FINITE ORBIT STRUCTURE ON LOCALLY COMPACT MANIFOLDS

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1. INTRODUCTION

The following conjecture was raised by D. Montgomery. *If a compact Lie transformation group* G *operates on a compact manifold* X, *then there are only a finite number of distinct conjugate classes of isotropy subgroups* G_x $(x \in X)$. The conjecture was answered in the affirmative through the efforts of Floyd [3] and Mostow [6]. It is also known that if the manifold X is only locally compact, then *locally* there are but a finite number of classes of isotropy subgroups [1, VII]. However, in the case where the manifold X is not compact, a counterexample due to Montgomery exists for the conjecture. In this counterexample [8], a circle group operates on a 3-manifold having infinitely generated integral cohomology, and the action produces an infinite number of distinct isotropy subgroups. It is precisely this counterexample which suggests the main result of this present paper: the conjecture still holds when X is an orientable cohomology manifold over the integers with finitely generated integral cohomology (Theorem 3.5). Alexander-Spanier cohomology with compact supports is used in the above.

Our techniques are based upon those of Floyd's paper [3], in that we establish a main lemma of the form (3.3) in Section 3. The author is grateful to Professor Floyd for bringing this problem to his attention. The results of this paper will be used, in a later paper, to investigate the action of compact Lie groups on complexes. The question of whether the orientability of X may be dropped in (3.5) is still open.

2. DEFINITIONS AND PRELIMINARY RESULTS

X will always be considered as a locally compact Hausdorff space, and L as a principal ideal domain. $H_c^i(X; L)$ will denote the i-th Alexander-Spanier cohomology group of X with compact supports. In all applications, L will be either Z, the ring of integers, or Z_p, the prime field of characteristic p. $H_c^*(X; L)$ will denote the direct sum of the groups $H_c^i(X; L)$, and reduced cohomology will be used for the 0-dimensional groups.

For an open subset U of X and a closed subset A of X, we have the standard homomorphisms

$$j_{XU}^*: H_c^*(U; L) \to H_c^*(X; L) \qquad \text{and} \qquad r_{XA}^*: H_c^*(X; L) \to H_c^*(A; L).$$

In fact, we have the following exact cohomology sequence for U open in X:

$$(2.1) \qquad \to H_c^i(U; L) \xrightarrow{j_{XU}^i} H_c^i(X; L) \xrightarrow{r_{X,X-U}^i} H_c^i(X - U; L) \to H_c^{i+1}(U; L) \to \cdots.$$

Received June 12, 1961.

Presented to the American Mathematical Society, January 26, 1961. The author is an ONR Research Associate.

(2.2) DEFINITION. We shall say that the *cohomology dimension of* X *with respect to* L *is less than or equal to* n (notation: $\dim_L X \leq n$, if $H_c^i(U; L) = 0$ for all open $U \subset X$ and all $i \geq n + 1$.

(2.3) DEFINITION. X is said to be *cohomologically locally connected over* L (notation: clc over L) at a point $x \in X$ if, corresponding to each compact neighborhood B of x, there exists a compact neighborhood A of x ($A \subset B$) with image r^i_{BA} trivial for all i. If X is clc over L at each of its points, then X is said to be clc over L.

The above notation and definitions are discussed in detail in [1, I].

The following proposition is a special case of a result of Raymond [7, pp. 955-956]. It will allow us to interchange the conditions of finitely generated global cohomology and cohomological local connectedness at infinity, in many of our results. $\dot{X}$ will denote the one-point compactification $X \cup \{\omega\}$ of X, where ω is the point at infinity.

(2.4) PROPOSITION. *Let* X *be* clc *over* L, *and let* $\dim_L X$ *be finite. Then the following two conditions are equivalent.*

(i) $H_c^*(X; L)$ *is finitely generated.*

(ii) $\dot{X}$ *is* clc *over* L *at* ω.

Proof. We show first that (ii) implies (i). If (ii) holds, then $\dot{X}$ is clc at each of its points. Since $\dot{X}$ is also compact, it follows that $H^*(\dot{X}; L)$ is finitely generated [2, (3.5)], which in turn implies (i).

Suppose now that (i) holds. Let U be an open neighborhood of ω in $\dot{X}$. Then $B = X - U$ is compact in X. Choose A' and A compact in X in such a way that $B \subset \text{int } A' \subset A' \subset \text{int } A$. Then $V' = \dot{X} - A'$ and $V = \dot{X} - A$ are neighborhoods of ω in $\dot{X}$ with $V \subset V' \subset U$. Since X is clc over L, it follows easily from [2, (3.5)] that image $r^i_{AA'}$ and image $r^i_{A'B}$ are finitely generated for each i. Consider then the following commutative diagram, where each row is of the form (2.1).

$$
\begin{array}{ccccccccc}
 & & & \cdots \longrightarrow & H_c^i(V) & \longrightarrow & H^i(\dot{X}) & \longrightarrow \cdots \\
 & & & & \downarrow {\scriptstyle j^i_{VV'}} & & \downarrow & \\
\cdots \longrightarrow & H^{i-1}(A') & \longrightarrow & & H_c^i(V') & \longrightarrow & H^i(\dot{X}) & \longrightarrow \cdots \\
 & \downarrow {\scriptstyle r^{i-1}_{A'B}} & & & \downarrow {\scriptstyle j^i_{V'U}} & & & \\
\cdots \longrightarrow & H^{i-1}(B) & \longrightarrow & & H_c^i(U) & \longrightarrow \cdots & &
\end{array}
\tag{2.5}
$$

Since $H^i(\dot{X})$ is finitely generated, an inspection of (2.5) shows that

$$\text{image } j^i_{VU} = \text{image } j^i_{V'U}\, j^i_{VV'}$$

is finitely generated. Thus the local Betti numbers of $\dot{X}$ around ω are at most "increasingly infinite" [1, I, (2.1)] and it follows from [1, I, (2.2)] that $\dot{X}$ is clc at ω.

(2.6) DEFINITION. A space X with $\dim_L X$ finite is said to be a *Wilder* n-*manifold over* L if the local Betti numbers around each point are equal to 1 in dimension n and equal to 0 otherwise [1, I]. A connected Wilder n-manifold is *orientable* if $H_c^n(X; L) \cong L$. It is *locally orientable* if each point has an open connected orientable neighborhood. A *cohomology* n-*manifold over* L (notation: n-cm over L) is a connected locally orientable Wilder n-manifold over L.

It follows from [1, I, (2.2)] that an n-cm over L is clc over L.

(2.7) DEFINITION. Let $\{Y_j\}$ be a sequence of closed subsets of the space X and let $\{L_j\}$ be a sequence of principal ideal domains. Following Floyd in [3] or [1, VI] we say that $\{Y_j\}$ *converges regularly* to Y over $\{L_j\}$ if, given $y \in Y$ and a compact neighborhood B of y, there exists a compact neighborhood $A \subset B$ of y and an integer J such that the homomorphisms

$$H^*(B \cap Y_j; L_j) \to H^*(A \cap Y_j; L_j),$$

$$H^*(B \cap Y; L_j) \to H^*(A \cap Y; L_j)$$

are trivial for all $j \geq J$.

Note the connection between regular convergence and clc as defined in (2.3). Lemma (2.8) below is required to demonstrate in (3.3) that if G is a toral group and the order of each G_j is the power of a prime, say $p_j^{\alpha_j}$, then the one-point compactifications of the fixed point sets of the G_j converge regularly over $\{Z_{p_j}\}$ to the one-point compactification of the fixed point set of G. Its proof is entirely analogous to that of a corresponding lemma of Floyd in [1, VI] and it will therefore be omitted.

(2.8) LEMMA. *Let* $\dot{X}$ *be the one-point compactification of the space* X *such that* $\dot{X}$ *is* clc *over* Z_p (p *fixed, prime*), *and such that each component* C *of* X *is a* cm *over* Z_p *with* $\dim_p C \leq n$. *Now let* G *be a toral group or an abelian transformation group of order* p^α *on* $\dot{X}$ (*leaving* ω *fixed*), *and suppose that* $A_0 \supset A_1 \supset \cdots \supset A_{(n+1)!}$ *is a sequence of compact invariant subsets of* $\dot{X}$ *with* $r^*_{A_jA_{j+1}} = 0$ *over* Z_p *for each* j. *Then*

$$r^*_{\dot{F}\cap A_0, \dot{F}\cap A_{(n+1)!}} = 0,$$

where $\dot{F}$ *is the fixed point set of* G *on* $\dot{X}$.

(2.9) LEMMA. *Let* X *be an* n-cm *over* Z_2 *such that* $H_c^*(X; Z_2)$ *is finitely generated, and let* G *be a toral group on* X. *Then* F(G) *has a finite number of components.*

Proof. Let $G_j = \{g \in G \mid \text{order } g \text{ divides } 2^j\}$. Then the $F(G_j)$ converge to F(G). It will be sufficient to show that the number of components of each $F(G_j)$ is bounded by a fixed integer.

For a transformation group H of order 2 on a locally compact Hausdorff space Y of finite cohomological dimension over Z_2, it follows from [1, III] that $\dim H_c^*(F(H); Z_2) \leq \dim H_c^*(Y; Z_2)$. By induction, this result can easily be extended to an abelian transformation group of order 2^α. We see then that

$$\dim H_c^*(F(G_j); Z_2) \leq \dim H_c^*(X; Z_2) = I$$

for each j, where I is some non-negative integer. Now each component of $F(G_j)$ is a cm over Z_2, and in fact each contributes a Z_2 factor in its top dimension cohomology group. Hence each $F(G_j)$ consists of at most I components, and (2.9) follows.

3. MAIN RESULTS

(3.1) DEFINITION. X will be said to have the *strengthened form of* clc at a point $x \in X$ if, corresponding to each compact neighborhood B of x, there exists a compact neighborhood A of x ($A \subset B$) such that $r^*_{BA} = 0$ over all Z_p (p prime).

(3.2) REMARK. If X is clc over Z at $x \in X$, then X has the strengthened form of clc at x.

(3.3) *Let* X *be an orientable* n-cm *over* Z *such that* $\dot{X}$ *has the strengthened form of* clc *at* ω. *Let* G *be an elementary group operating on* X, *and* G_j *a sequence of closed subgroups of* G *with* $\lim G_j = G$. *Then* $F(G_j) = F(G)$ *for* j *sufficiently large.*

Proof. If we define $g\omega = \omega$ for all $g \in G$, then G operates on $\dot{X}$. It may first be verified that if a toral group H operates on X and $\dot{X}$, all the essential properties of X and $\dot{X}$ are inherited by the fixed point sets F(H) and $\dot{F}(H)$. By [1] or [4], each component of F(G) is an orientable cm over Z, and by (2.8) it follows easily that $\dot{F}(H)$ has the strengthened form of clc at ω. Therefore the procedures of Floyd in [3] may be employed to reduce (3.3) to the case where G is a toral group, and the order of each G_j is a power of a prime, say $p_j^{\alpha_j}$.

By (2.4), $H^*_c(X; Z_p)$ is finitely generated for each prime p. Therefore, by (2.9), F(G) has a finite number of components. Let $x \in F$, and let C be the component of $F = F(G)$ containing x. Then $x \in C \subset C_j$ for each j, where C_j is the component of $F_j = F(G_j)$ containing x. Now C is open in F. Therefore $C = F \cap U$, for some open $U \subset X$. Now $x \in C_j \cap U \supset C \cap U = F \cap U$. By [1, VI] or [4], there exists an open neighborhood V of x such that $F \cap V = F_j \cap V$ for j large, say $j \geq J_1$. We have then $x \in C_j \cap U \cap V \supset C \cap U \cap V = F \cap U \cap V = F_j \cap U \cap V \supset C_j \cap U \cap V$ for $j \geq J_1$. Therefore, for $V' = U \cap V'$ and $j \geq J_1$,

$$C_j \cap V' = C \cap V'.$$

Fix $j \geq J_1$. Now C is an r-cm ($r < n$) over Z. Therefore C is an r-cm over Z_{p_j}, and $C \cap V'$, an open subset of C, has the local Betti numbers of S^r over Z_{p_j}. Therefore $C_j \cap V'$ has the same local Betti numbers over Z_{p_j}. But $C_j \cap V'$ is an open subset of C_j which is an r_j-cm over Z_{p_j}. Therefore $r_j = r$. If $C \neq C_j$, then C is a proper closed subset of C_j, which leads to a contradiction since C and C_j are of the same cohomology dimension over Z_{p_j} [1, I, (4.6)]. Therefore $C = C_j$ for $j \geq J_1$.

Since C consists of just a finite number of components, we could simply consider all $j \geq \max_s J_s$ (where s is the number of components of F) and obtain our result, if it were not for the possibility that some sequence of non-empty components of the F_j's might converge to an empty limit. We shall show through a regular convergence argument that this unpleasant situation cannot occur.

We consider now $\dot{X}$ and use (2.8) to show that $\{\dot{F}_j\}$ converges regularly to $\dot{F}$ over $\{Z_{p_j}\}$. The strengthened form of clc of $\dot{X}$ is of course required for this argument. By a technique of Floyd in [3, (4.4)], we may employ closed coverings to obtain

$$H^i(\dot{F}_j; Z_{p_j}) \cong H^i(\dot{F}; Z_{p_j})$$

for all i and for all sufficiently large j. We outline this argument, using the characterization of *strong refinements* and *determining pairs* for cohomology with compact supports which is given in [5, (5.1)]. Because of the regular convergence of $\dot{F}_j$ to $\dot{F}$, we can obtain a sequence $\alpha_0, \alpha_1, \cdots, \alpha_{2n+1}$ of finite closed covers of $\dot{X}$ with $\alpha_k \cap \dot{F}_j$ strongly refining $\alpha_{k-1} \cap \dot{F}_j$ (coefficient group Z_{p_j}), and with $\alpha_k \cap \dot{F}$ strongly refining $\alpha_{k-1} \cap \dot{F}$ (all coefficient groups Z_{p_j}), for $k = 1, 2, \cdots, 2n+1$ and for each j. Then by [5, (5.1)],

$$\alpha_{n+1} \cap \dot{F}_j \quad \text{and} \quad \alpha_{2n+1} \cap \dot{F}_j$$

determine $H^*(\dot{F}_j, Z_{p_j})$ up to n, and

$$\alpha_{n+1} \cap \dot{F} \quad \text{and} \quad \alpha_{2n+1} \cap \dot{F}$$

determine $H^*(\dot{F}; Z_{p_j})$ for each j up to n. Since, however, $\dot{F}_j$ converges to $\dot{F}$, we may identify the nerves of $\alpha_k \cap \dot{F}_j$ and $\alpha_k \cap \dot{F}$ for j large. It follows therefore that $H^i(\dot{F}_j; Z_{p_j}) \cong H^i(\dot{F}; Z_{p_j})$ for all i and for all sufficiently large j, say $j \geq K$.

We consider from now on only $j \geq \max(K, J)$, where $J = \max_s J_s$ (as defined previously). Let $\widetilde{F}_j$ represent the set-theoretic union of $\{\omega\}$ and all components of F_j that contain a component of F. From our previous considerations, we see that $\widetilde{F}_j = \dot{F}$. To conclude the proof of (3.3), it will be sufficient to show that $\dot{F}_j - \widetilde{F}_j$ is empty. Let C_j^* be a bounded component of $\dot{F}_j - \widetilde{F}_j$. Then, since

$$H^0(\widetilde{F}_j; Z_{p_j}) = H^0(\dot{F}; Z_{p_j}) \cong H^0(\dot{F}_j; Z_{p_j})$$

and since $H^0(\dot{F}_j; Z_{p_j})$ counts the number of bounded components of F_j, it follows that C_j^* must be empty. We suppose therefore that there exists an unbounded component C_j^* of $\dot{F}_j - \widetilde{F}_j$ and, since

$$H_c^i(\widetilde{F}_j - \{\omega\}; Z_{p_j}) \cong H_c^i(\dot{F}_j - \{\omega\}; Z_{p_j})$$

for $i \geq 1$, it is a simple matter to show that $H_c^i(C_j^*; Z_{p_j}) = 0$ for $i \geq 1$. But since X is orientable, C_j^* is an orientable r_j-cm over Z_{p_j}. Therefore $r_j = 0$ and C_j^* is at most a point. But it can't be just a point, since we had assumed that it is unbounded. Therefore $\dot{F}_j - \widetilde{F}_j = \phi$, and (3.3) follows.

(3.4) REMARK. It is interesting to note that the orientability of X was really used for the first time in this paper at the very end of the proof of (3.3). However, its contribution to the proof of (3.3) appears to be essential, since the C_j^* could conceivably consist of real projective spaces (minus a point) treated as cohomology manifolds over Z_p (p an odd prime).

(3.5) THEOREM. *Let* G *be a compact Lie group acting on an orientable* cm X *over* Z, *having finitely generated cohomology over* Z. *Then there exist only a finite number of non-conjugate isotropy subgroups* G_x $(x \in X)$.

Proof. We first consider the case where G is connected and abelian, that is, a toral group, and show that there exist but a finite number of isotropy subgroups in X. It is easy to verify that X satisfies the conditions of (3.3). Suppose that there exists a convergent sequence of points $\{x_j\}$ in $\dot{X}$ such that $G_{x_r} \neq G_{x_s}$ for $r \neq s$.

Let x_0 be the limit of the x_j. We may suppose that the G_{x_j} converge to an elementary subgroup H of G_{x_0}. Applying (3.3), we see that $\dot{F}(G_{x_r}) = \dot{F}(G_{x_s})$, for large r and s, from which a contradiction easily follows. Theorem (3.5) now follows for an arbitrary compact Lie group G, by the technique of Mostow in [6].

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