A NOTE ON ε-MAPS ONTO MANIFOLDS

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1. Let X be an n-dimensional, compact, metric absolute neighborhood retract and suppose that for every $\varepsilon > 0$ there exists a map f_ε of X onto a closed n-dimensional manifold Y_ε such that each of the sets $f_\varepsilon^{-1}(y)$, $y \in Y_\varepsilon$, has diameter less than ε. Then, as is shown in [4], X has the homotopy type of a closed n-dimensional manifold, its separation properties by closed subsets are similar to those of closed n-manifolds, and, if n = 2, X is homeomorphic to a closed surface. Therefore, it is natural to ask whether such a space X is necessarily homeomorphic to a manifold. The purpose of this note is to produce an example of a compact, metric, 3-dimensional absolute neighborhood retract that may be mapped onto the 3-sphere with arbitrarily small counter-images, but that fails to be a manifold.

2. Let X denote the quotient space obtained from the 3-sphere S^3 by shrinking the wild arc described in Example 1.1 of [3] to a point; let $\phi: S^3 \to X$ denote the identification map, and let $a = \phi(A)$, where A denotes the wild arc. Clearly, X is a 3-dimensional, compact, metrizable space, and, according to the Borsuk-Whitehead theorem [1], [5], it is an absolute neighborhood retract. That X is not a manifold, since it fails to be locally Euclidean at the point a, has already been pointed out in [2, p. 156].

Let d be any distance function in X, and note that for a compact space, the property of admitting maps onto S^3 with arbitrarily small counter-images does not depend on the particular choice of d. Now, let $\varepsilon > 0$ be given. The continuity of ϕ yields an $\eta > 0$ such that any subset $E \subset S^3$ satisfies the inequality

$$\text{diam}\, \phi(E) < \varepsilon/2 \quad \text{if} \quad \text{diam}\, E < \eta. \tag{1}$$

Let U and V be open cubical neighborhoods of the end points p and q of A such that $\overline{U} \cap \overline{V} = \emptyset$ and

$$\text{diam}\, U < \eta, \qquad \text{diam}\, V < \eta. \tag{2}$$

Select points r and s on $A - (p \cup q)$ such that the subarcs B and C of A, joining p with r and s with q, be entirely contained in U and V, respectively. Also, define maps

$$h': (A \cap \overline{U}) \cup (\overline{U} - U) \to \overline{U} \quad \text{and} \quad k': (A \cap \overline{V}) \cup (\overline{V} - V) \to \overline{V}$$

by

$$h'(x) = r \text{ if } x \in B, \quad h'(x) = x \text{ otherwise},$$

$$k'(x) = s \text{ if } x \in C, \quad k'(x) = x \text{ otherwise}.$$

Since the closed cubes $\overline{U}$ and $\overline{V}$ are absolute retracts, we may extend the maps h' and k' to maps

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$$h\colon \overline{U} \to \overline{U} \quad \text{and} \quad k\colon \overline{V} \to \overline{V}.$$

We now define a map $g\colon S^3 \to S^3$ by

$$g(x) = \begin{cases} x & \text{if } x \in S^3 - (U \cup V), \\ h(x) & \text{if } x \in \overline{U}, \\ k(x) & \text{if } x \in \overline{V}. \end{cases}$$

Since any map of a cube into itself which is the identity on the boundary is a map onto itself, $h(\overline{U}) = \overline{U}$ and $k(\overline{V}) = \overline{V}$, so that

$$g(S^3) = S^3. \tag{3}$$

Let Y denote the quotient space obtained from S^3 by shrinking the subset $g(A)$ to a point, let $\psi\colon S^3 \to Y$ denote the identification map, and let $b = \psi \circ g(A)$. As is easily seen in [3], the subarc $g(A)$ of A, whose end points are r and s, is tame. Thus Y is homeomorphic to S^3. The map

$$f = \psi \circ g \circ \phi^{-1}\colon X \to Y$$

is single-valued and hence continuous; also, by (3), f is a map onto Y. Next, if $y \in Y - b$, then $\psi^{-1}(y)$ consists of a single point $x \in S^3$, and $g^{-1}(x)$ is either a point or a subset of one of the cubes $\overline{U}$, $\overline{V}$, so that, by (2) and (1), diam $\phi \circ g^{-1}(x) < \varepsilon/2$. Also, $\psi^{-1}(b) = g(A)$ and $g^{-1} \circ \psi^{-1}(b)$ is certainly contained in $A \cup \overline{U} \cup \overline{V}$; since

$$\phi(A) = a \in \phi(\overline{U}) \cap \phi(\overline{V}), \tag{4}$$

we obtain the inclusion relation

$$\phi \circ g^{-1} \circ \psi^{-1}(b) \subset \phi(\overline{U}) \cup \phi(\overline{V})$$

and, by (4), (2), and (1), we conclude that diam $\phi(\overline{U}) \cup \phi(\overline{V}) < \varepsilon$. Thus, we have proved that, for each $y \in Y$, the subset $f^{-1}(y)$ of X has diameter less than ε.

3. The space X described above has been used by Curtis and Wilder [2] in order to prove the existence of certain 3-gcms in S^4 which can be distinguished from ordinary manifolds by local homotopy properties: $X - a$ is not 1-LC at a. Our result shows that a 3-gcm may fail to be a manifold even if it admits maps with arbitrarily small counter-images onto a fixed 3-manifold. However, we do not know whether an n-dimensional, compact, metric absolute neighborhood retract that may be mapped with arbitrarily small counter-images onto n-manifolds is necessarily an n-gcm.

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