# NON-NORMAL SUMS AND PRODUCTS OF UNBOUNDED NORMAL FUNCTIONS

Peter Lappan

## 1. INTRODUCTION

In their paper concerning normal meromorphic functions, Lehto and Virtanen [5, p. 53] remark that the sum of a "normal" function and a bounded function (which is necessarily normal) is a normal function. It is natural to ask whether the sum of two unbounded normal functions is normal. It shall be shown here that this is not in general the case.

The following notation shall be used throughout. The open unit disk will be denoted by D, and the unit circle by C. The non-Euclidean hyperbolic distance in D between z and z' will be denoted by $\rho(z, z')$ [2, Chapter 2]. The cluster set of the function f(z) in D will be denoted by $\mathscr{C}(f)$ [4, p. 84].

It is easy to show that the property of being normal depends only upon the behavior of the function near the boundary. Thus the remark of Lehto and Virtanen can be extended: if f(z) and g(z) are normal meromorphic functions in D, and if $\infty \notin \mathscr{C}(f)$, then $h(z) = f(z) + g(z)$ is a normal function in D.

In Section 2, normal meromorphic functions with an infinity of poles are discussed. In Section 3, similar results are obtained for unbounded normal holomorphic functions. The general normal meromorphic function with $\infty \in \mathscr{C}(f)$ is considered in Section 4, and the results are summarized in Theorem 5. In Section 5, the results are applied to functions omitting three values.

## 2. NORMAL FUNCTIONS WITH AN INFINITY OF POLES

We first formulate a sufficient condition for the non-normality of a meromorphic function.

LEMMA 1. *Let* f(z) *be a meromorphic function in* D, *and let* $\{z_n\}$ *and* $\{z_n'\}$ *be two sequences of points in* D *such that* $\lim_{n\to\infty} \rho(z_n, z_n') = 0$ *and* $\lim_{n\to\infty} |z_n| = 1$. *If*

$$\lim_{n\to\infty} f(z_n) = \alpha \quad \textit{and} \quad \lim_{n\to\infty} f(z_n') = \beta \qquad (\alpha \neq \beta),$$

*then* f(z) *is not a normal function.*

This is an immediate consequence of [1, Lemma 1, p. 10].

THEOREM 1. *Let* f(z) *be a normal meromorphic function in* D *with an infinity of poles. Then there exists a Blaschke product* $B_f(z)$ *such that* $h(z) = f(z)B_f(z)$ *is not a normal function.*

*Proof.* Let $\{z_n'\}$ be a sequence of poles of f(z) such that

Received May 26, 1961.

$$\sum_{n=1}^{\infty} (1 - |z_n'|) < \infty .$$

Choose a sequence of points $\{z_n\}$ in D such that $|z_n| > |z_n'|$, $f(z_n) \neq \infty$, and $\lim_{n\to\infty} \rho(z_n, z_n') = 0$. Let $B_f(z)$ be the Blaschke product defined by

$$B_f(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}.$$

Define $h(z) = f(z) B_f(z)$. Then for all $n \geq 1$, we have $h(z_n) = 0$ and $h(z_n') = \infty$, and by Lemma 1, $h(z)$ is not normal.

LEMMA 2. *Let* $f(z)$ *be a normal meromorphic function in* D, *and let* $g(z)$ *be a holomorphic function in* D *such that*

$$0 < M_1 < |g(z)| < M_2 ,$$

*where* $M_1$ *and* $M_2$ *are finite constants. Then the function* $h(z) = f(z)\, g(z)$ *is a normal meromorphic function.*

The proof consists of direct verification that the function $h(z)$ satisfies the definition of a normal function. The details are omitted here.

THEOREM 2. *Let* $f(z)$ *be a normal meromorphic function in* D *with an infinity of poles. Then there exists a normal meromorphic function* $g(z)$ *in* D *such that* $h(z) = f(z) + g(z)$ *is not a normal function.*

*Proof.* Let $B_f(z)$ be the Blaschke product described in the proof of Theorem 1. Set $g(z) = \frac{1}{2}(B_f(z) - 2)\, f(z)$. Since

$$1 < |B_f(z) - 2| < 3 ,$$

$g(z)$ is normal, by Lemma 2. But

$$h(z) = f(z) + g(z) = \frac{1}{2} B_f(z)\, f(z)$$

is not normal, by Theorem 1 and Lemma 2. Thus the theorem is proved.

## 3. UNBOUNDED NORMAL HOLOMORPHIC FUNCTIONS

As before, we desire a sufficient condition for the non-normality of a holomorphic function.

LEMMA 3. *Let* $f(z)$ *be a holomorphic function in* D, *and let* $\{z_n\}$ *and* $\{z_n'\}$ *be two sequences of points in* D *such that* $\lim_{n\to\infty} z_n = z_0$, *where* $z_0 \in C$. *If* $\rho(z_n, z_n') < M$, *where* M *is a positive constant,*

$$\lim_{n\to\infty} f(z_n) = 0, \quad \textit{and} \quad \lim_{n\to\infty} f(z_n') = \infty ,$$

*then* $f(z)$ *is not a normal function.*

*Proof.* Suppose $f(z)$ is a normal function. Let

$$S_n(z) = \frac{z + z_n}{1 + \bar{z}_n z}.$$

Since $S_n(z)$ is a linear transformation of D onto itself, the sequence of functions $\{f(S_n(z))\}$ forms a normal family. Then there exists a subsequence $\{f(S_{n_i}(z))\}$ which converges uniformly on each compact subset of D either to a function holomorphic in D or to the constant $\infty$ [3, p. 195]. Since, by hypothesis,

$$\lim_{i \to \infty} f(S_{n_i}(0)) = 0,$$

the limit of the sequence $\{f(S_{n_i}(z))\}$, which shall be denoted by $F(z)$, must be a function holomorphic in D. Hence there exists a positive constant L such that $|F(z)| < L$ in the disk $\rho(0, z) \leq M$. Then there exists a natural number N such that

$$|f(S_{n_i}(z))| < L + 1 \tag{1}$$

for all $n_i > N$ and all z in the disk $\rho(0, z) \leq M$.

However, setting $z''_n = S_n^{-1}(z'_n)$, we have

$$\rho(0, z''_n) = \rho(S_n(0), S_n(z''_n)) = \rho(z_n, z'_n) < M$$

as well as $f(S_n(z''_n)) = f(z'_n)$. Thus, by hypothesis,

$$\lim_{i \to \infty} f(S_{n_i}(z''_{n_i})) = \infty,$$

which contradicts (1). Thus $f(z)$ is not normal.

The following is similar to a result of Seidel and Walsh [6, Corollary 1, p. 197] concerning holomorphic functions that omit two finite values.

LEMMA 4. *Let* $f(z)$ *be a normal holomorphic function in* D. *Let* $\{z_n\}$ *and* $\{z'_n\}$ *be two sequences of points in* D, *and let* $M'$ *be a constant such that* $\rho(z_n, z'_n) < M'$. *If* $\lim_{n \to \infty} f(z_n) = \infty$, *then* $\lim_{n\to\infty} f(z'_n) = \infty$.

The proof, similar to that of Lemma 3, is omitted here.

The next lemma is a restatement of a result of Bagemihl and Seidel [1, Example 4, pp. 11-13].

LEMMA 5. *There exists a Blaschke product* $B(z)$ *in* D, *together with positive constants* L *and* M *and two sequences* $\{x_n\}$ *and* $\{x'_n\}$ *of positive real numbers such that*

$$x_n < x'_n < x_{n+1} < 1, \tag{2}$$

$$\rho(x_n, x_{n+1}) < M, \tag{3}$$

$$B(x_n) = 0 \quad \textit{and} \quad |B(x'_n)| > L. \tag{4}$$

Note that $x_n$ above corresponds to $z_{2n}$ in the notation of Bagemihl and Seidel, and similarly $x_n'$ corresponds to $x_{2n}$ in their notation. The Blaschke product here is the product of the even-numbered factors of the Blaschke product of Bagemihl and Seidel.

Note also that

$$|B(x_n')| = \prod_{j=1}^{\infty} \left| \frac{x_n' - x_j}{1 - \bar{x}_j x_n'} \right| . \tag{5}$$

LEMMA 6. *If* $z_0$, $z_1$, $z_2$, *and* $z_3$ *are points in* D *such that* $\rho(z_0, z_1) > \rho(z_2, z_3)$, *then*

$$\left| \frac{z_0 - z_1}{1 - \bar{z}_1 z_0} \right| > \left| \frac{z_2 - z_3}{1 - \bar{z}_3 z_2} \right| .$$

The lemma is a consequence of the definition of $\rho(z, z')$.

THEOREM 3. *Let* $f(z)$ *be a normal holomorphic function unbounded in* D. *Then there exists a Blaschke product* $B_f(z)$ *in* D *such that* $g(z) = f(z)B_f(z)$ *is not a normal function.*

*Proof.* Let $\{z_n''\}$ be a sequence of points in D such that

$$\lim_{n\to\infty} f(z_n'') = \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} (1 - z_n'') < \infty .$$

Choose a subsequence $\{z_n\}$ of $\{z_n''\}$ such that for each $j < n$, $\rho(z_j, z_n) > 3(n - j)M'$, where $M' > M$ and $M$ is the constant in Lemma 5. Then choose $\{z_n'\}$ such that $\rho(z_n, z_n') = M'$. Let $B_f(z)$ denote the Blaschke product with zeros at $z_n$. Then

$$|B_f(z_n')| = \prod_{j=1}^{\infty} \left| \frac{z_n' - z_j}{1 - \bar{z}_j z_n'} \right| . \tag{6}$$

We wish to compare the factors in (5) with the factors in (6) by means of Lemma 6. For $n = j$, we have, using (2) and (3),

$$\rho(z_n', z_n) = M' > M > \rho(x_n', x_n) . \tag{7}$$

For $n \neq j$, we have

$$\begin{aligned} \rho(z_n', z_j) &\geq \rho(z_n, z_j) - \rho(z_n, z_n') \geq 3|n - j|M' - M' \\ &\geq (|n - j| + 1)M' > (|n - j| + 1)M \\ &\geq \rho(x_n, x_j) + \rho(x_n', x_n) \geq \rho(x_n', x_j) . \end{aligned} \tag{8}$$

Combining (7) and (8), we see that, for each $n$ and every $j$, $\rho(z_n', z_j) > \rho(x_n', x_j)$. Then, by Lemma 6, each factor of (6) is greater than the corresponding factor of (5). Hence by (4), $|B_f(z_n')| > L$.

Define $g(z) = f(z) B_f(z)$. By construction, $B_f(z_n) = 0$ and thus $g(z_n) = 0$. By hypothesis, $\lim_{n\to\infty} f(z_n) = \infty$ and hence, by our construction and Lemma 4, $\lim_{n\to\infty} f(z_n') = \infty$. Therefore $\lim_{n\to\infty} g(z_n') = \infty$. Then, by Lemma 3, $g(z)$ is not normal. Thus the theorem is proved.

THEOREM 4. *Let* $f(z)$ *be a normal holomorphic function unbounded in* D. *Then there exists a normal holomorphic function* $g(z)$ *in* D *such that* $h(z) = f(z) + g(z)$ *is not a normal function.*

The proof is the same as that of Theorem 2, except that Theorem 3 is cited in place of Theorem 1.

## 4. UNBOUNDED NORMAL FUNCTIONS

If $f(z)$ is a normal meromorphic function such that $f(z)$ has only a finite number of poles and $\infty \in \mathscr{C}(f)$, then the constructions in the proofs of Theorems 3 and 4 are valid for $f(z)$, since $f(z)$ is holomorphic near the boundary. Thus, incorporating Theorems 2 and 4, we have the following result.

THEOREM 5. *Let* $f(z)$ *be a normal meromorphic function with* $\infty \in \mathscr{C}(f)$. *Then there exists a normal meromorphic function* $g(z)$ *in* D *such that* $h(z) = f(z) + g(z)$ *is not a normal function.*

## 5. FUNCTIONS OMITTING THREE VALUES

The following example shows that the sum of two holomorphic functions in D, each omitting the values 0 and 1, need not be normal.

Let $f(z) = 2 + 2/(1 - z)$ and $g(z) = [B(z) - 2][1 + 1/(1 - z)]$, where $B(z)$ is the Blaschke product described in Lemma 5. An easy calculation shows that $|f(z)| > 3$ and $|g(z)| > 3/2$. Thus $f(z)$ and $g(z)$ omit the values 0 and 1. Let

$$h(z) = f(z) + g(z) = B(z)[1 + 1/(1 - z)].$$

A modification of the argument in the proof of Theorem 3, using $z_n = x_n$ and $z_n' = x_n'$, shows that $B(z)[1 + 1/(1 - z)]$ is not a normal function. Hence $h(z)$ is not a normal function.

## REFERENCES


1. F. Bagemihl and W. Seidel, *Sequential and continuous limits of meromorphic functions*, Ann. Acad. Sci. Fenn. Ser. A.I 280 (1960), 1-17.

2. C. Carathéodory, *Conformal representation*, Second Edition, Cambridge, 1952.

3. ———, *Theory of functions*, Vol. I, New York, 1954.

4. E. F. Collingwood and M. L. Cartwright, *Boundary theorems for a function meromorphic in the unit circle*, Acta Math. 87 (1952), 83-146.

5. O. Lehto and K. L. Virtanen, *Boundary behaviour and normal meromorphic functions*, Acta Math. 97 (1957), 47-65.

6. W. Seidel and J. L. Walsh, *On the derivatives of functions analytic in the unit circle and their radii of univalence and of* p-*valence*, Trans. Amer. Math. Soc. 52 (1942), 128-216.

University of Notre Dame