ON NORMAL EPr MATRICES

Martin Pearl

1. INTRODUCTION

In an earlier paper [1], the author considered the problem of characterizing normal matrices and EPr matrices with elements from an arbitrary field F with an involutory automorphism $\lambda: a \leftrightarrow \bar{a}$. A matrix A of order n and rank r with elements from F is called an EPr matrix if it satisfies the condition

$$\sum_{i=1}^{n} a_i A_i = 0 \text{ if and only if } \sum_{i=1}^{n} \bar{a}_i A^i = 0 \qquad (a_i \in F), \tag{1}$$

where A_i is the i^{th} row of A and A^i is the i^{th} column of A. In [1], it was shown that if A is a normal matrix and A has the same rank as AA^*, then the matrix A is an EPr matrix. Furthermore, the following theorem was proved.

THEOREM. *Let* A *have the same rank as* AA^*. *Then a necessary and sufficient condition that* A *be normal is that* $A^* = NA = AN$ *for some nonsingular matrix* N.

The main purpose of this paper is to show (Section 3) that the condition that A have the same rank as AA^* can be eliminated. It will be assumed that the reader is familiar with [1], and the definitions and notation introduced there will also be used here.

2. PRELIMINARY RESULTS

For the proof of the main theorem we shall need the following two theorems.

THEOREM 1 (Schwerdtfeger [2, p. 51]). *The crossing matrix (that is, the matrix formed by the intersection) of* r *linearly independent columns with* r *linearly independent rows of an* $m \times n$ *matrix of rank* r $(m \geq r,\ n \geq r)$ *has rank* r.

THEOREM 2. *Let* B, C *and* D *be* $s \times s$, $t \times s$ *and* $s \times t$ *matrices, respectively. A necessary and sufficient condition that there exist a* $t \times t$ *matrix* X *such that*

$$N(X) = \begin{bmatrix} B & D \\ C & X \end{bmatrix} \tag{2}$$

is nonsingular is that the rank of each of the matrices

$$\begin{bmatrix} B \\ C \end{bmatrix} \quad \textit{and} \quad [B \quad D]$$

be s.

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Proof. That the condition is necessary is clear.

The proof of the sufficiency is by an induction on t. First we shall show that if B is nonsingular then the theorem is true. When B is nonsingular, we choose $X = I + CB^{-1}D$, since

$$N(I + CB^{-1}D) = \begin{bmatrix} B & D \\ C & I + CB^{-1}D \end{bmatrix} = \begin{bmatrix} B & 0 \\ C & I \end{bmatrix}\begin{bmatrix} I & B^{-1}D \\ 0 & I \end{bmatrix} \tag{3}$$

and both matrices in the last expression are nonsingular.

We shall henceforth assume that B is singular. Now, let $t = 1$, and let $\begin{bmatrix} B \\ C \end{bmatrix}$ and $[B \ \ D]$ each have rank s. If s were the rank of N, then, by Theorem 1, the crossing matrix B of $\begin{bmatrix} B \\ C \end{bmatrix}$ and $[B \ \ D]$ would have the rank s. Since, by supposition, the rank of B is less than s, the rank of N (which cannot be less than s), must be greater than s; that is, the rank of N is $s + 1$. As an $(s + 1) \times (s + 1)$ matrix, N is therefore nonsingular for every choice of the 1×1 matrix X.

Now, we suppose that the theorem is true when X is a $(t - 1) \times (t - 1)$ matrix. Again, it follows from Theorem 1 that for every choice of X, the rank of N(X) is greater than s. We momentarily set $X = 0$. We may then assume that the first row of $[C \ \ 0]$ together with the rows of $[B \ \ D]$ are linearly independent. Similarly, we may assume that the first column of $\begin{bmatrix} D \\ 0 \end{bmatrix}$ together with the columns of $\begin{bmatrix} B \\ C \end{bmatrix}$ are linearly independent. We now repartition N(0) by choosing B' to be the $(s + 1) \times (s + 1)$ matrix in the upper left-hand corner of N(0), C' to be the $(t - 1) \times (s + 1)$ matrix in the lower left-hand corner of N(0), and D' to be the $(s + 1) \times (t - 1)$ matrix in the upper right-hand corner of N(0), that is,

$$N(0) = \begin{bmatrix} B & D \\ C & 0 \end{bmatrix} = \begin{bmatrix} B' & D' \\ C' & 0 \end{bmatrix} \quad \text{and} \quad N(X) = \begin{bmatrix} B' & D' \\ C' & X' \end{bmatrix} \tag{4}$$

where X' is a $(t - 1) \times (t - 1)$ matrix. This is equivalent to setting the first row and first column of X equal to zero. If B' is nonsingular, then we set

$$X' = I + C'(B')^{-1}D'.$$

On the other hand, if B' is singular, the existence of a matrix X' for which N(X') is nonsingular follows from the induction hypothesis.

3. THE MAIN THEOREM

We are now prepared to prove the main theorem.

THEOREM 3. *A normal matrix* A *is an* EPr *matrix if and only if there exists a nonsingular matrix* N *such that* $A^* = NA = AN$.

Proof. It is clear that the condition is sufficient (see [1, Theorems 1 (v), 2]).

Now, let A be an $n \times n$ normal EPr matrix. Since A is an EPr matrix, its rank is r. Moreover (see [1, Theorem 1 (vi)]), there exists a nonsingular $r \times r$ matrix D and an $(n - r) \times r$ matrix X and a permutation matrix T such that

$$A = T\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & X^* \\ 0 & I \end{bmatrix}T^*, \tag{5}$$

where X^* and T^* are the conjugate (with respect to the involutory automorphism λ) transpose of X and T respectively. Taking the conjugate transpose of each matrix in equation (5) and reversing the order of the matrices on the right-hand side of the equation, we have

$$A^* = T\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}\begin{bmatrix} D^* & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & X^* \\ 0 & I \end{bmatrix}T^*. \tag{6}$$

Finally, substituting equations (5) and (6) into the equation

$$AA^* = A^*A \tag{7}$$

and simplifying, we see that the normality of A is equivalent to the condition

$$D(I + X^*X)D^* = D^*(I + X^*X)D. \tag{8}$$

With the help of equations (5) and (6), the general solution of the equation $A^* = NA$ can be shown to be

$$N = T\begin{bmatrix} D^*D^{-1} - PX & P \\ XD^*D^{-1} - QX & Q \end{bmatrix}T^*, \tag{9}$$

where P and Q are arbitrary $r \times (n - r)$ and $(n - r) \times (n - r)$ matrices, respectively. If we now substitute the expressions for A, A^*, and N given by equations (5), (6), and (9) into the equation $A^* = NA$ and simplify, we obtain

$$\begin{bmatrix} D^* & D^*X^* \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} D & DX^* \\ 0 & 0 \end{bmatrix}\begin{bmatrix} D^*D^{-1} - PX & P \\ XD^*D^{-1} - QX & Q \end{bmatrix}. \tag{10}$$

Thus the existence of a matrix N satisfying $A^* = NA = AN$ is reduced to finding matrices P and Q which satisfy the simultaneous equations

$$D^* = DD^*D^{-1} - DPX + DX^*XD^*D^{-1} - DX^*QX, \tag{11}$$

$$D^*X = DP + DX^*Q. \tag{12}$$

Substitution of (12) into (11) yields

$$D^* = DD^*D^{-1} + DX^*XD^*D^{-1} - DX^*X, \tag{11'}$$

which is equivalent to equation (8). Moreover, this process is reversible, that is, equation (11) follows from equations (8) and (12). Hence, to complete the proof of the theorem it is only necessary to show the existence of matrices P and Q which satisfy equation (12) and for which N, as given by equation (9) is nonsingular.

Since D is nonsingular, we may solve equation (12) for P:

$$P = D^{-1}D^*X - X^*Q. \tag{13}$$

Substituting this expression for P into equation (9), we have

$$N = T\begin{bmatrix} D^*D^{-1} - D^{-1}D^*X^*X + X^*QX & D^{-1}D^*X^* - X^*Q \\ XD^*D^{-1} - QX & Q \end{bmatrix}T^*. \tag{14}$$

We now set

$$M = \begin{bmatrix} I & X^* \\ 0 & I \end{bmatrix} T^*NT \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} = \begin{bmatrix} D^*D^{-1} + X^*XD^*D^{-1} & D^{-1}D^*X^* \\ XD^*D^{-1} & Q \end{bmatrix} \tag{15}$$

and show that Q can always be chosen so that M (and consequently N) is nonsingular. It follows from equation (15) that M may be expressed as

$$M = \begin{bmatrix} I + X^*X & D^{-1}D^*X^* \\ X & Q \end{bmatrix}\begin{bmatrix} D^*D^{-1} & 0 \\ 0 & I \end{bmatrix}. \tag{16}$$

However,

$$U = \begin{bmatrix} I & X^* \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ X & I \end{bmatrix} = \begin{bmatrix} I + X^*X & X^* \\ X & I \end{bmatrix} \tag{17}$$

is nonsingular. Hence the rank of the $n \times r$ matrix $\begin{bmatrix} I + X^*X \\ X \end{bmatrix}$ is r, and it follows that the first r columns of M are linearly independent. Similarly, the substitution of equation (8) into equation (15) yields another expression for M,

$$M = \begin{bmatrix} D^{-1}D^* + D^{-1}D^*X^*X & D^{-1}D^*X \\ XD^*D^{-1} & Q \end{bmatrix}, \tag{15'}$$

which may be rewritten in the form

$$M = \begin{bmatrix} D^{-1}D^* & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I + X^*X & X^* \\ XD^*D^{-1} & Q \end{bmatrix}. \tag{16'}$$

As before, it follows from equation (16') that the rank of the $r \times n$ matrix $[I + X^*X \;\; X^*]$ is r, and that the first r rows of M are linearly independent. By Theorem 2, there exists an $(n - r) \times (n - r)$ matrix Q such that M (and consequently N) is nonsingular; this completes the proof of the theorem.

We are now prepared to prove the theorem described in the Introduction.

THEOREM 4. *Let* A *be a matrix of order* n *and rank* r. *A necessary and sufficient condition that* A *be a normal* EPr *matrix is that* $A^* = NA = AN$ *for some nonsingular matrix* N.

Proof. If $A^* = NA = AN$, then $AA^* = ANA = A^*A$, and hence A is normal. Furthermore, the rank of A is r and, by Theorem 1 (v) of [1], A is an EPr matrix.

Conversely, if A is a normal EPr matrix, then the existence of a nonsingular matrix N for which $A^* = NA = AN$ is guaranteed by Theorem 3.

REFERENCES

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2. H. Schwerdtfeger, *Introduction to linear algebra and the theory of matrices*, P. Noordhoff, Groningen, 1950.

University of Maryland
College Park, Maryland