APPROXIMATION OF ALGEBRAIC NUMBERS BY ALGEBRAIC NUMBERS

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Let $P(x) = \Sigma_{i=0}^{n} a_i x^i$ be a polynomial with arbitrary complex coefficients whose leading coefficient a_n is not 0. We call n the *degree*, $h = \max_{i \leq n} |a_i|$ the *height*, and $s = \Sigma_{i=0}^{n} |a_i|$ the *size* of the polynomial P(x). To every algebraic number α there corresponds a polynomial P(x) of lowest degree with $P(\alpha) = 0$ and such that its coefficients are rational integers without a common divisor. The degree, the height, and the size of this polynomial are called the *degree*, the *height*, and the *size* of α, respectively. We denote the set of all polynomials with rational integral coefficients whose degrees, heights, and sizes are $n > 0$, $h > 0$, and $s > 0$, respectively, by $\mathfrak{P}(n, h, s)$, and the set of all algebraic numbers satisfying the same conditions by $\mathfrak{A}(n, h, s)$. By $\mathfrak{P}^*(n, h, s)$ we denote the corresponding set of polynomials with arbitrary complex coefficients. In order to have a simple way of stating the theorems, we shall make use of these symbols even if not all of the numbers n, h, and s are actually needed.

It is well known that, for an algebraic number $\alpha \in \mathfrak{A}(m, h, s)$, the value of a polynomial $P(x) \in \mathfrak{P}(n, k, t)$ for which $P(\alpha) \neq 0$ cannot be arbitrarily small. In T. Schneider's *Einführung in die transzendenten Zahlen* we find the proof of the following theorem [10, Theorem 3]:

Let $\alpha \in \mathfrak{A}(m, h, s)$ *be an algebraic number whose leading coefficient is* a, *and let* $P(x) \in \mathfrak{P}(n, k, t)$ *be a polynomial for which* $P(\alpha) \neq 0$. *Then*

$$|P(\alpha)| > |a|^{-nm} (n+1)^{-n(m-1)} (h+1)^{-n(m-1)} k^{-(m-1)}. \tag{1}$$

A similar theorem holds for polynomials in several variables. N. I. Feldman ([3], Lemma 6; [4], Lemma 2) proved the following result:

Let

$$A(x_1, \cdots, x_m) = \sum_{i_1=0}^{N_1} \sum_{i_2=0}^{N_2} \cdots \sum_{i_m=0}^{N_m} a_{i_1 i_2 \cdots i_m} x^{i_1} x^{i_2} \cdots x^{i_m}$$

be a polynomial in m *variables* x_i, *of degrees* N_i *in* x_i $(i = 1, 2, \cdots, m)$, *with rational integral coefficients satisfying the inequality* $|a_{i_1 i_2 \cdots i_m}| \leq h$. *Let* $\alpha_i \in \mathfrak{A}(n_i, h_i, s_i)$ $(i = 1, 2, \cdots, m)$ *be* m *algebraic numbers for which*

$$A(\alpha_1, \alpha_2, \cdots, \alpha_m) \neq 0;$$

and let q *be the degree of the field* $R(\alpha_1, \alpha_2, \cdots, \alpha_m)$ *over the field* R *of rational numbers. Then*

$$|A(\alpha_1, \alpha_2, \cdots, \alpha_m)| \geq (8^{N_1+N_2+\cdots+N_m} h\, h_1^{N_1/n_1} h_2^{N_2/n_2} \cdots h_m^{N_m/n_m})^{-q}. \tag{2}$$

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(I proved Theorem 6 of the present paper in November, 1960, without knowing of the papers [3] and [4] of Feldman.)

Related to the problem of finding lower bounds for the values of polynomials at algebraic points is the following question: If $\alpha_1 \in \mathfrak{A}(n_1, h_1, s_1)$ and $\alpha_2 \in \mathfrak{A}(n_2, h_2, s_2)$ are different algebraic numbers, what can be said about their difference in terms of their degrees, heights and sizes? Clearly, an inequality of the type (2) can be applied to this problem (one need only consider the polynomial $A(x_1, x_2) = x_1 - x_2$). E. Bombieri [1] found, for such pairs of numbers, the bound

$$(3) \qquad |\alpha_1 - \alpha_2| > (4n_1 n_2)^{-3n_1n_2} h_1^{-n_2} h_2^{-n_1}.$$

While Bombieri considered only algebraic numbers, K. Mahler [7] generalized these theorems to the zeros of two polynomials with arbitrary complex coefficients. He proved the following theorem:

Let γ *be any zero of the polynomial* $P(x) \in \mathfrak{P}^*(m, h, s)$, *let* δ *be any zero of the polynomial* $Q(x) \in \mathfrak{P}^*(n, k, t)$, *and let* R *denote the resultant of* $P(x)$ *and* $Q(x)$. *Then either*

$$(4) \quad |\gamma - \delta| \geq 1 \quad or \quad |\gamma - \delta| \geq \{(m+1)^{2n}(n+1)^{2m}4^{mn}c(P, Q)^{mn} h^n k^m\}^{-1} |R|,$$

where $c(P, Q)$ *is* 1 *when at least one of the polynomials has only real zeros, and where otherwise* $c(P, Q) = \min(m+1, n+1)$.

Mahler derived also a lower bound for the difference of distinct zeros γ_1 and γ_2 of a single polynomial. His result is as follows:

Let γ_1 *and* γ_2 *be two different zeros of the polynomial* $P(x) \in \mathfrak{P}^*(m, h, s)$ *with discriminant* D. *Then*

$$(5) \qquad |\gamma_1 - \gamma_2| \geq \{(m+1)^{4m} 4^{m^2} c(P)^{m^2} h^{2m-1}\}^{-1} |aD|,$$

where a *is the leading coefficient of* $P(x)$ *and where* $c(P)$ *is* 1 *when all zeros of* $P(x)$ *are real, while otherwise* $c(P)$ *has the value* $m + 1$. (Prof. Mahler told me the estimates (4), (5) and (6) on July 8, 1960. At that time I had proved weaker theorems than these stated here, because I had used, instead of the inequality (6) of Mahler, the inequality $|\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_m}| \leq 2^n(n+1)h/|a|$ of N. I. Feldman [2, Lemma 2].)

Inequalities of the type (1), (2), (3), (4), and (5) are important in the theory of transcendental numbers (see, besides the references mentioned above, the book by A. O. Gelfond [5, Section 2, Lemmas 2 and 3]). Clearly, an inequality such as (1) can be used to construct transcendental numbers. It was the idea of Mahler [6] to use the accuracy with which polynomials in transcendental numbers γ approximate the number 0 as a means of classifying the transcendental numbers. Similarly, J. F. Koksma [9] subdivided the transcendental numbers into different classes according to the accuracy with which they can be approximated by algebraic numbers.

The purpose of this note is to give new proofs for these inequalities and to sharpen them. The first four theorems deal with polynomials with arbitrary complex coefficients and their zeros. The last four theorems are applications of the first four theorems to polynomials with rational integral coefficients and to algebraic numbers. The improvement of the inequality (1) is contained in Theorem 5. Theorem 6 is a generalization of Theorem 5 to polynomials in m variables, and it represents an improvement of Feldman's inequality (2). The improvements of Mahler's inequalities are given in Theorem 3 and Theorem 4.

We use in an essential way the following result of Mahler [8], which is a sharpening of a lemma of Feldman [2, Lemma 2]

LEMMA 1. *Let* $P(x) \in \mathfrak{P}^*(n, h, s)$ *have the leading coefficient* a *and the zeros* $\gamma_1, \gamma_2, \cdots, \gamma_n$. *Then, if the subscripts* $i_1, i_2, \cdots, i_m$ *are distinct,*

(6)
$$|\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_m}| \leq s/|a| \leq (n+1)h/|a|.$$

The following is an immediate consequence.

LEMMA 2. *Under the conditions of Lemma* 1,

(7)
$$\max(1, |\gamma_{i_1}|)\max(1, |\gamma_{i_2}|)\cdots\max(1, |\gamma_{i_m}|) \leq s/|a| \leq (n+1)h/|a|.$$

LEMMA 3. *Let* γ *be a root of the polynomial* $P(x) = a(x-\gamma)(x-\gamma_2)\cdots(x-\gamma_n)$ *in* $\mathfrak{P}^*(n, h, s)$, *and let* δ *be any complex number* ($|\delta| \leq 1$). *Then*

(8)
$$|\gamma - \delta| \geq \frac{|P(\delta)|}{2^{n-1}s}.$$

Proof. We establish first the inequality

$$|a||(\delta-\gamma_2)(\delta-\gamma_3)\cdots(\delta-\gamma_n)| \leq 2^{n-1}s.$$

When we write out the product on the left side, we get a sum of 2^{n-1} terms. By Lemma 1, the products of numbers γ_i in each of these terms are bounded by $s/|a|$. Since $|\delta| \leq 1$, each term is bounded by $|a|(s/|a|)\cdot 1 = s$. This implies that

$$|\gamma - \delta| = \frac{|P(\delta)|}{|a||(\delta-\gamma_2)(\delta-\gamma_3)\cdots(\delta-\gamma_n)|} \geq \frac{|P(\delta)|}{2^{n-1}s}.$$

LEMMA 4. *Let* γ *be a zero of the polynomial* $P(x) \in \mathfrak{P}^*(n, h, s)$, *and let* a *be the leading coefficient of* $P(x)$. *Then* $|\gamma| \leq 2h/|a|$.

Proof. The proof of the first lemma of Schneider's book [10] can be used. One has only to replace α by γ and a_0 by a. From Schneider's result, $|\gamma| \leq h/|a| + 1$, Lemma 4 follows since $h \geq |a|$.

We are now ready to prove

THEOREM 1. *Let* $P(x) \in \mathfrak{P}^*(n_1, h_1, s_1)$ *and* $Q(x) \in \mathfrak{P}^*(n_2, h_2, s_2)$ *be two polynomials, and let* γ *be one of the zeros of* $Q(x)$. *Then*

(9)
$$|P(\gamma)| \geq \frac{|R|}{s_1^{n_2-1}s_2^{n_1}} \geq \frac{|R|}{\{(n_1+1)h_1\}^{n_2-1}\{(n_2+1)h_2\}^{n_1}},$$

where R *denotes the resultant of the polynomials* $P(x)$ *and* $Q(x)$. *For* $n_2 = 2$,

(10)
$$|P(\gamma)| \geq \frac{|R|}{(2h_2)^{n_1}s_1} \geq \frac{|R|}{(2h_2)^{n_1}(n_1+1)h_1}.$$

Proof. Let $\gamma = \gamma_1, \gamma_2, \cdots, \gamma_{n_2}$ be the zeros of $Q(x)$. Let the absolute value of the leading coefficient of $Q(x)$ be denoted by c. Then

$$|R| = c^{n_1} \prod_{i=1}^{n_2} |P(\gamma_i)|.$$

Hence

(11) $$|P(\gamma)| = \frac{|R|}{c^{n_1} \prod_{i=2}^{n_2} |P(\gamma_i)|}.$$

For $i = 2, 3, \cdots, n_2$, we have

$$|P(\gamma_i)| \le s_1 \max(1, |\gamma_i|^{n_1}).$$

It follows that

$$\left| \prod_{i=2}^{n_2} P(\gamma_i) \right| \le s_1^{n_2-1} \prod_{i=2}^{n_2} \max(1, |\gamma_i|^{n_1}) = s_1^{n_2-1} \left(\prod_{i=2}^{n_2} \max(1, |\gamma_i|) \right)^{n_1}.$$

By applying Lemma 2 to the last product, we get

$$\left| \prod_{i=2}^{n_2} P(\gamma_i) \right| \le s_1^{n_2-1} (s_2/c)^{n_1}.$$

Hence, by (11), we obtain

$$|P(\gamma)| \ge \frac{|R|}{c^{n_1} s_1^{n_2-1} (s_2/c)^{n_1}} = \frac{|R|}{s_1^{n_2-1} s_2^{n_1}}.$$

The inequality (10) is obtained in the following way: By Lemma 4, $|\gamma_2| \le 2h_2/c$. Thus

$$|P(\gamma_2)| \le s_1 (2h_2/c)^{n_1}.$$

Since $|R| = c^{n_1} |P(\gamma)| \, |P(\gamma_2)|$, we get

$$|P(\gamma)| \ge \frac{|R|}{c^{n_1} s_1 (2h_2/c)^{n_1}} = \frac{|R|}{(2h_2)^{n_1} s_1} \ge \frac{|R|}{(2h_2)^{n_1} (n_1 + 1) h_1}.$$

This completes the proof of Theorem 1.

We now prove the following generalization of Theorem 1:

THEOREM 2. *Let*

$$C(x_1, x_2, \cdots, x_m) = \sum_{i_1=0}^{N_1} \sum_{i_2=0}^{N_2} \cdots \sum_{i_m=0}^{N_m} a_{i_1 i_2 \cdots i_m} x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}$$

be a polynomial with arbitrary complex coefficients in m *variables* x_i, *of degrees* N_i *in* x_i $(i = 1, 2, \cdots, m)$, *and of size*

$$s = \sum_{i_1=0}^{N_1} \sum_{i_2=0}^{N_2} \cdots \sum_{i_m=0}^{N_m} |a_{i_1 i_2 \ldots i_m}|;$$

and let $\bar{s} = \max(1, s)$. *Let* $P_i(x) \in \mathfrak{P}^*(n_i, h_i, s_i)$ *be* m *polynomials which have the zeros* $\gamma_i = \gamma_{i1}, \gamma_{i2}, \cdots, \gamma_{in_i}$, *and let* b_i *denote the absolute value of the highest coefficient of* $P_i(x)$. *If* $C(\gamma_1, \gamma_2, \cdots, \gamma_m) \neq 0$, *then*

$$|C(\gamma_1, \gamma_2, \cdots, \gamma_m)|$$
$$\geq \bar{s}\left(\bar{s}(s_1/b_1)^{N_1/n_1} (s_2/b_2)^{N_2/n_2} \cdots (s_m/b_m)^{N_m/n_m}\right)^{-n_1 n_2 \cdots n_m} |S|,$$

where

$$S = \prod_{i_1=1}^{n_1} \prod_{i_2=1}^{n_2} \cdots \prod_{i_m=1}^{n_m} C(\gamma_{1i_1}, \gamma_{2i_2}, \cdots, \gamma_{mi_m}).$$

S *can be replaced by any product of factors* $C(\gamma_{1i_1}, \gamma_{2i_2}, \cdots, \gamma_{mi_m})$ *that contains at least the factor* $C(\gamma_1, \gamma_2, \cdots, \gamma_m)$ *and in which any two factors differ in at least one subscript. In particular, all vanishing factors can be omitted.*

Proof. Let $\tilde{S}$ be any product of factors $C(\gamma_{1i_1}, \gamma_{2i_2}, \cdots, \gamma_{mi_m})$ with the mentioned properties. For brevity, put $C = C(\gamma_1, \gamma_2, \cdots, \gamma_m)$. Then evidently

$$\tilde{S}/C \leq T/C, \tag{12}$$

where T arises from $\tilde{S}$ on replacement of each factor $C(\gamma_{1i_1}, \gamma_{2i_2}, \cdots, \gamma_{mi_m})$ of $\tilde{S}$ by $\max\{1, |C(\gamma_{1i_1}, \gamma_{2i_2}, \cdots, \gamma_{mi_m})|\}$. Since the subscripts of two factors of the product $\tilde{S}$ differ in at least one subscript, it is evident that

$$\frac{T}{C} \leq \prod_{\substack{i_1=1 \\ (i_1,\cdots,i_m)\neq(1,\cdots,1)}}^{n_1} \prod_{i_2=1}^{n_2} \cdots \prod_{i_m=1}^{n_m} \max\{1, |C(\gamma_{1i_1}, \gamma_{2i_2}, \cdots, \gamma_{mi_m})|\}. \tag{13}$$

We now use the inequality

$$\max\{1, |C(\gamma_{1i_1}, \gamma_{2i_2}, \cdots, \gamma_{mi_m})|\} \leq \bar{s} \prod_{j=1}^{m} \max(1, |\gamma_{ji_j}|^{N_j}),$$

which holds for each factor in (13). It follows that

$$\prod_{i_1=1}^{n_1} \max\{1, |C(\gamma_{1i_1}, \cdots, \gamma_{mi_m})|\} \leq \bar{s}^{n_1} \prod_{i_1=1}^{n_1} \max(1, |\gamma_{1i_1}|^{N_1}) \prod_{j=2}^{m} \max(1, |\gamma_{ji_j}|^{N_j n_1})$$
$$\leq \bar{s}^{n_1} (s_1/b_1)^{N_1} \prod_{j=2}^{m} \{\max(1, |\gamma_{ji_j}|)\}^{N_j n_1},$$

where the last inequality is obtained by Lemma 2. In the same way, we find that

$$\prod_{i_1=1}^{n_1}\prod_{i_2=1}^{n_2} \max\{1, |C(\gamma_{1i_1}, \gamma_{2i_2}, \cdots, \gamma_{mi_m})|\}$$

$$\le \bar{s}^{n_1 n_2}(s_1/b_1)^{N_1 n_2}(s_2/b_2)^{n_1 N_2}\prod_{j=3}^{m}\{\max(1, |\gamma_{ji_j}|)\}^{N_j n_1 n_2},$$

and finally

$$\prod_{i_1=1}^{n_1}\prod_{i_2=1}^{n_2}\cdots\prod_{\substack{i_m=1\\(i_1,\cdots,i_m)\neq(1,\cdots,1)}}^{n_m} \max\{1, |C(\gamma_{1i_1}, \gamma_{2i_2}, \cdots, \gamma_{mi_m})|\} \tag{14}$$

$$\le \bar{s}^{n_1 n_2\cdots n_m - 1}\left((s_1/b_1)^{N_1/n_1}(s_2/b_2)^{N_2/n_2}\cdots(s_m/b_m)^{N_m/n_m}\right)^{n_1 n_2\cdots n_m}.$$

The theorem now follows from the inequalities (12), (13) and (14).

THEOREM 3. *Let* $P(x) \in \mathfrak{P}^*(n_1, h_1, s_1)$ *and* $Q(x) \in \mathfrak{P}^*(n_2, h_2, s_2)$ *have zeros* γ *and* δ, *respectively. Then*

$$|\gamma - \delta| > \frac{|R|}{2^{\max(n_1,n_2)-1}\, s_1^{n_2}\, s_2^{n_1}}, \tag{15}$$

where R *denotes the resultant of* $P(x)$ *and* $Q(x)$.

Proof. We distinguish three cases.

1) If $|\gamma| \le 1$, then, according to Lemma 3,

$$|\gamma - \delta| \ge \frac{|Q(\gamma)|}{2^{n_2-1}\, s_2}.$$

By Theorem 1 it follows that

$$|\gamma - \delta| \ge \frac{|R|}{2^{n_2-1}\, s_2\, s_1^{n_2}\, s_2^{n_1-1}} \ge \frac{|R|}{2^{\max(n_1,n_2)-1}\, s_1^{n_2}\, s_2^{n_1}}.$$

2) If $|\gamma| > 1$ and $|\delta| \le 1$, we find, as in part 1, that

$$|\gamma - \delta| \ge \frac{|R|}{2^{n_1-1}\, s_1^{n_2}\, s_2^{n_1}} \ge \frac{|R|}{2^{\max(n_1,n_2)-1}\, s_1^{n_2}\, s_2^{n_1}}.$$

3) If $|\gamma| > 1$ and $|\delta| > 1$, we consider $1/\gamma$ and $1/\delta$. Let $\tilde{P}(x) = x^{n_1} P(1/x)$. Then $\tilde{P}(1/\gamma) = 0$. Since the degree, the height and the size of $\tilde{P}(x)$ are the same as those of $P(x)$, we see that $\tilde{P}(x) \in \mathfrak{P}^*(n_1, h_1, s_1)$. Similarly,

$$\tilde{Q}(x) = x^{n_2} Q(1/x) \in \mathfrak{P}^*(n_2, h_2, s_2).$$

Since $|1/\gamma| < 1$, the inequality (15) holds when γ and δ are replaced by $1/\gamma$ and $1/\delta$. But

$$|\gamma - \delta| = |\gamma\delta| \left|\frac{1}{\gamma} - \frac{1}{\delta}\right| > \left|\frac{1}{\gamma} - \frac{1}{\delta}\right|.$$

This concludes the proof of Theorem 3.

Theorem 3 does not give an estimate for the difference of zeros of one polynomial, since in this case $R = 0$. Therefore we prove

THEOREM 4. *Let* $P(x) \in \mathfrak{P}^*(n, h, s)$ *have zeros* $\gamma_1, \gamma_2, \cdots, \gamma_n$. *Then, for any subscripts* i_0 *and* i_1 $(i_0 \neq i_1)$,

$$|\gamma_{i_0} - \gamma_{i_1}| \geq \frac{|aD|^{1/2}}{(4n)^{(n-2)/2}\, s^{(2n-1)/2}},$$

where a *denotes the leading coefficient and* D *the discriminant of* P(x).

Proof. From

$$D = a^{2n-2} \prod_{i<k} (\gamma_i - \gamma_k)^2 \qquad \text{and} \qquad P'(\gamma_i) = a \prod_{\substack{k=1 \\ k\neq i}}^{n} (\gamma_i - \gamma_k),$$

where $P'(x)$ stands for the derivative of $P(x)$ with respect to x, it follows that

$$D = a^{n-2} \prod_{i=1}^{n} P'(\gamma_i).$$

Without loss of generality, we can assume that $i_0 = 1$ and $i_1 = 2$. Since

$$P'(\gamma_1) = a(\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3) \cdots (\gamma_1 - \gamma_n)$$

and

$$P'(\gamma_2) = a(\gamma_2 - \gamma_1)(\gamma_2 - \gamma_3) \cdots (\gamma_2 - \gamma_n),$$

we obtain

$$|\gamma_1 - \gamma_2|^2 = \frac{|P'(\gamma_1)\, P'(\gamma_2)|}{a^2\, |(\gamma_1 - \gamma_3)(\gamma_1 - \gamma_4) \cdots (\gamma_1 - \gamma_n)(\gamma_2 - \gamma_3) \cdots (\gamma_2 - \gamma_n)|}$$

$$= \frac{|D|}{|a^n (\gamma_1 - \gamma_3)(\gamma_1 - \gamma_4) \cdots (\gamma_1 - \gamma_n)(\gamma_2 - \gamma_3) \cdots (\gamma_2 - \gamma_n) \prod_{i=3}^{n} P'(\gamma_i)|}.$$

For the factors in the denominator we have the bounds

$$|(\gamma_1 - \gamma_3) \cdots (\gamma_1 - \gamma_n)(\gamma_2 - \gamma_3) \cdots (\gamma_2 - \gamma_n)|$$

$$\leq 4^{n-2} \max(1, |\gamma_1|^{n-2}) \max(1, |\gamma_2|^{n-2}) \prod_{i=3}^{n} \max(1, |\gamma_i|^2)$$

and

$$\left| \prod_{i=3}^{n} P'(\gamma_i) \right| \leq (ns)^{n-2} \prod_{i=3}^{n} \max(1, \gamma_i)^{n-1}.$$

The last inequality holds because $P'(x) \in \mathfrak{P}^*(n-1, nh, ns)$. Therefore we get

$$|\gamma_1 - \gamma_2|^2 \geq \frac{|D|}{|a|^n (4ns)^{n-2} \Pi_{i=1}^n \{\max(1, |\gamma_i|)\}^{n+1}}.$$

It follows from Lemma 2 that

$$|\gamma_1 - \gamma_2|^2 \geq \frac{|D|}{|a|^n (4ns)^{n-2} (s/|a|)^{n+1}} = \frac{|aD|}{(4n)^{n-2} s^{2n-1}}.$$

By taking square roots, we obtain Theorem 4.

We now apply these theorems to algebraic numbers and polynomials with rational integral coefficients.

THEOREM 5. *Let* $\alpha \in \mathfrak{A}(m, h, s)$ *and* $P(x) \in \mathfrak{P}(n, k, t)$ *be such that* $P(\alpha) \neq 0$. *Then*

$$|P(\alpha)| \geq \frac{1}{s^n t^{m-1}} \geq \frac{1}{\{(m+1)h\}^n \{(n+1)k\}^{m-1}}. \tag{16}$$

If $m = 2$, *then*

$$|P(\alpha)| \geq \frac{1}{(2h)^n t} \geq \frac{1}{(2h)^n (n+1)k}. \tag{17}$$

Proof. Let $Q(x) \in \mathfrak{P}(m, h, s)$ be the polynomial for which $Q(\alpha) = 0$, and let $\alpha_1, \alpha_2, \cdots, \alpha_m$ be the conjugates of α. $Q(x)$ is irreducible, since according to the definition of $\mathfrak{A}(m, h, s)$, it is the polynomial of lowest degree that vanishes for α. Hence it follows from $P(\alpha) \neq 0$ that $P(\alpha_i) \neq 0$ for $i = 2, 3, \cdots, m$. This implies that the resultant R of $P(x)$ and $Q(x)$ does not vanish, and since R is an integer, $|R| \geq 1$. From Theorem 1 we now obtain the inequalities (16) and (17).

THEOREM 6. *Let*

$$A(x_1, x_2, \cdots, x_m) = \sum_{i_1=0}^{N_1} \sum_{i_2=0}^{N_2} \cdots \sum_{i_m=0}^{N_m} a_{i_1 i_2 \cdots i_m} x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}$$

be a polynomial in m *variables* x_i, *of degrees* N_i *in* x_i $(i = 1, \cdots, m)$, *of size* s, *and with rational integral coefficients. Let* $\alpha_i \in \mathfrak{A}(n_i, h_i, s_i)$ $(i = 1, 2, \cdots, m)$ *be* m

algebraic numbers for which $A(\alpha_1, \alpha_2, \cdots, \alpha_m) \neq 0$, *and let* q *be the degree of the finite algebraic extension which is obtained by the adjunction of the algebraic numbers* $\alpha_1, \alpha_2, \cdots, \alpha_m$ *to the rational number field. Then*

$$|A(\alpha_1, \alpha_2, \cdots, \alpha_m)| \geq s(s_1^{N_1/n_1} s_2^{N_2/n_2} \cdots s_m^{N_m/n_m})^{-q}.$$

Proof. Let $\alpha_{j1}, \alpha_{j2}, \cdots, \alpha_{jq}$ be the field conjugates of α_j. Since

$$A = A(\alpha_1, \alpha_2, \cdots, \alpha_m)$$

is a nonzero algebraic number, we know that its norm $N(A)$ does not vanish, and we have

$$N(A) = \prod_{i=1}^{q} A(\alpha_{1i}, \alpha_{2i}, \cdots, \alpha_{mi}). \tag{18}$$

From this it follows that

$$A(\alpha_1, \alpha_2, \cdots, \alpha_m) = \frac{N(A)}{\Pi_{i=2}^{q} A(\alpha_{1i}, \alpha_{2i}, \cdots, \alpha_{mi})}. \tag{19}$$

For each $i = 2, 3, \cdots, q$, we have the inequality

$$A(\alpha_{1i}, \alpha_{2i}, \cdots, \alpha_{mi}) \leq s \prod_{j=1}^{m} \alpha_{ji}^{N_j} \leq s \prod_{j=1}^{m} \max(1, |\alpha_{ji}|)^{N_j}.$$

Hence we obtain

$$\prod_{i=2}^{q} A(\alpha_{1i}, \alpha_{2i}, \cdots, \alpha_{mi}) \leq s^{q-1} \prod_{i=1}^{q} \prod_{j=1}^{m} \max(1, |\alpha_{ji}|)^{N_j}.$$

Now the set of the q field conjugates of α_j consists of q/n_j sets of the conjugates of α_j. Hence, by applying Lemma 2 separately to each set of conjugates of α_j, we get the inequality

$$\prod_{i=2}^{q} A(\alpha_{1i}, \alpha_{2i}, \cdots, \alpha_{mi}) \leq s^{q-1} \prod_{j=1}^{m} \left(\frac{s_j}{a_j}\right)^{N_j q/n_j}, \tag{20}$$

where a_j is the absolute value of the leading coefficient of the polynomial $P_j(x) \in \mathfrak{P}(n_j, h_j, s_j)$ for which $P(\alpha_j) = 0$.

We now prove the inequality

$$N(A) \geq \prod_{j=1}^{m} \left(\frac{1}{a_j}\right)^{N_j q/n_j}. \tag{21}$$

From (18) we have

$$N(A) = \prod_{i=1}^{q} \sum_{i_1=0}^{N_1} \sum_{i_2=0}^{N_2} \cdots \sum_{i_m=0}^{N_m} a_{i_1 i_2 \cdots i_m} \alpha_{1i}^{i_1} \alpha_{2i}^{i_2} \cdots \alpha_{mi}^{i_m}. \tag{22}$$

As i assumes the values 1, 2, ⋯, q, every conjugate α_{ji} of α_j ($j = 1, 2, \cdots, m$) is assumed q/n_j times. Therefore the exponent of α_{ji} in each term of (22) is not greater than $N_j q/n_j$. This means that

$$N(A) \prod_{j=1}^{m} a^{N_j q/n_j}$$

is an algebraic integer, and since N(A) is a rational number, it follows that it is a rational integer. From $N(A) \neq 0$ we finally deduce the inequality (21). Now the inequality of the theorem follows immediately from (19), (20) and (21).

COROLLARY. *Let* $\alpha_i \in \mathfrak{A}(n_i, h_i, s_i)$ *be different algebraic numbers for* $i = 1, 2$. *Then*

$$|\alpha_1 - \alpha_2| \geq 2^{1-q}(s_1^{1/n_1} s_2^{1/n_2})^{-q},$$

where q *is the degree of the field* $R(\alpha_1, \alpha_2)$.

Proof. Take $A(x_1, x_2) = x_1 - x_2$ in Theorem 6.

In the case $q = n_1 n_2$, the following theorem gives a sharpening of this corollary.

THEOREM 7. *Let* $\alpha \in \mathfrak{A}(n_1, h_1, s_1)$ *and* $\beta \in \mathfrak{A}(n_2, h_2, s_2)$ *be nonconjugate algebraic numbers. Then*

$$|\alpha - \beta| \geq 2^{1-\max(n_1,n_2)} s_1^{-n_2} s_2^{-n_1} \geq 2^{1-\max(n_1,n_2)} \{(n_1+1)h_1\}^{-n_2} \{(n_2+1)h_2\}^{-n_1}.$$

Proof. The suppositions about α and β imply that the resultant R in Theorem 3 is a nonvanishing integer.

For conjugate algebraic numbers we find the following result:

THEOREM 8. *If* $\alpha_1 \in \mathfrak{A}(n, h, s)$ *and* $\alpha_2 \in \mathfrak{A}(n, h, s)$ *are conjugate algebraic numbers, then*

$$|\alpha_1 - \alpha_2| \geq (4n)^{-(n-2)/2} s^{-(2n-1)/2} \geq (4n)^{-(n-2)/2} \{(n+1)h\}^{-(2n-1)/2}. \tag{23}$$

Proof. Let $P(x) \in \mathfrak{P}(n, h, s)$ be the polynomial for which $P(\alpha_1) = P(\alpha_2) = 0$. Then P(x) is irreducible. Hence the discriminant D of P(x) does not vanish. Since the coefficients of P(x) are integers, it follows that $|D| \geq 1$ and that $|a| \geq 1$. By using Theorem 4, we obtain the inequality (23).

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