A SIMPLIFIED PROOF OF THE PARTITION FORMULA

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We give a short proof of the celebrated Hardy-Ramanujan formula

$$p(n) \sim \frac{\exp \pi \sqrt{2n/3}}{4n\sqrt{3}},$$

where p(n) represents the number of ways in which n can be written as a sum of positive integers (see [1, p. 79]).

This proof, like the old one, utilizes the circle method, with the difference that we have only one major arc and one minor arc. Another difference is that we do not use any results on theta functions or modular functions. Of course, our error term suffers as a result; but we do obtain the asymptotic formula.

Suppose, as usual, that we define $p(0) = 1$ and write

$$f(z) = \sum_{n=0}^{\infty} p(n) z^n = \prod_{m=1}^{\infty} (1 - z^m)^{-1} \qquad (|z| < 1).$$

We also write

$$\phi(z) = \left(\frac{1 - z}{2\pi}\right)^{1/2} \exp\left[\frac{\pi^2}{12}\left(-1 + \frac{2}{1 - z}\right)\right].$$

The crux of our proof is the establishment of the two estimates

$$\text{(I)} \qquad |f(z)| < \exp\left(\frac{1}{1 - |z|} + \frac{1}{|1 - z|}\right),$$

$$\text{(II)} \qquad f(z) = \phi(z)[1 + O(1 - z)] \qquad (|z| < 1,\ |1 - z| \le 2(1 - |z|)).$$

Proof of (I). Taking logarithms, we obtain the well-known identity

$$\text{(1)} \qquad \log f(z) = \sum \frac{1}{n} \frac{z^n}{1 - z^n}.$$

Hence,

$$|\log f(z)| \le \frac{|z|}{|1 - z|} + \sum_{n=2}^{\infty} \frac{1}{n} \frac{|z|^n}{1 - |z|^n}$$

$$< \frac{1}{|1 - z|} + \frac{1}{1 - |z|} \sum_{n=2}^{\infty} \frac{1}{n^2} \frac{n}{|z|^{-1} + |z|^{-2} + \cdots + |z|^{-n}}$$

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$$< \frac{1}{|1 - z|} + \frac{1}{1 - |z|} \sum_{n=2}^{\infty} \frac{1}{n^2} < \frac{1}{|1 - z|} + \frac{1}{1 - |z|},$$

and (I) follows directly.

Proof of (II). Set $z = e^{-w}$, where w is chosen so that $|\Im w| \leq \pi$. Since z is restricted to the region $|1 - z| \leq 2(1 - |z|)$, there exists a K such that $|\arg w| < K < \pi/2$. In terms of w, the identity (1) becomes

$$\log f(z) = \sum \frac{1}{n(e^{nw} - 1)}.$$

Adding and subtracting $\frac{1}{2} \log(1 - e^{-w}) + \pi^2/6w$ on the right-hand side, we obtain the formula

$$\log f(z) = \frac{\pi^2}{6w} + \frac{1}{2} \log(1 - e^{-w}) + w \sum \left(\frac{1}{nw(e^{nw} - 1)} - \frac{1}{n^2 w^2} + \frac{e^{-nw}}{2nw} \right).$$

Now (see [3, p. 37]), if $t > 0$ and $g(u)$ has total variation V on $[0, \infty))$, then

$$\left| t \sum g(nt) - \int_0^{\infty} g(u)\, du \right| \leq tV.$$

By rotation, the same inequality can be asserted for any ray L from the origin; namely, if $w \in L$ and g is of bounded variation on L, then

$$|w \sum g(nw) - \int_L g(u)\, du| \leq |w|\, V_L, \tag{2}$$

where V_L is the total variation of g on L. Apply this conclusion to the function

$$g(u) = \frac{1}{u(e^u - 1)} - \frac{1}{u^2} + \frac{e^{-u}}{2u}$$

and the ray L on which $\arg u = \arg w$. First of all, by Cauchy's Theorem,

$$\int_L g(u)\, du = \int_0^{\infty} g(u)\, du,$$

and (see [2])

$$\int_0^{\infty} g(u)\, du = -\frac{1}{2} \log 2\pi. \tag{3}$$

Next,

$$V_L = \int_L |g'(u)| \cdot |du|,$$

and there exists a positive number M' such that, for $|\arg u| < K$,

$$|g'(u)| \leq M'/|u|^3.$$

Also, $g(u)$ is analytic for $|u| < 2\pi$, and thus there exists a constant M'' such that $|g'(u)| \leq M''$ for $|u| < 1$. Therefore, V_L is bounded provided $|\arg w| < K$. This fact and the results (2) and (3) yield the estimate

$$w \sum g(nw) = -\frac{1}{2} \log 2\pi + O(w).$$

Returning to $\log f(z)$, we find that

$$\begin{aligned} \log f(z) &= \frac{\pi^2}{6w} + \frac{1}{2} \log \frac{1 - e^{-w}}{2\pi} + O(w) \\ &= \frac{\pi^2}{6(1 - z)} - \frac{\pi^2}{12} + \frac{1}{2} \log \frac{1 - z}{2\pi} + O(1 - z) \\ &= \log \phi(z) + O(1 - z). \end{aligned}$$

This completes the proof of (II).

We now write $\phi(z) = \Sigma\, q(n) z^n$ and prove

(III) $$p(n) = q(n) + O(n^{-5/4} \exp \pi\sqrt{2n/3}).$$

This will reduce the problem simply to the computation of the coefficients of the elementary function $\phi(z)$.

Proof of (III). Clearly

$$p(n) - q(n) = \frac{1}{2\pi i} \int_C \frac{f(z) - \phi(z)}{z^{n+1}} dz,$$

where C is the circle $|z| = 1 - \pi/\sqrt{6n}$. We split C into

$$A = \{z \in C: |1 - z| < \pi\sqrt{2/3n}\}$$

and $B = C - A$. By (I), we obtain the estimate

(4) $$\begin{aligned} \int_B \frac{f(z) - \phi(z)}{z^{n+1}} dz &= O\left(\int_B |z|^{-n} \left\{\exp \frac{\pi^2}{6|1 - z|} + \exp[|1 - z|^{-1} + (1 - |z|)^{-1}]\right\} |dz|\right) \\ &= O\left(\int_B \exp \pi\sqrt{n/6} \left\{\exp \frac{\pi}{6}\sqrt{3n/2} + \exp \frac{1}{\pi}[\sqrt{3n/2} + \sqrt{6n}]\right\} |dz|\right) \\ &= O(\exp a\sqrt{n}), \end{aligned}$$

where $a < \pi\sqrt{2/3}$. On the other hand, the estimate (II) is applicable on A. Also, the length of A is $O(n^{-1/2})$. Thus,

$$\int_A \frac{f(z) - \phi(z)}{z^{n+1}} dz = O\left(\int_A |z|^{-n} |1 - z|^{3/2} \exp\left(\frac{\pi}{6} \cdot \frac{1}{1 - |z|}\right) |dz|\right)$$

$$(5) \qquad = O\left(\frac{\exp \pi\sqrt{n/6}}{\exp(-\pi\sqrt{n/6})} \cdot n^{-3/4}\right) \cdot O(n^{-1/2})$$

$$= O(n^{-5/4} \exp \pi\sqrt{2/3n}) .$$

The estimates (4) and (5) imply (III).

Finally, we compute the $q(n)$. To begin with, we have the identity

$$\pi\sqrt{2}\, e^{\pi^2/12} \phi(z) = (1 - z) \int_{-\infty}^{\infty} \exp[\pi t\sqrt{2/3} - (1 - z)t^2]\, dt .$$

The power series in z of the right-hand member must be identical with $\pi\sqrt{2} \exp(\pi^2/12) \Sigma\, q(n) z^n$. Therefore,

$$\pi\sqrt{2}\, e^{\pi^2/12} q(n) = \int_{-\infty}^{\infty} \exp[\pi t\sqrt{2/3} - t^2] \left(\frac{t^{2n}}{n!} - \frac{t^{2n-2}}{(n-1)!}\right) dt$$

$$\sim \frac{\exp \pi\sqrt{2/3n}}{\sqrt{2\pi}\, n} \int_{-\infty}^{\infty} s \exp\{\pi\sqrt{2/3}\, s - s^2 - 2\sqrt{n}\, s\} \left(1 + \frac{s}{\sqrt{n}}\right)^{2n-2} \left(2 + \frac{s}{\sqrt{n}}\right) ds ,$$

where we have set $t = s + \sqrt{n}$ and have used Stirling's formula for $n!$. Now

$$\lim_{n \to \infty} e^{-2\sqrt{n}s} \left(1 + \frac{s}{\sqrt{n}}\right)^{2n-2} \left(2 + \frac{s}{\sqrt{n}}\right) = 2e^{-s^2} .$$

Moreover, the integrand is dominated by the function

$$F(s) = \begin{cases} s \exp\{\pi\sqrt{2/3}\, s - s^2\} (2 + s) & (s \geq 0) , \\ |s| \exp\{\pi\sqrt{2/3}\, s - s^2\} (2 - s) \exp(s^2 + 1) & (s \leq 0) . \end{cases}$$

The integral $\int_{-\infty}^{\infty} F(s)\, ds$ converges. Therefore, by the theorem on dominated convergence, the limit may be taken under the integral sign, and we conclude that

$$\pi\sqrt{2}\, e^{\pi^2/12} q(n) \sim \frac{\exp \pi\sqrt{2n/3}}{\sqrt{2\pi}\, n} \int_{-\infty}^{\infty} 2s \exp(\pi\sqrt{2/3}\, s - 2s^2)\, ds$$

$$= \frac{\pi}{2\sqrt{6}\, n} \exp(\pi^2/12 + \pi\sqrt{2n/3}) ,$$

or

$$q(n) \sim \frac{\exp \pi\sqrt{2n/3}}{4n\sqrt{3}}.$$

The proof is now complete.

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