ON SOME METRIC PROPERTIES OF POLYNOMIALS WITH REAL ZEROS, II

Ch. Pommerenke

1. Let

$$f(z) = \prod_{\nu=1}^{n} (z - x_\nu) \qquad (x_\nu \text{ real}), \tag{1}$$

and let E: $|f(z)| \leq 1$. A closed disk that has a segment $[a, b]$ $(a \leq b)$ of the real axis X as diameter will be called the *orthogonal circle over* $[a, b]$. By $\mathfrak{C}$ we shall denote the class of all closed bounded sets F such that for every $z \in F$ there exists an orthogonal circle K with $z \in K$ and $K \subset F$. Every E: $|f(z)| \leq 1$ belongs to $\mathfrak{C}$ [3] and has cap E = 1. Later on (see Remark 2 after Theorem 2) we shall show that there are sets $F \in \mathfrak{C}$ with cap F = 1 that cannot be approximated by lemniscate domains $|f(z)| \leq 1$, where f(z) has only real zeros.

THEOREM 1. *Let* $F \in \mathfrak{C}$. *If* Λ *and* d *are the sums of the perimeters and diameters of the components of* F, *then*

$$\Lambda \leq \pi d \leq 4\pi \operatorname{cap} F.$$

Remark. If we take F_m as the union of the orthogonal circles

$$|z - (2k - 1)/m| \leq 1/m \quad (k = 1, \cdots, 2m),$$

we have $d_m = 4$, $\Lambda_m = 4\pi$. Because F_m is contained in the rectangle

$$\{0 \leq \Re z \leq 4, \ |\Im z| \leq 1/m\},$$

it follows that cap $F_m \to 1$. The example shows that the inequalities

$$\Lambda \leq 4d \leq 4\pi \operatorname{cap} F$$

and $\Lambda \leq 4\pi \operatorname{cap} F$ are best possible (except that perhaps $\Lambda < 4\pi \operatorname{cap} F$).

COROLLARY. *If the polynomial* f(z) *has the form* (1) *and if* Λ *is the length of the lemniscate* $|f(z)| = 1$, *then* $\Lambda \leq 4\pi$.

LEMMA. *Let* K_1 *and* K_2 *be orthogonal circles over* $[a_j, b_j]$ $(j = 1, 2)$, *and let* $a_1 < a_2 < b_1 < b_2$. *Let* L_j *denote the arc of the periphery of* K_j *which lies between the points of intersection with the periphery of* K_{3-j} *and contains the point* a_j. *If* l_j *is the length of* L_j $(j = 1, 2)$ *and* l_0 *is the perimeter of the orthogonal circle over* $[a_1, a_2]$, *then* $l_1 \leq l_0 + l_2$.

Proof. Let K_0^* be the orthogonal circle over $[a_2, b_1]$, of perimeter l_0^*, let L_1^* denote the complement of L_1 relative to the periphery of K_1, and let l_1^* denote the length of L_1^*. The convex curve $L_2 \cup L_1^*$ contains the (convex) circle K_0^* in its closed interior. Hence, by a classical theorem, $l_0^* \leq l_2 + l_1^*$. Since

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$$l_1 + l_1^* = \pi(b_1 - a_1) = \pi(a_2 - a_1) + \pi(b_1 - a_2) = l_0 + l_0^*,$$

we obtain

$$l_1 = l_0 + l_0^* - l_1^* \leq l_0 + (l_2 + l_1^*) - l_1^* = l_0 + l_2.$$

Proof of Theorem 1. (1.1) Suppose $F = \bigcup_{\mu=1}^m K_\mu$, where K_μ denotes the orthogonal circles over $[a_\mu, b_\mu]$. We may assume that $a_1 < \cdots < a_m$ and that no K_μ is contained in the union of the rest (otherwise, we can delete this K_μ). We have $d = \text{meas}\,(F \cap X)$, where X denotes the real axis. We shall prove that $\Lambda \leq \pi d$, by induction. For $m = 1$, we have $\Lambda = \pi d$. For $m > 1$, there are two cases:

(i) $b_1 \leq a_2$: Here K_1 is contained in $\Re z \leq a_2$, and $F_1 = \bigcup_{\mu=2}^m K_\mu$ in $\Re z \geq a_2$. Therefore $\Lambda = \pi(b_1 - a_1) + \Lambda_1$, where Λ_1 is the perimeter of F_1, and by the induction hypothesis, $\Lambda_1 \leq \pi d_1$ (d_1 = diameter-sum for F_1), hence

$$\Lambda \leq \pi(b_1 - a_1) + \pi d_1 = \pi d.$$

(ii) $a_2 < b_1$: We apply the lemma to K_1 and K_2. If Λ_1 denotes the length of the part of the boundary of F that belongs to F_1, then $\Lambda = l_1 + \Lambda_1$. The part of the boundary of F_1 that is contained in K_1 belongs entirely to K_2 (otherwise, $K_2 \subset K_1 \cup K_{\mu'}$ for some μ'). If l_2 is its length, and if l_0 is again the perimeter of the orthogonal circle K_0 over $[a_1, a_2]$, it follows from the lemma that

$$\Lambda = l_1 + \Lambda_1 \leq l_0 + l_2 + \Lambda_1,$$

and the last quantity is equal to the perimeter Λ^* of $F^* = K_0 \cup F_1$. The diameter-sum d^* is equal to d. We have case (i) for F^*, and therefore $\Lambda \leq \Lambda^* \leq \pi d$.

(1.2) Let F be an arbitrary connected set in $\mathfrak{C}$. The part of the boundary of F that lies in $\Im z \geq 0$ is a simple curve of the form

$$C = \{ z = x + iy(x) \;:\; x_0 \leq x \leq x_0 + d \},$$

where $y(x)$ is a single-valued real function. The length of C is $\Lambda/2$. We choose points $z_\mu \in C$ ($\mu = 1, \cdots, m$) with $x_1 < \cdots < x_m$ such that

$$\sum_{\mu=1}^{m} |z_\mu - z_{\mu-1}| > \Lambda/2 - \varepsilon \qquad (\text{or} > 1/\varepsilon \text{ if } \Lambda = \infty). \tag{2}$$

For each μ, let K_μ be an orthogonal circle through z_μ with $K_\mu \subset F$. The set

$$\tilde{F} = (X \cap F) \cup \bigcup_\mu F_\mu$$

is connected and has again the diameter d. Let $\tilde{C}$ be the part of the boundary of $\tilde{F}$ contained in $\Im z \geq 0$. If the segments of X contained in $\tilde{C}$ have the total length d', the length of $\tilde{C}$ is

$$\tilde{\Lambda}/2 = d' + \Lambda''/2,$$

where $\tilde{\Lambda}$ and Λ'' are the perimeters of $\tilde{F}$ and $\bigcup K_\mu$. Applying part (1.1), we obtain

(3) $$\tilde{\Lambda} \leq 2d' + \pi d'' \leq \pi(d' + d'') = \pi d.$$

Because

$$\tilde{C} = \{z = x + i\tilde{y}(x) : x_0 \leq x \leq x_0 + d\},$$

with single-valued $\tilde{y}(x)$, and $z_\mu \in \tilde{C}$, $x_1 < \cdots < x_m$, we have

$$\tilde{\Lambda}/2 \geq \sum |z_\mu - z_{\mu-1}|,$$

and therefore, by (2) and (3),

$$\Lambda < \pi d + 2\varepsilon \qquad (\text{or } \Lambda < 2/\varepsilon < \pi d \text{ if } \Lambda = \infty)$$

for every $\varepsilon > 0$, hence $\Lambda < \infty$, $\Lambda \leq \pi d$.

(1.3) If $F \in \mathfrak{C}$ is not connected, we obtain $\Lambda \leq \pi d$ by adding the corresponding inequalities for the components of F. Finally we have $d \leq 4 \operatorname{cap} F$ (this inequality was first proved by Pólya; see also [3]), and hence $\Lambda \leq \pi d \leq 4\pi \operatorname{cap} F$.

2. Again, let E: $|f(z)| \leq 1$, where f(z) is a polynomial of the form (1). Let ρ be the radius of the largest (orthogonal) circle contained in E, and b the width of E. Since there exists an orthogonal circle $K \subset E$ through each $z \in E$, we have (with $z = x + iy$)

(4) $$b = 2\rho = 2 \max_{z \in E} |y|.$$

It is easy to see [3] that $b \leq 2$ (and that this also holds for all sets F in $\mathfrak{C}$ whose capacity is ≤ 1), with equality for the set E: $|z| \leq 1$. Using a parameter, I shall give a sharper upper bound for b, for the case where E is symmetric with respect to the point 0.

THEOREM 2. *Let the distribution of the zeros* x_ν *of* f(z) *be symmetrical with respect to* 0, *and let* $a = |f(0)|^{1/n}$. *Then*

$$b \leq \begin{cases} 1/a & \text{if } \frac{1}{2}\sqrt{2} \leq a \leq \infty, \\ 2(1-a^2)^{1/2} & \text{if } 0 \leq a \leq \frac{1}{2}\sqrt{2} \text{ and } |x_\nu| \leq \frac{1}{2}\sqrt{2}, \end{cases}$$

with equality for $f_0(z) = z^2 - a^2$.

Remarks. 1. The first inequality $b \leq 1/a$ remains true for all $a > 0$, but it is not longer the best bound (at least for $a \leq \frac{1}{2}$). For $0 \leq a \leq \frac{1}{2}\sqrt{2}$, the inequality $b \leq 2(1-a^2)^{1/2}$ probably holds even without the restriction $|x_\nu| \leq \frac{1}{2}\sqrt{2}$.

2. The function

$$z = \left(\frac{i}{2} \log \frac{iw + 1}{iw - 1}\right)^{-1} = w + \cdots$$

maps $|w| > 1$ conformally onto the complementary region of the union F_0^* of the orthogonal circles $|z \pm 2/\pi| \leq 2/\pi$. Consequently $\operatorname{cap} F_0^* = 1$. Since $0.6 < 2/\pi$, there exists a $c > 2/\pi$ such that the union F_0 of the orthogonal circles $|z \pm c| \leq 0.6$ has again capacity 1. Suppose that F_0 could be approximated arbitrarily closely by lemniscate domains $|f(z)| \leq 1$ determined by polynomials with real zeros. Then there would exist a polynomial $f(z) = z^n + \cdots$ such that

$$|f(\pm c + 0.55i)| < 1, \qquad |f(0)| > 1,$$

because $0 \notin F_0$. The polynomial

$$g(z) = (-1)^n f(z) f(-z) = z^{2n} + \cdots$$

would have real zeros symmetrical to 0, and it would satisfy the conditions

$$|g(c + 0.55i)| < 1, \qquad |g(0)| > 1.$$

Therefore the width of the set $|g(z)| \leq 1$ would be greater than 1.10, whereas the width is at most 1, by Theorem 2.

Proof. We may assume, by [3, Theorem 2], that $f(0) \neq 0$, that is, $x_\nu \neq 0$. Since the zeros are symmetrically distributed with respect to 0, we can group them in pairs x_ν, $-x_\nu$ ($\nu = 1, \cdots, m$), where $m = n/2$. Then

$$(5) \qquad |f(z)|^2 = \prod_{\nu=1}^{m} |z^2 - x_\nu^2|^2 = \prod_{\nu=1}^{m} (x^4 + 2x^2y^2 + y^4 - 2x_\nu^2 x^2 + 2x_\nu^2 y^2 + x_\nu^4).$$

Therefore

$$|f(z)|^2 = \prod_{\nu=1}^{m} [(x^2 + y^2 - x_\nu^2)^2 + 4x_\nu^2 y^2] \geq \prod_{\nu=1}^{m} 4x_\nu^2 y^2 = 4^m \left(\prod_{\nu=1}^{m} x_\nu^2 \right) y^{2m}.$$

If $z \in E$, then $1 \geq 2^n |f(0)| \cdot |y|^n$ and

$$|y| \leq \frac{1}{2} |f(0)|^{-1/n} = 1/(2a),$$

and because of (4) we have proved the first inequality of Theorem 2.

Let $|x_\nu| \leq \frac{1}{2}\sqrt{2}$. Then $a = \left(\prod_{1}^{m} x_\nu^2 \right)^{1/n} \leq \frac{1}{2}\sqrt{2}$.

From (5) we obtain

$$|f(z)|^2 = \prod_{\nu=1}^{m} [(y^2 + x_\nu^2)^2 + x^4 + 2x^2(y^2 - x_\nu^2)].$$

Suppose there exists a $z \in E$ with $y^2 > 1 - a^2$. Then

$$y^2 - x_\nu^2 > 1 - a^2 - x_\nu^2 \geq 0,$$

because $a^2 \leq \frac{1}{2}$ and $x_\nu^2 \leq \frac{1}{2}$, and

$$1 \geq |f(z)|^{1/m} \geq \prod_{\nu=1}^{m} (y^2 + x_\nu^2)^{1/m}.$$

We apply the inequality [2, p. 55]

$$\prod_{\nu=1}^{m} (a_\nu + b_\nu)^{1/m} \geq \prod_{\nu=1}^{m} a_\nu^{1/m} + \prod_{\nu=1}^{m} b_\nu^{1/m}$$

(for $a_\nu \geq 0$, $b_\nu \geq 0$) and obtain (with $n = 2m$)

$$1 \geq y^2 + \left(\prod_{\nu=1}^{m} x_\nu^2 \right)^{2/n} = y^2 + a^2,$$

and therefore $y^2 \leq 1 - a^2$, contrary to our hypothesis.

Finally, let $f_0(z) = z^2 - r^2$ $(r \geq 0)$ and E_0: $|z^2 - r^2| \leq 1$. Computation shows that

$$b_0 = 2 \max_{z \in E_0} |y| = \begin{cases} 2(1 - r^2)^{1/2} & \text{for } 0 \leq r \leq \frac{1}{2}\sqrt{2}, \\ 1/r & \text{for } \frac{1}{2}\sqrt{2} \leq r < \infty. \end{cases} \tag{6}$$

Because $a = |f_0(0)|^{1/2} = r$, this proves the statement about equality.

THEOREM 3. *If* $|x_\nu| \leq r$, *then*

$$b \geq \begin{cases} 2(1 - r^2)^{1/2} & \textit{for } 0 \leq r \leq \frac{1}{2}\sqrt{2}, \\ 1/r & \textit{for } \frac{1}{2}\sqrt{2} \leq r \leq 1, \end{cases}$$

with equality for $f_0(z) = z^2 - r^2$.

Remark. If $r < 2$, the segment $[-r, +r]$ has capacity $r/2 < 1$. Therefore, as Erdös, Herzog and Piranian proved [1, Theorem 6], there exists a $\rho(r) > 0$ (independent of $f(z)$) such that E contains a disk of radius $\rho(r)$ if $|x_\nu| \leq r$. Hence $b = 2\rho \geq 2\rho(r)$. Theorem 3 gives the best lower bound for b and therefore for $\rho(r)$, if $r \leq 1$. A similar method yields $b \geq (2 - r^2)^{1/2}$ for $1 < r < \sqrt{2}$, but this is not the best estimate, at least if r is near $\sqrt{2}$.

Proof. Let $|z_\nu| \leq r$ and $r \leq \frac{1}{2}\sqrt{2}$. Then

$$|f(i(1 - r^2)^{1/2})|^2 = \prod_{\nu=1}^{n} |i(1 - r^2)^{1/2} - x_\nu|^2 = \prod_{\nu=1}^{n} (1 - r^2 + x_\nu^2) \leq 1.$$

Therefore the point $i(1 - r^2)^{1/2}$ belongs to E, and $b \geq 2(1 - r^2)^{1/2}$.

If $\frac{1}{2}\sqrt{2} \leq r \leq 1$, let $\xi = (r^2 - (2r)^{-2})^{1/2}$. Then

$$|f(-\xi + i(2r)^{-1})\, f(+\xi + i(2r)^{-1})|^2 = \prod \left| \left(-\xi + \frac{i}{2r} - x_\nu\right) \left(\xi + \frac{i}{2r} - x_\nu\right) \right|^2$$

$$= \prod \left[\left(\xi^2 + \frac{1}{4r^2} - x_\nu^2\right)^2 + x_\nu^2 r^{-2} \right]$$

$$= \prod (r^4 - 2x_\nu^2 r^2 + x_\nu^2 r^{-2} + x_\nu^4)$$

$$\leq \max(r^{4n}, 1) \leq 1$$

(note that the factor occurring under the last product sign is a quadratic function of x_ν^2). Hence one of the two points $z = \pm\xi + i(2r)^{-1}$ belongs to E, and $b \geq 2(2r)^{-1} = r^{-1}$. Equation (6) shows that we have equality for $f_0(z)$.

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Universität Göttingen