_i.$$

Now $\alpha_i \cap C_i'$ is a 1-cell that differs from a 1-cell γ_i in Bd C_i' by a move across a 2-cell in C_i'. The ends of γ_i may be joined by a 1-cell δ_i in Bd $C_i' - A$, since Bd C_i' is (n - 1)-dimensional and $H_{n-2}(A \cap \text{Bd } C_i'; Z) = 0$. Note that, since γ_i and δ_i both lie in Bd C_i', they differ by a move across a 2-cell of C_i'. Hence α_i and $[\alpha_i - \text{Int } C_i'] \cup \delta_i$ differ by two cellular moves. Thus, we can find the desired homeomorphism h_i throwing α_i onto $[\alpha_i - \text{Int } C_i'] \cup \delta_i$, where $h_i(\alpha_i) \cap A = \emptyset$. This completes the proof.

4. PROOF OF THEOREM 1

Let $K_1, K_2, \cdots$ be a special sequence for B (see Lemma 2) and $H_1, H_2, \cdots$ a special sequence for A. We may assume that dim $A \leq m - 2$, so that, by Lemma 3, each H_i collapses to an (m - 2)-dimensional subcomplex H_i' of H_i. By Lemma 2 there is no loss in generality if we assume that $H_1 \times K_1$, $H_2 \times K_2$, $\cdots$ is a special sequence for $A \times B$ relative to $M \times N$.

Let $M \times N$ be triangulated as a combinatorial (m + n)-manifold. We show that there exists a combinatorial (m + n)-cell $\triangle$ such that $A \times B \subset \triangle \subset H_i \times K_i$.

Since K_{i+1} has nonempty boundary, it collapses onto an (n - 1)-dimensional complex K_{i+1}'. Hence $H_{i+1} \times K_{i+1}$ collapses onto $H_{i+1}' \times K_{i+1}'$, which has dimension m + n - 3. By Lemma 1 there exists a combinatorial (m + n)-cell $\triangle'$ such that $H_{i+1}' \times K_{i+1}' \subset \triangle' \subset H_i \times K_i$. As was shown in Lemma 1 of [8], there exists a piecewise linear homeomorphism h of $M \times N$ onto itself, fixed outside of $H_i \times K_i$, such that $H_{i+1} \times K_{i+1} \subset h(\triangle') = \triangle$. This completes the proof.

5. AN APPLICATION

If A is a compact absolute retract in euclidean n-space R^n, then the quotient space R^n/A will be called a *pinched space*. If dim A = k, we shall call R^n/A an *(n - k)-pinched space*.

THEOREM 2. *If* X *and* Y *are* p-*pinched and* q-*pinched spaces, respectively, then* $X \times Y$ *is a homotopy manifold provided that either* p *or* q *is at least* 2.

Proof. If a set K of R^n is combinatorially cellular, then its complement is homeomorphic to the complement of a point. For we see that in the one-point compactification S^n of R^n, the complement of K is the union of open n-cells. Hence the complement is an open n-cell, by a theorem due to Brown [4]. Thus the complement of K in R^n is an open n-cell with one point removed.

Hence we may apply a theorem due to Kwun [7]. Kwun proved that if $f: S^n \to L$ is such that each $f^{-1}(x)$ has a complement homeomorphic to the complement of a point, then L is a homotopy manifold. By Theorem 1, the quotient space map of $R^n \times R^m$ onto $X \times Y$ has this property. (Note that the sets $A \times y$ and $x \times B$ are cellular by Theorem 1, since a point is an absolute retract.) Theorem 2 is proved.

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