RETRACTS AND EXTENSION SPACES FOR PERFECTLY NORMAL SPACES

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Let Q be a class of topological spaces, and n a nonnegative integer. A topological space Y is called an n-AR(Q) [n-ANR(Q)] if

(a) Y is in Q and

(b) whenever Z is in Q and Y is imbedded as a closed subset of Z with $\dim(Z - Y) \leq n$, then Y is a retract of Z [Y is a retract of some neighborhood of Y in Z].

Y is called an AR(Q) [ANR(Q)] if it satisfies (a) and the statement (b') obtained from (b) by omitting "with $\dim(Z - Y) \leq n$." A space Y is called an n-ES(Q) [n-NES(Q)] if

(a) Y is in Q and

(b) whenever X is in Q, C is a closed subset of X with $\dim(X - C) \leq n$, and $f: C \to Y$ is a continuous mapping, then f has a continuous extension over X [over some neighborhood of C in X] with respect to Y.

Finally, Y is called an ES(Q) [NES(Q)] if Y satisfies (a) and the statement (b') obtained from (b) by omitting "with $\dim(X - C) \leq n$." In the above definitions, dim X means the dimension of X defined in terms of finite open coverings.

A normal space X is called *perfectly normal* if every closed subset of X is a G_δ. Every metric space is perfectly normal, and every perfectly normal space is countably paracompact [1, p. 221]. Some justification for our interest in the class of perfectly normal spaces is provided by the following theorem of M. Katětov [7].

THEOREM. *Let* B *be a separable Banach space,* K *a convex subset of* B, *and* C *a closed set of type* G_δ *in a normal space* X. *Then every continuous mapping* $f: C \to K$ *has a continuous extension* $F: X \to K$ *with*

$$\dim F(X - C) \leq \min[\dim C + 1, \dim f(C) + 1, \dim X].$$

The object of this paper is to prove the following five theorems.

THEOREM 1. *Let* Y *be a separable metric space. Then the following implications hold between the statements listed below:* (a) *is equivalent to* (d) *and* (b) *is equivalent to* (c); *moreover,* (b) *implies* (a) *and* (c) *implies* (d).

(a) Y *is* LC^{n-1}.

(b) Y *is an* n-ANR (perfectly normal).

(c) Y *is an* n-NES (perfectly normal).

(d) *If* X *is perfectly normal,* $\dim X \leq n$, *and* C *is closed in* X, *then any continuous* $f: C \to Y$ *has a continuous extension over some neighborhood of* C *in* X *with respect to* Y.

Received April 19, 1961.

THEOREM 2. *Let* Y *be a separable metric space. Then the following implications hold between the statements listed below:* (a') *is equivalent to* (d') *and* (b') *is equivalent to* (c'); *moreover,* (b') *implies* (a'), *and* (c') *implies* (d').

(a') Y *is* LC^{n-1} *and* $\pi_i(Y) = 0$ $(i = 0, \cdots, n - 1)$.

(b') Y *is an* n-AR (*perfectly normal*).

(c') Y *is an* n-ES (*perfectly normal*).

(d') *If* X *is perfectly normal,* $\dim X \leq n$, *and* C *is closed in* X, *then any continuous* $f: C \rightarrow Y$ *has a continuous extension over* X.

THEOREM 3. *Let* Y *be an* n-*dimensional separable metric space. Then*

(i) Y *is an* ANR (*perfectly normal*) *if and only if* Y *is* LC^n,

(ii) Y *is an* AR (*perfectly normal*) *if and only if* Y *is* LC^n *and* $\pi_i(Y) = 0$ $(i = 0, 1, \cdots, n)$.

THEOREM 4. *Let* Y *be a perfectly normal space. Then*

(i) Y *is an* ANR (*perfectly normal*) *if and only if* Y *is an* NES (*perfectly normal*),

(ii) Y *is an* AR (*perfectly normal*) *if and only if* Y *is an* ES (*perfectly normal*).

THEOREM 5. *Let* Y *be a separable metric space. Then*

(i) Y *is an* ANR (*perfectly normal*) *if and only if* Y *is an* ANR (*metric*),

(ii) Y *is an* AR (*perfectly normal*) *if and only if* Y *is an* AR (*metric*).

Kuratowski [10, p. 265] proved that if Q is the class of all separable metric spaces, then the statements of Theorem 1 are equivalent, likewise the statements of Theorem 2. Kodama later generalized Kuratowski's results to the case where Y is metric and Q is the class of all metric spaces [8]. Theorem 3 was also proved by Kuratowski for the case where Q is the class of separable metric spaces [10, p. 289]. Theorem 4 has been proved by several authors under various hypotheses [3, 5]. Dowker [2, p. 313] proved Theorem 5 for the case where Y is metric and Q is the class of completely normal perfectly normal spaces.

Before proceeding to the proofs of the theorems we need some preparatory remarks.

Let Y be a metric space, and B the Banach space of all bounded, continuous, real-valued functions defined on Y. Kuratowski [9] showed that Y is imbedded isometrically in B. Wojdysławski [11] subsequently proved that in Kuratowski's imbedding, Y is a closed subset of the convex hull K of Y, and moreover that B is separable whenever Y is separable. This is the Banach space B that will be used in the application of Katětov's theorem.

Another imbedding space that is useful in the non-metric cases is the *adjunction space* [6, p. 9], which was first used by O. Hanner in problems of this type [4, p. 376]. Let X and Y be spaces, C a closed subset of X, and $f: C \rightarrow Y$ a continuous mapping. Let Z be the adjunction space (called by Hanner the identification space) obtained from the free union $X \cup Y$ of X and Y by identifying each $x \in C$ with $f(x) \in Y$. There are two natural mappings $j: Y \rightarrow Z$ and $k: X \rightarrow Z$, and a set V is open in Z if and only if $j^{-1}(V)$ and $k^{-1}(V)$ are open. The mapping j is a homeomorphism; and therefore we may assume that Y is a subspace of Z. Note that $k | X - C$ is a homeomorphism onto $Z - Y$. Moreover, k is an extension of f over X with respect to Z.

Later, we shall need the following lemma.

LEMMA 1. *Let* X *and* Y *be perfectly normal spaces,* C *a closed subset of* X, *and* $f: C \to Y$ *a continuous mapping. Then the adjunction space* Z *is also perfectly normal.*

Proof. Since X and Y are normal, it follows that Z is normal [4, p. 376]. Hence it remains to be shown that every closed subset of Z is of type G_δ. Let F be a closed subset of Z. Then $F \cap Y$ a closed subset of Y. Since Y is perfectly normal, $F \cap Y$ is of type G_δ. Therefore

$$F \cap Y = \bigcap_{i=1}^{\infty} U_i, \tag{1}$$

where the U_i are open sets of Y. Since Y is closed in Z, it follows that for each i, $\overline{U}_i \subset Y$, closure being taken in Z. Therefore the sets $F \cup \overline{U}_i$ are closed in Z, and hence each of the sets $k^{-1}(F \cup \overline{U}_i)$ is closed in X. Since X is perfectly normal, each of the sets $k^{-1}(F \cup \overline{U}_i)$ is of type G_δ, and thus

$$k^{-1}(F \cup \overline{U}_i) = \bigcap_{j=1}^{\infty} V_{i,j} \qquad (i = 1, 2, 3, \cdots), \tag{2}$$

where the $V_{i,j}$ are open sets of X. Now Hanner has proved [4, p. 377] that the sets

$$G_{i,j} = k(V_{i,j} - C) \cup U_i$$

are open in Z. We shall prove that the intersection of the countable collection

$$\{G_{i,j}\}_{i,j=1}^{\infty}$$

is equal to F. Since $k|X - C$ is a homeomorphism, it follows that for fixed i

$$\bigcap_{j=1}^{\infty} k(V_{i,j} - C) = (F \cup \overline{U}_i) \cap (Z - Y) = F \cap (Z - Y).$$

Therefore

$$\bigcap_{j=1}^{\infty} G_{i,j} = [F \cap (Z - Y)] \cup U_i$$

for each i, and hence

$$\bigcap_{i,j=1}^{\infty} G_{i,j} = \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} G_{i,j} = [F \cap (Z - Y)] \cup \bigcap_{i=1}^{\infty} U_i = [F \cap (Z - Y)] \cup [F \cap Y] = F.$$

This shows that F is of type G_δ, and completes the proof of the lemma.

Proof of Theorem 1. The propositions that (b) implies (a), (d) implies (a), (c) implies (b), and (c) implies (d) were proved by Kuratowski [10, p. 265]. Therefore we need prove only that (a) implies (d) and that (b) implies (c).

Proof that (a) implies (d). Let X be a perfectly normal space of dimension at most n, C a closed subset of X, and $f: C \to Y$ a continuous mapping. By Kuratowski's imbedding we may assume that Y is a closed subset of a convex subset K of a separable Banach space B. Since X is perfectly normal, C is a closed set of type G_δ in X. Therefore we can apply Katětov's theorem to obtain a continuous extension $F: X \to K$ of f with $\dim F(X - C) \leq n$. Since

$$F(X - C) \supset F(X) - F(C) = F(X) - f(C) \supset F(X) - Y,$$

it follows that $\dim [F(X) - Y] \leq n$. By Kuratowski's version of Theorem 1, (a) implies that Y is an n-ANR (separable metric). But B is separable metric, so that $Y \cup F(X)$ is also separable metric. Since Y is closed in K, Y is also closed in $Y \cup F(X)$, and moreover

$$\dim (Y \cup F(X) - Y) = \dim[F(X) - Y] \leq n.$$

Hence Y is a retract of some neighborhood V of Y in $Y \cup F(X)$. Let $r: V \to Y$ be the retraction, and define $U = F^{-1}(V)$. U is a neighborhood of C in X. Define $f^*: U \to Y$ by $f^*(x) = r\,F(x)$ for $x \in U$. Then f^* is a continuous extension of f over U; this completes the proof that (a) implies (d).

Proof that (b) implies (c). Let X be a perfectly normal space, C a closed subset of X such that $\dim (X - C) \leq n$, and $f: C \to Y$ a continuous mapping. We shall show that f has a continuous extension over some neighborhood U of C. In this case Katětov's theorem will not give us the result, since we have no control over $\dim F(X - C)$. Instead, let us use Hanner's method. Let Z be the adjunction space of $X \cup Y$ described above. Z is perfectly normal, by Lemma 1, and since $Z - Y$ is homeomorphic to $X - C$, it follows that $\dim (Z - Y) \leq n$. Hence, by (b), Y is a retract of some neighborhood V of Y in Z. Let $r: V \to Y$ be the retraction. Then $U = k^{-1}(V)$ is a neighborhood of C in X, and the mapping $f^*: U \to Y$ defined by $f^*(x) = r\,k(x)$ for $x \in U$ is the required extension of f over U. This completes the proof of Theorem 1.

Proof of Theorem 2. The proof that (a') implies (d') is obtained from the proof that (a) implies (d) by replacing V by $Y \cup F(X)$ and U by X. The proof that (b') implies (c') is obtained from the proof that (b) implies (c) by replacing V by Z and U by X. The remaining implications were proved by Kuratowski [10, p. 266].

Proof of Theorem 3. (i) If Y is separable metric and an ANR (perfectly normal), then Y is an ANR (separable metric). Hence Kuratowski's theorem [10, p. 289] shows that Y is LC^n.

Conversely, if $\dim Y = n$ and Y is LC^n, then Y is an ANR (separable metric) [10, p. 289]. Let Y be imbedded in a perfectly normal space Z as a closed subset. By Kuratowski's imbedding theorem, we can assume that Y is a closed subset of a convex subset K of a separable Banach space. Define a mapping $f: Y \to Y \subset K$ by $f(y) = y$ for each $y \in Y$. By Katětov's theorem, f has a continuous extension $F: Z \to K$. Since Y is an ANR (separable metric), Y is a retract of some neighborhood V of Y in K. Let $r: V \to Y$ be the retraction, and define $U = F^{-1}(V)$. Then U is a neighborhood of Y in Z, and for $z \in U$, $r\,F(z)$ is a retraction of U onto Y. Therefore Y is an ANR (perfectly normal).

The proof of (ii) is so similar to that of (i) that it need not be given.

Proof of Theorem 4. (i) It is easy to see that if Y is an NES (perfectly normal) then Y is an ANR (perfectly normal).

To establish the converse, we need only mention that by Lemma 1 the adjunction space Z is perfectly normal, so that Hanner's proof [5, p. 325] is valid in this case also.

The same remarks apply to the proof of (ii).

Proof of Theorem 5. (i) If Y is an ANR (perfectly normal), then, being metric, Y is an ANR (metric).

Suppose now that Y is an ANR (metric). Then we can use the same argument as in (i) of Theorem 3 to show that Y is an ANR (perfectly normal).

The proof of (ii) is similar to that of (i), and it will not be given.

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