ON THE MAXIMAL DOMAIN OF A "MONOTONE" FUNCTION

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1. INTRODUCTION

Let $\mathfrak{X}$ be a Hilbert-space. In $\mathfrak{X} \times \mathfrak{X}$, we define the M-relation by

$$(x_1, y_1)\, M\, (x_2, y_2) \qquad \text{provided } \Re \langle x_1 - x_2, y_1 - y_2 \rangle \geq 0.$$

(The symbol $\Re$ may be dropped if the scalars are real.) This relation has been studied in previous papers (for example [2], [3]). In harmony with these papers, we shall say that a set $E \subset \mathfrak{X} \times \mathfrak{X}$ is *totally-M-related* provided $(x_1, y_1), (x_2, y_2) \in E$ implies $(x_1, y_1)\, M\, (x_2, y_2)$. We shall say that a map F from a subset of $\mathfrak{X}$ into $\mathfrak{X}$ is *monotone* provided its graph in $\mathfrak{X} \times \mathfrak{X}$ is totally-M-related.

A *real linear subspace* is a subset $\mathfrak{X}_0 \subset \mathfrak{X}$ such that β_1, β_2 real and $x_1, x_2 \in \mathfrak{X}_0$ imply $\beta_1 x_1 + \beta_2 x_2 \in \mathfrak{X}_0$. A *real affine variety* is a translate of a real linear subspace. We call a set $Q \subset \mathfrak{X}$ *almost-convex* provided it contains the interior of its convex hull $K(Q)$, where the "interior" is taken relative to the smallest real affine variety containing Q (or equivalently, $K(Q)$).

2. THE THEOREM

THEOREM. *Let $\mathfrak{X}$ be a finite-dimensional Hilbert-space, with real or complex scalars, and let $E \subset \mathfrak{X} \times \mathfrak{X}$ be a maximal totally-M-related set. Let P be the projection $P(x, y) = x$. Then $P(E)$ is an almost-convex set.*

Proof. Our object is to show that $\text{int } K[P(E)] \subset P(E)$. For the moment, we restrict attention to the case where the scalars of $\mathfrak{X}$ are real. Let $x_0 \in \text{int } K[P(E)]$. Without loss of generality, we shall assume that x_0 is the zero-vector θ; for if this does not hold, the translation $x \to x - x_0$ (leaving the y's fixed) will carry E into a new maximal totally-M-related set E', and x_0 into θ, and so forth. Thus the "affine variety" of the theorem becomes "linear subspace."

Furthermore, we lose no generality by assuming that the "interior" is taken relative to the space $\mathfrak{X}$. Suppose that $\mathfrak{X}_0$, the subspace spanned by $K[P(E)]$, is of dimension less than that of $\mathfrak{X}$, and let $\mathfrak{X}_0^{\perp}$ be the orthogonal complement of $\mathfrak{X}_0$. Then each vector y can be resolved as $y = y^0 + y^1$, where $y^0 \in \mathfrak{X}_0$ and $y^1 \in \mathfrak{X}_0^{\perp}$, and it is easily seen that $(x, y) \in E$ if and only if $(x, y^0) \in E$, and that the image E_0 of E under the map $(x, y) \to (x, y^0)$ is a maximal totally-M-related set in $\mathfrak{X}_0 \times \mathfrak{X}_0$ such that $P(E_0) = P(E)$.

With these assumptions, we proceed. Let S be a sphere with center θ such that $S \subset K[P(E)]$. It is easy to find a finite set F of vectors of S which generate $\mathfrak{X}$ (considered as a convex cone). Each vector of F, in turn, is a finite linear combination, with positive coefficients, of vectors of $P(E)$, so that we can find a finite set $x_1, \cdots, x_m$ of vectors of $P(E)$ which generate $\mathfrak{X}$.

Consider now the polyhedral convex set

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$$C = \bigcap_{i=1}^{m} \{y: \langle x_i - \theta, y_i - y \rangle \geq 0\} .$$

(It will be seen later that C is nonvacuous, but at the moment we wish to show only that C is compact, so that we can ignore the vacuous case.) The set

$$C' = \bigcap_{i=1}^{m} \{y: \langle x_i, y \rangle \leq 0\}$$

consists only of the vector θ; for suppose that $y \neq \theta$. Then $y = \sum_{i=1}^{m} c_i x_i$, where $c_i \geq 0$, and thus

$$0 < \|y\|^2 = \sum_{i=1}^{m} \langle c_i x_i, y \rangle = \sum_{i=1}^{m} c_i \langle x_i, y \rangle ,$$

so that at least one of the $\langle x_i, y \rangle$ is positive. Hence, by the Resolution Theorem for Polyhedral Convex Sets (see [1, Theorem 1]), C is the sum of a *bounded* convex polyhedral set and the vector θ; that is, C is bounded. Being closed, it is compact.

Now, it is clear that if a family of closed sets has the properties

(i) the intersection of some finite subfamily is compact, and

(ii) the intersection of *any* finite subfamily is nonempty,

then the whole family has nonempty intersection. We have just shown that the family

$$\{y: \langle x_\alpha - \theta, y_\alpha - y \rangle \geq 0\} \qquad ((x_\alpha, y_\alpha) \in E)$$

has property (i); the fact that it has property (ii) is precisely the main theorem of [2]. Let y denote a point of the common intersection. We now have $(x_\alpha, y_\alpha)\, M\, (\theta, y)$ for all $(x_\alpha, y_\alpha) \in E$, and hence $(\theta, y) \in E$ and $\theta \in P(E)$.

The case where the scalars of $\mathfrak{X}$ are complex is now taken care of by the fact that, with $[x, y] = \Re \langle x, y \rangle$, $\mathfrak{X}$ becomes a Hilbert-space with *real* scalars and twice the former dimension.

3. REMARKS

By the symmetry of the definition of the M-relation, it is clear that if we define $P_1(x, y) = y$, then $P_1(E)$ is also an almost-convex set.

An interesting related fact, trivial to prove, is that if P is regarded as a map from E to $\mathfrak{X}$, the inverse-image of a point is a convex set in $\mathfrak{X} \times \mathfrak{X}$.

The theorem could be rewritten as a theorem on the *domain* of a monotone function with maximal domain. For it is clear that if the domain is already maximal, the further extension of the *graph* to be a maximal totally-M-related set does not add any further points to the domain. If the theorem is rephrased in this way, it loses the symmetry mentioned above, unless one knows that the graph is already maximal —see [3] for sufficient conditions.

In the view of some readers, the theorem may take a more natural form if the M-relation is defined on $\mathfrak{X} \times \mathfrak{Y}$, where $\mathfrak{X}$ is a finite-dimensional linear space and $\mathfrak{Y}$ is the dual-space. (A Hilbert-space structure can always be imposed on $\mathfrak{X}$, and then $\mathfrak{Y}$ can be identified with $\mathfrak{X}$.)

REFERENCES

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