PARTNERSHIP GAMES WITH SECRET SIGNALS PROHIBITED

Martin Fox and Herman Rubin

The ethics of bridge prohibit the use of secret signals by either partnership. This is explicitly stated in Law 5 of *The Laws of Duplicate Contract Bridge* (see [2], pp. 223-224). Two game-theoretical formalizations of this rule are:

1. *Whenever an agent of either player is required to make a bid or to play a card as defender, he must announce his behavioral strategy as well as the bid or play which results from the randomization required by that behavioral strategy.*

2. *Instead of announcing the behavioral strategy to all other agents, the agent who is moving announces it to a referee. The referee announces to the other agents their a posteriori probabilities for the hands unseen by them under the assumption of the previous sequence of bids and plays.*

These formalizations are never equivalent, as considerably more information is divulged in the first than in the second. In Section 4, we give a sufficient condition that, from the standpoint of the value of the game, the information given in Formalization 2 is as good as that in Formalization 1.

In Section 1, an abstract version of bridge will be introduced.

In Section 2 it will be shown that abstract bridge with all agents always required to announce behavioral strategies is a perfect information game according to the definition by Blackwell and Girshick [1, Definition 1.7.1]. Hence, with this rule the game has a pure value.

In actual bridge, a declarer is not required to announce his behavioral strategies. Thus, in abstract bridge we may wish to allow an agent to stop announcing his behavioral strategies at some point in the game when he knows that his partners are not required to make any further choices. An example due to Fox [3] shows that in this case the game may not have a value.

In Section 3 it is shown that the game does have a value provided that, if an agent is no longer required to announce behavioral strategies, then his actions at each of his moves are restricted to a finite set.

In Section 4 we again assume that all behavioral strategies are announced. It is shown that if each agent has perfect information about the other agents' actions, then ε-good strategies exist in which the behavioral strategies depend only on the a posteriori probabilities for the deal.

In Section 5 it is shown that both players have good strategies provided the action-spaces are all finite. This is true even if behavioral strategies are not always announced.

Thompson [7] has given an example of a simplified bridge game in which the optimal strategies differ according to whether secret signals are or are not permitted. His example is valid even for our more general strategy space, in the case of no secret signals.

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1. ABSTRACT BRIDGE

In this section abstract bridge will be introduced.

Let Ω be the space of possible deals. Assume Ω is countable. With suitable regularity conditions, Ω can be taken to be arbitrary and many of the results of this paper will hold. Let $P(\omega)$ be the probability that the deal is ω. We assume P is known to all agents.

Let the total number of agents of the two players be k. When the deal is ω, the j-th agent $(j = 1, 2, \cdots, k)$ observes the value of $f_{0j}(\omega)$, where f_{0j} is some function on Ω.

Assume the game contains only a finite number, say m, of personal moves. For $i = 1, 2, \cdots, m$, let A_i be the set of all actions which might be legal for the agent making the i-th personal move. Due to the past history of the game, some of the elements of A_i may be illegal moves at the time the i-th move is to be made. It will be assumed, without loss of generality, that the agent who will make this move will not depend on the past history of the game.

If the deal is ω and actions $a_1, a_2, \cdots, a_i$ have been taken, then the agent who takes the $(i + 1)$st action observes the value of $f_i(\omega; a_1, a_2, \cdots, a_i)$, where f_i is some function on $\Omega \times A_1 \times \cdots \times A_i$.

If the deal is ω and actions $a_1, a_2, \cdots, a_m$ are taken, then the payoff is $g(\omega; a_1, a_2, \cdots, a_m)$. If one or more of the a_i are replaced by randomized behavioral strategies, then the value of the function g obtained is the expected value of g under these behavioral strategies.

2. ABSTRACT BRIDGE AS A PERFECT INFORMATION GAME

Assume that the behavioral strategies are announced for all personal moves in an abstract bridge game. It will be shown that abstract bridge is then a perfect information game, according to the definition by Blackwell and Girshick. From this it follows that the game has a pure value.

Without loss of generality, assume that agent 1 makes the first personal move and that he is an agent of player I.

A pure strategy for player I is given by a pair (μ_1, z), where μ_1 is a behavioral strategy on A_1 and z is a pure strategy in abstract bridge with $m - 1$ personal moves on $A_2, A_3, \cdots, A_m$. The deal in the later game occurs in $\Omega \times A_1$, according to the probability measure P' for which P is the marginal measure on Ω and $\mu_1(a_1; f_{01})$ is the conditional probability measure on A_1, given f_{01}. Thus, for every $\omega \in \Omega$ and $a_1 \in A_1$, we have $P'(\omega, a_1) = P(\omega)\mu_1(a_1; f_{01}(\omega))$. Since P is known and μ_1 is announced, the measure P' is known.

A pure strategy for player II is a function y on the space of behavioral strategies on A_1, where $y(\mu_1)$ is a pure strategy in the $(m - 1)$-personal-move abstract bridge game of the previous paragraph.

Let M be the payoff function in the original abstract bridge game and M_{μ_1} the payoff function in the game with $m - 1$ personal moves. Then,

$$M((\mu_1, z), y) = M_{\mu_1}(z, y(\mu_1)).$$

However, abstract bridge with zero personal moves has a constant payoff, so that it is a perfect information game of order zero. We have shown that if every abstract bridge game with $m - 1$ personal moves is a perfect information game of order $m - 1$, then every abstract bridge game with m personal moves is a perfect information game of order m.

3. THE CASE OF A PLAYER WHO IS REDUCED TO A SINGLE AGENT

In this section we consider the case of a player who is reduced to a single agent. In this case we may wish to permit that player to stop announcing behavioral strategies. However, such a game may not have a value. We shall show here that the game has a value provided that the first player to stop announcing behavioral strategies has only a finite number of possible actions for any of the moves on which he does not announce.

We first consider the case in which player I never announces his behavioral strategies. It is sufficient to prove existence of a value when Ω is a finite set.

Assume player I makes moves $m_1, m_2, \cdots, m_r$. Then a pure strategy for player I is a vector $x = (x_1, x_2, \cdots, x_r)$, where x_α is a function taking values in A_{m_α}. There are $m_\alpha - \alpha + 2$ arguments for x_α. Of these, $m_\alpha - \alpha$ are the behavioral strategies μ_i $(i < m_\alpha)$ for player II, while the remaining are the values of f_{01} (assuming player I is agent 1) and of $f_{m_\alpha - 1}$. Thus, given the μ_i for

$$i \neq m_1, m_2, \cdots, m_r,$$

the remaining arguments can only take a finite number of values.

Fix a pure strategy y for player II (the μ_i for $i \neq m_1, m_2, \cdots, m_r$). Let $x^{(\beta)}$ be a convergent net of pure strategies for player I (convergent in the product topology with the topology in each coordinate space taken to be discrete). For discussion of these topological notions see Kelley [4], pages 37 (discrete topology), 65-66 (nets and Moore-Smith convergence), and 90 (product topology). Let x be the limit of this net. The action from A_{m_1} which player I takes depends on only a finite number of components of his strategy. Let β_1 be such that if $\beta \geq \beta_1$ ($\geq$ is the relation directing the net), then these components of $x^{(\beta)}$ are the same as those of x. (Eventually they are equal, since they converge and take only a finite number of possible values.)

Now consider the components giving the actions of player I chosen from A_{m_2}. There exists $\beta_2 \geq \beta_1$ such that if $\beta \geq \beta_2$, then these components of $x^{(\beta)}$ are the same as those of x. Continuing in this way, we see that there exists β^* such that if $\beta \geq \beta^*$, then those components of $x^{(\beta)}$ which determine actions taken are all equal to the corresponding components of x.

Letting M be the payoff function, we see that for $\beta \geq \beta^*$ we have

$$M(x^{(\beta)}, y) = M(x, y).$$

Hence, M is continuous in x for each fixed y. But Sion [5] has shown that any game $G = (X, Y, M)$ has a value if X is a compact Hausdorff space and $M(x, y)$ is upper semicontinuous in x for each fixed y. The result by Sion actually follows directly from a theorem of his in [6]. This completes the proof.

It should be noted that when Sion's theorem applies, player I has a good strategy. The infimum of a family of linear functions is a concave function. Hence, player I is trying to maximize a concave function over a compact convex set. But the maximum of such a function always exists on such a set.

We now consider abstract bridge with the assumptions of this section. By the introduction of moves not affecting the payoff, we can assume that one of the players ceases announcing behavioral strategies on the i_0-th move of the game. We have shown that the game has a value if $i_0 = 1$. Using the induction of Section 2, we can show that the game has a value if $i_0 = 2, 3, \cdots, m$.

4. BEHAVIORAL STRATEGIES AS FUNCTIONS OF A POSTERIORI PROBABILITIES

In this section we show that if $f_i(\omega; a_1, a_2, \cdots, a_i) = (a_1, a_2, \cdots, a_i)$ for some i, then the agent making the $(i + 1)$st move suffers no loss if he uses behavior strategies dependent only on his a posteriori probabilities. It is assumed that this agent knows the behavioral strategies used on the first i moves. Thus, we see that the second statement in the Introduction can be used when these conditions hold for all $i = 1, 2, \cdots, m - 1$ and that the game will have the same value.

It is sufficient to prove this for $i = 1$. Assume that agents 1 and 2 make the first and second moves, respectively. The assumptions of the paragraph above are that agent 2 is told μ_1, the behavioral strategy for agent 1 for the first move and that $f_2(\omega; a_1) = a_1$.

Suppose behavioral strategies $\mu_1, \mu_2, \cdots, \mu_m$ are used. Then, the expected payoff is

$$\sum_{\omega}\sum_{a_1} g(\omega; a_1, \mu_2, \cdots, \mu_m)\, \mu_1(a_1; f_{01}(\omega))\, P(\omega)$$

$$= \sum_{\omega}\sum_{a_1} g(\omega; a_1, \mu_2, \cdots, \mu_m)\, P'(\omega, a_1),$$

where P' is the joint frequency function on $\Omega \times A_1$. But $P'(\omega, a_1) = \rho(a_1)\,\nu(\omega; a_1)$, where ρ is the marginal frequency function on A_1, and ν is the a posteriori frequency function on Ω. Hence, the expected payoff depends on μ_1 only through ν and, without suffering any loss, agent 2 can play the game as if he were told only ν.

5. EXISTENCE OF OPTIMAL STRATEGIES

We have seen in Section 3 that, under the conditions of that section, a player who never announces his behavioral strategies has a good strategy. In this section we assume that all the A_i are finite and we show that both players have good strategies. No assumptions are made about which behavioral strategies are announced, except that a player not reduced to a single agent always must announce.

By the result in Section 3, the game has a value, since the A_i are finite. Hence, by the spy-proof property of optimal strategies, we may assume that all behavioral strategies are announced, so that our game is of the form considered in Section 2.

Let $M^{(P)}$ be the payoff function when P is the measure used on the space Ω of deals. Let x and y be strategies for players I and II respectively. We wish to show, for fixed x and y, that $M^{(P)}(x, y)$ is continuous in P. Let P' be another measure on Ω for which $\Sigma_{\omega \in \Omega} |P(\omega) - P'(\omega)| < \varepsilon$. Assume that when the strategies x and y are used, the sequence of behavioral strategies becomes $\mu_1, \mu_2, \cdots, \mu_m$. Then,

$$M^{(P)}(x, y) = \sum_{\omega \in \Omega} P(\omega) g(\omega; \mu_1, \mu_2, \cdots, \mu_m),$$

so that

$$|M^{(P)}(x, y) - M^{(P')}(x, y)| \leq \sum_{\omega \in \Omega} |P(\omega) - P'(\omega)| \, |g(\omega; \mu_1, \mu_2, \cdots, \mu_m)|$$

$$\leq K \sum_{\omega \in \Omega} |P(\omega) - P'(\omega)| < K\varepsilon,$$

where

$$K = \sup_{\omega; \mu_1, \mu_2, \cdots, \mu_m} |g(\omega; \mu_1, \mu_2, \cdots, \mu_m)| .$$

Hence, $M^{(P)}(x, y)$ is continuous as a function of P for fixed x and y. Thus, v(P), the value of the game when P is the measure used on Ω, is also a continuous function.

Assume that the players have good strategies in every (m - 1) personal move game in which the A_i are finite. This is certainly true if m = 1. Let M be the payoff function in an m-personal-move game. We carry out the induction of Section 2, so that $M((\mu_1, z), y) = M_{\mu_1}(z, y(\mu_1))$, where M_{μ_1} is the payoff function in an (m - 1)-personal-move game which has value $v(\mu_1)$.

By the induction hypothesis, for each μ_1, there exist $z(\mu_1)$ and $y^*(\mu_1)$ such that

$$M_{\mu_1}(z(\mu_1), y(\mu_1)) \geq c(\mu_1) \qquad \text{for all } y(\mu_1),$$

$$M_{\mu_1}(z, y^*(\mu_1)) \leq v(\mu_1) \text{ for all } z.$$

Let $v = \sup_{\mu_1} v(\mu_1)$. Let $\{\mu_{1n}\}$ be a sequence of behavioral strategies on A_1 such that $v(\mu_{1n}) \uparrow v$. Let μ_1^* be the limit of a convergent subsequence. But $v(\mu_1)$ is a continuous function of μ_1, so that $v(\mu_1^*) = v$. Thus,

$$M((\mu_1^*, z(\mu_1^*)), y) \geq v \qquad \text{for all } y,$$

$$M((\mu_1, z), y^*) \leq v \qquad \text{for all } \mu_1 \text{ and } z.$$

Hence, $(\mu_1^*, z(\mu_1^*))$ and y^* are good strategies for players I and II, respectively.

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Michigan State University