ON THE COEFFICIENTS OF CLOSE-TO-CONVEX FUNCTIONS

Ch. Pommerenke

1. Let K_α $(0 \leq \alpha \leq 1)$ be the class of all functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

analytic in $|z| < 1$ that satisfy $f'(z) \neq 0$ and

$$\int_{\theta_1}^{\theta_2} \frac{\partial}{\partial\theta} \arg[e^{i\theta} f'(re^{i\theta})]\, d\theta > -\pi\alpha$$

for all $\theta_1 < \theta_2$ and $0 \leq r < 1$. The class K_1 is the class of close-to-convex functions [4], and the classes K_α are subclasses of K_1. Hence all functions $f \in K_\alpha$ $(0 \leq \alpha \leq 1)$ are univalent. The class K_0 consists of the convex functions. A function of the form (1) belongs to K_α if and only if there exists a function

$$g(z) = \sum_{n=1}^{\infty} b_n z^n, \tag{2}$$

starlike in $|z| < 1$, such that (see [4] and [12])

$$\left|\arg \frac{zf'(z)}{g(z)}\right| < \frac{\pi}{2}\alpha .$$

M. O. Reade [12] has proved that

$$|a_n| \leq 1 - \alpha + n\alpha . \tag{3}$$

For $\alpha = 0$ and $\alpha = 1$, this reduces to the sharp inequalities

$$|a_n| \leq 1 \qquad (f \in K_0) \tag{4}$$

and [11]

$$|a_n| \leq n \qquad (f \in K_1). \tag{5}$$

For $n = 2$, inequality (3) is best possible for every α. On the other hand, it will be shown that

$$a_n = O(n^\alpha) \qquad (n \to \infty).$$

For $0 < \alpha < 1$ and large n, this estimate is better than (3). For a function f of boundary rotation not greater than $2\pi + \pi\alpha$ (which implies $f \in K_\alpha$), Rényi has proved that $|a_n| \leq n^\alpha$.

Received January 30, 1962.

More generally, we shall consider m-fold symmetric functions of class K_α (by definition, every function is 1-fold symmetric). We shall first derive estimates for the length $L(r)$ of the image curve of $|z| = r$, from which estimates of the coefficients will follow.

THEOREM 1. *Let*

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}$$

be a function of class K_α $(m = 1, 2, \cdots;\ 0 \leq \alpha \leq 1)$, *and let*

$$L(r) = r \int_0^{2\pi} |f'(re^{i\theta})|\, d\theta \qquad (0 \leq r < 1).$$

If $\alpha + \frac{2}{m} > 1$ *then, as* $r \to 1$,

$$L(r) \leq \left\{ 2^{2-2/m} \pi \Gamma\left(\alpha + \frac{2}{m} - 1\right) \left[\Gamma\left(\frac{\alpha}{2} + \frac{1}{m}\right)\right]^{-2} + o(1) \right\} (1 - r^m)^{1-\alpha-2/m}, \tag{6}$$

and if $\alpha + \frac{2}{m} = 1$, *then*

$$L(r) \leq (2^{2-2/m} + o(1)) \log \frac{1}{1-r}. \tag{7}$$

Hence for $\alpha + \frac{2}{m} > 1$, *as* $n \to \infty$,

$$|a_n| \leq [A_m(\alpha) + o(1)]\, n^{\alpha - 2 + 2/m}, \tag{8}$$

with

$$A_m(\alpha) \leq 2^{1-2/m} \Gamma\left(\alpha + \frac{2}{m} - 1\right) \left[\Gamma\left(\frac{\alpha}{2} + \frac{1}{m}\right)\right]^{-2} [e/(m\alpha + 2 - m)]^{\alpha - 2 + 2/m},$$

and for $\alpha + \frac{2}{m} = 1$,

$$|a_n| = O(n^{-1} \log n). \tag{9}$$

Remarks. 1. It will be shown later that inequalities (6) and (7) are best possible. The exponent of n in (8) is best possible, but certainly not the given upper bound for $A_m(\alpha)$. In equation (9), the correct order of magnitude of a_n is probably $O(n^{-1})$.

2. For $m = 1$, inequalities (6) and (8) become

$$L(r) \leq \left\{ \pi\, \Gamma(\alpha + 1) \left[\Gamma\left(\frac{\alpha}{2} + 1\right)\right]^{-2} + o(1) \right\} (1 - r)^{-\alpha - 1}$$

and

(10) $$|a_n| \le [A_1(\alpha) + o(1)]n^{\alpha},$$

with

$$A_1(\alpha) \le \frac{1}{2}\Gamma(\alpha + 1)\left[\Gamma\left(\frac{\alpha}{2} + 1\right)\right]^{-2}[e/(\alpha + 1)]^{\alpha+1}.$$

This gives $A_1(0) \le e/2$ and $A_1(1) \le e^2/(2\pi)$. Hence (10) is less sharp than (4) and (5). For $\alpha = 1$, we shall later find the best possible bound $A_m(1) = m^{1-2/m}[\Gamma(2/m)]^{-1}$ (Theorem 3).

3. It is not difficult to show that under the assumptions of Theorem 1

(11) $$M(r) = \max_{|z|=r} |f(z)| = O((1 - r)^{1-\alpha-2/m})$$

if $\alpha + \frac{2}{m} > 1$. Since $f(z)$ is univalent, we could then appeal to the following general result: If $f(z)$ is univalent and if $\beta > \frac{1}{2}$, then

(12) $$M(r) = O((1 - r)^{-\beta}) \Rightarrow a_n = O(n^{\beta-1}),$$

[3, p. 46]. Thus (11) implies that $a_n = O(n^{\alpha-2+2/m})$ if $\alpha + \frac{2}{m} > \frac{3}{2}$. But this proof breaks down for $\frac{3}{2} \ge \alpha + \frac{2}{m} > 1$. Indeed, J. E. Littlewood [6] has shown that there exist a positive σ, an m_0, and a bounded m_0-fold symmetric function whose coefficients satisfy $|a_n| > n^{\sigma}$ for infinitely many n. Hence (12) does not hold for small β, and (8) is not trivial. (If $f(z)$ is starlike, then (12) holds for all $\beta \ge 0$ [10]. It is an interesting question whether this is also true for all close-to-convex functions.)

In Theorem 1 we considered the case $\alpha + \frac{2}{m} \ge 1$. The case $\alpha + \frac{2}{m} < 1$ is entirely different if $\alpha > 0$. Instead of (8) we only have $a_n = o(n^{-1})$, and this inequality cannot be improved.

THEOREM 2. *Let*

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}$$

be a function of class K_α $(m = 3, 4, \cdots;\ 0 \le \alpha < 1)$, *and let* $\alpha + \frac{2}{m} < 1$. *Then* $f'(z)$ *belongs to the Hardy class* H_γ *for* $\gamma < 1/\left(a + \frac{2}{m}\right)$. *Hence* $f(z)$ *is continuous in* $|z| \le 1$, *maps* $|z| < 1$ *onto a domain with rectifiable boundary, and satisfies*

(13) $$a_n = o(n^{-1}).$$

If also $1 \le \gamma \le 2$, *then*

(14) $$\sum_{n=1}^{\infty} n^{2\gamma-2}|a_n|^{\gamma} < \infty.$$

2. To prove our first two theorems, we need two lemmas.

LEMMA 1. *Let* $g(z) = b_1 z + \cdots$ *be starlike in* $|z| < 1$ *and* m*-fold symmetric* $(m = 1, 2, \cdots)$. *Then for every* $\lambda > 0$

$$r^{-\lambda} \int_0^{2\pi} |g(re^{i\theta})|^\lambda d\theta \leq \int_0^{2\pi} |1 - r^m e^{it}|^{-2\lambda/m} dt. \tag{15}$$

Proof. Golusin [2] has proved that

$$\frac{g(z)}{z} \prec b_1 (1 - z)^{-2/m}, \tag{16}$$

where $\prec$ denotes subordination [7, p. 163]. Since $z^{-1} g(z) = b_1 + b_{m+1} z^m + \cdots$, a slight generalization [9, Hilfssatz 5] of a theorem of Littlewood [7, p. 165] on subordination shows that (16) implies (15).

LEMMA 2. *As* $\rho \to 1$,

$$\frac{1}{2\pi}\int_0^{2\pi} |1 - \rho e^{it}|^{-\lambda} dt \sim \begin{cases} 2^{-\lambda+1}\Gamma(\lambda - 1)[\Gamma(\lambda/2)]^{-2} \dfrac{1}{(1-\rho)^{\lambda-1}} & \text{for } \lambda > 1, \\ \pi^{-1} \log \dfrac{1}{1-\rho} & \text{for } \lambda = 1. \end{cases}$$

For $\lambda < 1$, *the integral remains bounded in* $0 \leq \rho < 1$.

Proof. Let

$$\begin{aligned} \Phi(\rho) &= \frac{1}{2\pi}\int_0^{2\pi} |1 - \rho e^{it}|^{-\lambda} dt = \frac{1}{2\pi}\int_0^{2\pi} |(1 - \rho e^{it})^{-\lambda/2}|^2 dt \\ &= \sum_{n=0}^{\infty} \binom{\lambda/2 + n - 1}{n}^2 \rho^{2n}. \end{aligned} \tag{17}$$

Also, let

$$\begin{aligned} \Psi(\rho) &= \frac{1}{(1-\rho^2)^{\lambda-1}} = \sum_{n=0}^{\infty} \binom{\lambda + n - 2}{n} \rho^{2n} \qquad \text{for } \lambda > 1, \\ \Psi(\rho) &= \log \frac{1}{1-\rho^2} = \sum_{1}^{\infty} \frac{\rho^{2n}}{n} \qquad \text{for } \lambda = 1. \end{aligned} \tag{18}$$

These functions have positive even-numbered coefficients, and the series diverge for $\rho = 1$. Hence [8, vol. I, p. 14], for $\lambda > 1$,

$$\lim_{\rho\to 1} \Phi(\rho)/\Psi(\rho) = \lim_{n\to\infty} \binom{\lambda/2 + n - 1}{n}^2 \binom{\lambda + n - 2}{n}^{-1} = \Gamma(\lambda - 1)\left[\Gamma\left(\frac{\lambda}{2}\right)\right]^{-2},$$

where we have used the relation

$$\binom{\gamma + n - 1}{n} \sim \frac{1}{\Gamma(\gamma)} n^{\gamma-1} \qquad (n \to \infty), \tag{19}$$

(valid for $\gamma > 0$). For $\lambda = 1$ we find that

$$\lim_{\rho\to 1} \Phi(\rho)/\Psi(\rho) = \lim_{n\to\infty} n\binom{n - 1/2}{n}^2 = \frac{1}{\pi}.$$

The first part of the lemma now follows from (17) and (18). The statement about the case $\lambda < 1$ follows at once from (17) and (19).

Proof of Theorems 1 and 2. a) An inspection of Kaplan's proof [4, Theorem 2] shows that we can choose the starlike function (2) to be m-fold symmetric. Also, we may assume that $|b_1| = 1$. Then

$$F(z) = \frac{zf'(z)}{g(z)} = \bar{b}_1 + c_m z^m + \cdots. \tag{20}$$

Since $|\arg F(z)| \leq \frac{\pi}{2}\alpha$, we obtain

$$F(z) \prec \left(\frac{e^{i\beta} + e^{-i\beta} z}{1 - z}\right)^{\alpha}$$

with $\beta = \alpha^{-1} \arg \bar{b}_1$. Because F(z) has the form (20), the version of Littlewood's theorem mentioned in the proof of lemma 1, shows that

$$\int_0^{2\pi} |F(re^{i\theta})|^{\lambda} d\theta \leq \int_0^{2\pi} \left|\frac{e^{i\beta} + e^{-i\beta} r^m e^{it}}{1 - r^m e^{it}}\right|^{\alpha\lambda} dt \tag{21}$$

for $\lambda > 0$.

b) Let $\gamma \geq 1$. We apply Hölder's inequality with $p = 1 + \frac{\alpha m}{2}$ and $q = 1 + \frac{2}{\alpha m}$ (with the usual interpretation if $p = 1$ and $q = \infty$). We find that

$$\int_0^{2\pi} |f'(re^{i\theta})|^{\gamma} d\theta = \int_0^{2\pi} |r^{-1} g(re^{i\theta})|^{\gamma} |F(re^{i\theta})|^{\gamma} d\theta$$

$$\leq \left(\int_0^{2\pi} |r^{-1} g(re^{i\theta})|^{\gamma p} d\theta\right)^{1/p} \left(\int_0^{2\pi} |F(re^{i\theta})|^{\gamma q} d\theta\right)^{1/q}.$$

Because of Lemma 1 and (21), this is not greater than

$$\left(\int_0^{2\pi} |1 - r^m e^{it}|^{-2\gamma p/m} dt\right)^{1/p} \left(2^{\alpha\gamma q} \int_0^{2\pi} |1 - r^m e^{it}|^{-\alpha\gamma q} dt\right)^{1/q}.$$

Since $2\gamma p/m = \alpha\gamma q = \gamma\left(\alpha + \frac{2}{m}\right)$, it follows that

(22) $$\int_0^{2\pi} |f'(re^{i\theta})|^\gamma d\theta \le 2^{\alpha\gamma} \int_0^{2\pi} |1 - r^m e^{it}|^{-\gamma(\alpha+2/m)} dt.$$

c) Let first $\alpha + 2/m \geq 1$. We take $\gamma = 1$. Then (6) and (7) follow from (22) and Lemma 2 (with $\lambda = \alpha + 2/m$). To prove the estimates for the coefficients, we use the easily established inequality

$$|a_n| \le \frac{L(r)}{2\pi n r^n}.$$

In the case $\alpha + 2/m > 1$, we take $r_n = (n/(n+\xi))^{1/m}$, where $\xi = m(\alpha + 2/m - 1) > 0$. Then, as $n \to \infty$,

$$\frac{1}{nr_n^n\left(1 - r_n^m\right)^{\xi/m}} = \frac{1}{n}\left(\frac{n+\xi}{n}\right)^{n/m}\left(\frac{n+\xi}{\xi}\right)^{\xi/m} \sim e^{\xi/m}\,\xi^{-\xi/m}\,n^{\xi/m-1}.$$

Therefore (8) follows from (6). In the case $\alpha + 2/m = 1$, we take $r_n = 1 - 1/n$ and apply (7). This completes the proof of Theorem 1.

d) Let $\alpha + \frac{2}{m} < 1$. If $\gamma < 1/\left(\alpha + \frac{2}{m}\right)$, then (22) and Lemma 2 show that

$$\int_0^{2\pi} |f'(re^{i\theta})|^\gamma d\theta$$

remains bounded in $0 \le r < 1$. Hence $f'(z)$ belongs to the Hardy class H_γ. If $1 \le \gamma \le 2$, then

$$f'(z) = \sum_{n=1}^{\infty} na_n z^{n-1} \in H_\gamma$$

implies

$$\sum_{n=1}^{\infty} (n|a_n|)^\gamma n^{\gamma-2} < \infty.$$

[15, vol. II, p. 110]. The other assertions of Theorem 2 follow from the fact that $f' \in H_1$ and from well-known properties of this class [15, Vol. I, p. 285].

3. We shall now study the question how far Theorems 1 and 2 can be improved.

a) Let $\alpha + 2/m \ge 1$. The function

(23) $$f(z) = \int_0^z (1+\zeta^m)^\alpha (1-\zeta^m)^{-\alpha-2/m} d\zeta$$

is m-fold symmetric and belongs to K_α (with $g(z) = z(1 - z^m)^{-2/m}$ and $F(z) = (1 + z^m)^\alpha (1 - z^m)^{-\alpha}$). It is easy to show that

$$L(r) = r \int_0^{2\pi} \frac{|1 + r^m e^{im\theta}|^\alpha}{|1 - r^m e^{im\theta}|^{\alpha+2/m}} d\theta \sim \int_0^{2\pi} \frac{2^\alpha}{|1 - r^m e^{it}|^{\alpha+2/m}} dt$$

as $r \to 1$. Hence Lemma 2 shows that equality holds in (6) or (7) for the function (23). Also, it can be shown that the coefficients of (23) satisfy

$$a_n \sim 2^\alpha m^{1-\alpha-2/m} \left[\Gamma\left(\alpha + \frac{2}{m}\right)\right]^{-1} n^{\alpha-2+2/m} \qquad (n \equiv 1 \pmod m),\ n \to \infty).$$

Hence the exponent of n in (8) cannot be replaced by a smaller one.

As an example, take $\alpha = \frac{1}{2}$, $m = 1$. Then the function (23) satisfies

$$a_n \sim 2^{5/2} 3^{-1} \pi^{-1/2} \sqrt{n} \qquad (2^{5/2} 3^{-1} \pi^{-1/2} \approx 1.064),$$

whereas (8) gives

$$|a_n| \leq \frac{\pi^{1/2} e^{3/2} 6^{1/2}}{18\left[\Gamma\left(1 + \frac{1}{4}\right)\right]^2} \sqrt{n} + o(\sqrt{n}) < 1.316\sqrt{n}$$

for large n.

b) Let $\alpha > 0$, $\alpha + 2/m < 1$. We shall prove that the estimate $a_n = o(n^{-1})$ cannot be improved. Let $\{\eta_n\}$ be any sequence with $\eta_n > 0$ and $n\eta_n \to 0$. We choose a subsequence $\{\eta_{n_k}\}$ with $n_k \equiv 1 \pmod m$ and $n_k > 1$ such that

$$\sum_{k=1}^{\infty} n_k \eta_{n_k} \leq \sin\frac{\pi}{2}\alpha. \tag{24}$$

Then the function

$$f(z) = z + \sum_{k=1}^{\infty} \eta_{n_k} z^{n_k}$$

is m-fold symmetric and satisfies

$$|f'(z) - 1| \leq \sum_{k=1}^{\infty} n_k \eta_{n_k} \leq \sin\frac{\pi}{2}\alpha$$

because of (24). Hence $zf'(z)/z = F(z)$ with $|\arg F(z)| \leq \frac{\pi}{2}\alpha$, so that $f \in K_\alpha$. Since (14) implies that $a_n = o(n^{2/\gamma-2})$, it also follows that (14) does not always hold for $\gamma > 2$.

4. If $f \in K_0$, that is, if f(z) is convex, and if $m > 2$, then the estimate $a_n = o(n^{-1})$ is no longer best possible. This follows from a theorem of Waadeland [14], who proved that every starlike m-fold symmetric function

$$g(z) = z + \sum_{k=1}^{\infty} b_{mk+1} z^{mk+1}$$

satisfies

$$|b_{mk+1}| \leq \binom{2/m + k - 1}{k}. \tag{25}$$

Since $g(z) = zf'(z)$ is starlike if $f(z)$ is convex, it follows from (25) that

$$|a_{mk+1}| \leq \frac{1}{mk+1}\binom{2/m + k - 1}{k} \sim \frac{1}{m\Gamma(2/m)} k^{2/m-2} \tag{26}$$

for every m-fold symmetric convex function. For $\alpha + 2/m < 1$, there is thus a surprising discontinuity between the case $\alpha > 0$, where only $a_n = o(n^{-1})$ holds, and the case $\alpha = 0$, where (26) holds.

We can easily generalize Waadeland's inequality (25) to obtain the sharp bounds for the coefficients of $f \in K_1$.

THEOREM 3. *Let* $f(z) = z + \cdots$ *be close-to-convex and* m*-fold symmetric in* $|z| < 1$. *Then*

$$|a_{mk+1}| \leq \binom{2/m + k - 1}{k} \sim \frac{1}{\Gamma(2/m)} k^{2/m-1}. \tag{27}$$

This inequality is best possible.

Proof. We may assume that $F(z) = zf'(z)/g(z)$ with

$$g(z) = \sum_{k=0}^{\infty} b_{mk+1} z^{mk+1}, \qquad F(z) = \sum_{k=0}^{\infty} c_{mk} z^{mk},$$

and that $|b_1| = 1$ and $|c_0| = 1$. Then we find

$$(nk+1) a_{nk+1} = \sum_{j=0}^{k} b_{mj+1} c_{m(k-j)}.$$

Since $|c_0| = 1$ and $\Re F(z) > 0$, it follows that $|c_{m\nu}| \leq 2$ for $\nu \geq 1$, hence that

$$(mk+1)|a_{mk+1}| \leq 2 \sum_{j=0}^{k-1} |b_{mj+1}| + |b_{mk+1}|. \tag{28}$$

From (25) we obtain

$$(mk+1)|a_{mk+1}| \leq 2 \sum_{j=0}^{k-1} \binom{2/m + j - 1}{j} + \binom{2/m + k - 1}{k} = (nk+1)\binom{2/m + k}{k+1}.$$

For the starlike function $f(z) = z(1 - z^m)^{2/m}$, we have equality in (27).

5. We can introduce classes of mappings of $|z| > 1$ onto domains containing ∞ that are analogous to the classes K_α. Let K_α^* denote the class of all functions

$$f(z) = z + \sum_{n=0}^{\infty} a_n z^{-n},$$

analytic in $1 < |z| < \infty$, for which there exists a function $g(z) = bz + b_0 + \cdots$, starlike in $|z| > 1$, such that

$$\left|\arg \frac{zf'(z)}{g(z)}\right| \leq \frac{\pi}{2}\alpha . \tag{29}$$

It is not difficult to show that the functions in K_α^* need not be univalent if $\alpha > 0$. The class K_α^* is again the class of convex functions (hence $f \in K_\alpha^*$ is univalent), and K_1^* is the class of functions close-to-convex in $|z| > 1$, introduced by Libera and Robertson [5] and also by the author [9]. It is known that

$$a_n = O(n^{-2}) \qquad (f \in K_0^*) \tag{30}$$

(this follows from a result of Clunie [1] and the fact that $zf'(z)$ is starlike if $f(z)$ is convex), and that

$$a_n = O(n^{-1}) \qquad (f \in K_1^*)$$

([9, Satz 3]; if $f(z)$ is univalent, the proposition also follows from [5]).

THEOREM 4. *Let* $f \in K_\alpha^*$ *and* $0 < \alpha < 1$. *Then* $f'(\zeta^{-1})$ *belongs to the Hardy class* H_γ, *if* $\gamma < 1/\alpha$. *Hence* $f(z)$ *is continuous on* $|z| = 1$, *maps* $|z| > 1$ *onto a domain with rectifiable boundary, and satisfies*

$$a_n = o(n^{-1}). \tag{31}$$

If also $1 \leq \gamma \leq 2$, *then*

$$\sum_1^{\infty} n^{2\gamma-2} |a_n|^\gamma < \infty.$$

Proof. Let $F(\zeta) = \zeta^{-1} f'(\zeta^{-1})/g(\zeta^{-1})$ $(|\zeta| < 1)$. Then, for $\rho < 1$,

$$\int_0^{2\pi} |f'(\rho^{-1} e^{-it})|^\gamma dt = \rho^\gamma \int_0^{2\pi} |F(\rho^{-1} e^{-it})|^\gamma |g(\rho^{-1} e^{-it})|^\gamma dt.$$

We may assume that $|b| = 1$, $|F(0)| = 1$. Since $g(z)$ is univalent in $|z| > 1$, it follows that $|g(\rho^{-1} e^{-it})| \leq \rho^{-1} + 2 + \rho$ [8, Vol. II, p. 25]. Using (29), we find that

$$\int_0^{2\pi} |f'(\rho^{-1} e^{-it})|^\gamma dt \leq \rho^\lambda (\rho^{-1} + 2 + \rho)^\gamma \int_0^{2\pi} \frac{(1+\rho)^{\alpha\gamma}}{|1 - \rho e^{-it}|^{\alpha\gamma}} dt.$$

Since $\alpha\gamma < 1$, Lemma 2 shows that this expression remains bounded as $\rho \to 1$, and Theorem 4 follows as in the proof of Theorem 2.

We shall show that (31) is best possible, even in the class of univalent functions in K_α^* $(0 < \alpha < 1)$. Given $\{\eta_n\}$ with $n\eta_n \to 0$, we again choose $\{n_k\}$ so that

$$\sum_{k=1}^{\infty} n_k \eta_{n_k} \leq \sin \frac{\pi}{2} \alpha < 1. \tag{32}$$

Then the function

$$f(z) = z + \sum_{k=1}^{\infty} \eta_{n_k} z^{-n_k}$$

belongs to K_α^*. From (32) it follows that f(z) is starlike and univalent [9, Hilfssatz 4]. Again we notice the discontinuity between the case $\alpha = 0$, where (30) holds, and the case $0 < \alpha < 1$, where only (31) holds.

REFERENCES

1. J. Clunie, *On meromorphic schlicht functions*, J. London Math. Soc. 34 (1959), 215-216.

2. G. M. Golusin, *Einige Koeffizientenabschätzungen für schlichte Funktionen*, Mat. Sb. N.S. 3 (45) (1938), 321-330 (Russian, German summary).

3. W. K. Hayman, *Multivalent functions*, Cambridge University Press, Cambridge, 1958.

4. W. Kaplan, *Close-to-convex schlicht functions*, Michigan Math. J. 1 (1952), 169-185.

5. R. J. Libera and M. S. Robertson, *Meromorphic close-to-convex functions*, Michigan Math. J. 8 (1961), 167-175.

6. J. E. Littlewood, *On the coefficients of schlicht functions*, Quart. J. Math. 9 (1938), 14-20.

7. ———, *Lectures on the theory of functions*, Oxford University Press, Oxford, 1944.

8. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, J. Springer, Berlin, 1925.

9. Ch. Pommerenke, *Über einige Klassen meromorpher schlichter Funktionen*, Math. Z. (to appear).

10. ———, *On starlike and convex functions*, J. London Math. Soc. (to appear).

11. M. O. Reade, *On close-to-convex univalent functions*, Michigan Math. J. 3 (1955-56), 59-62.

12. ———, *The coefficients of close-to-convex functions*, Duke Math. J. 23 (1956), 459-462.

13. A. Rényi, *On the coefficients of schlicht functions*, Publ. Math. Debrecen 1 (1949-50), 18-23.

14. H. Waadeland, *Über* k-*fach symmetrische sternförmige schlichte Abbildungen des Einheitskreises,* Math. Scand. 3 (1955), 150-154.

15. A. Zygmund, *Trigonometric series*, Second Edition, Cambridge University Press, Cambridge, 1959.

The University of Michigan