

# IMAGES OF CONVEX DOMAINS UNDER CONVEX CONFORMAL MAPPINGS

Ch. Pommerenke

Let  $w = f(z)$  be a convex mapping of  $|z| < 1$ ; that is, let  $f(z)$  map  $|z| < 1$  conformally and one-to-one onto a convex domain. Study [1] has provided that  $f(z)$  maps every disk in  $|z| < 1$  onto a convex domain.

A circle in  $|z| \leq 1$  that touches  $|z| = 1$  is called an *oricycle*. Through each pair of points in  $|z| < 1$  there pass exactly two oricycles.

**THEOREM.** *A convex set  $C$  in  $|z| < 1$  is mapped onto a convex set by every convex mapping of  $|z| < 1$  if and only if, for each pair of points  $z_1$  and  $z_2$  in  $C$ , the arcs between  $z_1$  and  $z_2$  of the two oricycles through  $z_1$  and  $z_2$  also belong to  $C$ .*

*Proof.* 1. Let  $C$  be a convex set in  $|z| < 1$  that has the property just stated, and let  $z_1$  and  $z_2$  be two points in  $C$ . Let  $Z_1$  and  $Z_2$  be the two oricycles passing through both  $z_1$  and  $z_2$ . Let  $K_1$  and  $K_2$  be the closed interiors of  $Z_1$  and  $Z_2$ , and let  $K = K_1 \cap K_2$ . Then the boundary of  $K$  belongs to  $C$ , and therefore  $K \subset C$ .

Let  $K_1^*$ ,  $K_2^*$ ,  $K^*$ , and  $C^*$  be the images of  $K_1$ ,  $K_2$ ,  $K$ , and  $C$ . By Study's theorem, the sets  $K_1^*$  and  $K_2^*$  are convex; hence, the set  $K^* = K_1^* \cap K_2^*$  is also convex. Since  $K \subset C$ , we have  $K^* \subset C^*$ ; and because  $K^*$  is convex, the segment  $[f(z_1), f(z_2)]$  also belongs to  $C^*$ . Therefore  $C^*$  is convex.

2. Let the image  $C^*$  of  $C$  be convex for every convex mapping of  $|z| < 1$ . Let  $Z$  be an oricycle through the points  $z_1$  and  $z_2$  of  $C$ , and let  $Z$  touch the unit circle at  $z_0$ . The function

$$w = \frac{z_0 + z}{z_0 - z}$$

maps  $|z| < 1$  onto the half-plane  $\Re w > 0$ . Since the oricycle  $Z$  passes through  $z_0$ , it is mapped onto a straight line  $Z^*$ . Because  $C^*$  is convex, the segment

$$[f(z_1), f(z_2)]$$

of  $Z^*$  belongs to  $C^*$ . Hence the arc of  $Z$  between  $z_1$  and  $z_2$  belongs to  $C$ .

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## REFERENCE

1. E. Study, *Vorlesungen über ausgewählte Gegenstände der Geometrie*, II. Heft, Leipzig, 1913.

The University of Michigan

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