ON HYPERBOLIC SURFACES IN THREE-DIMENSIONAL EUCLIDEAN SPACE

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1. Let the function $f(x, y)$ be defined for all real (x, y). Then, if f possesses at least continuous partial derivatives up to second order, the surface defined by $t = f(x, y)$ in three-dimensional Euclidean space becomes in a natural manner a simply-connected Riemann surface. The question was raised by Loewner whether such surfaces can be of hyperbolic type. This question was answered in the affirmative by Osserman [4], [5], who gave two constructions of such a surface. The first type of surface was pieced together of plane pieces; thus it was not differentiable, and the proof that it was hyperbolic depended on a global property of the surface. The second construction provided an infinitely differentiable surface, but depended on considerations of imbedding Riemann covering surfaces in three-space. Recently, Huber [1] has given a much simpler and more natural construction of such a surface. His proof depends essentially on the estimation of the module of certain doubly-connected domains by a Dirichlet integral and on explicit analytic estimations of certain integrals involved.

In the present paper we shall give a construction of a hyperbolic surface in three-dimensional Euclidean space, built up of constituent pieces for which we require only certain rather general geometric properties. To prove that the surface obtained is hyperbolic we use only the notions of module of a quadrangle and quasi-conformal mapping.

2. Our basic building block is a portion of surface in three-space, lying simply over the $Z = X + iY$-plane, which we denote by $S(L, K, H, g, h)$, where L, K, H are positive constants with $L > 2(K + H)$, and where g, h refer to functions to be characterized below. Our portion of surface is given by an equation $U = F(X, Y)$, where F is defined for $0 \leq X \leq L$, $0 \leq Y \leq 1$. First,

$$F(X, Y) \equiv 0 \qquad (0 \leq X \leq K \text{ or } L - K \leq X \leq L,\ 0 \leq Y \leq 1).$$

Further,

$$F(X, Y) = g(Y) \qquad (K + H \leq X \leq L - (K + H),\ 0 \leq Y \leq 1),$$

where $g(Y)$ is an infinitely differentiable function satisfying

$$g(Y) = g(1 - Y) \quad (0 \leq Y \leq 1) \quad \text{and} \quad g(0) = g^{(n)}(0) = 0 \quad (n \geq 1).$$

Moreover, on $K \leq X \leq K + H$, $0 \leq Y \leq 1$, we require that $F(X, Y) = h(X, Y)$, where the latter function is chosen to satisfy $h(X, Y) = h(X, 1 - Y)$ $(0 \leq Y \leq 1)$ and to make $F(X, Y)$ infinitely differentiable on $K \leq X \leq K + H$, $0 \leq Y \leq 1$. For example, we may take $h(X, Y) = k(X)\,g(Y)$, where $k(X)$ is infinitely differentiable for

$$K \leq X \leq K + H$$

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and satisfies

$$k(K) = k^{(n)}(K) = 0, \quad k(K + H) = 1, \quad k^{(n)}(K + H) = 0 \quad (n \geq 1).$$

Evidently there are many such functions. Finally, we require that

$$F(X, Y) = h(L - X, Y) \quad (L - (K + H) \leq X \leq L - K, \ 0 \leq Y \leq 1).$$

The portion of surface thus obtained is evidently infinitely differentiable for $0 \leq X \leq L$, $0 \leq Y \leq 1$. To distinguish various functions obtained in this way, we denote this function by $F(X, Y; L, K, H, g, h)$.

3. We require several lemmas concerning the surface $S(L, K, H, g, h)$. The choice of the points $(0, 0, 0)$, $(L, 0, 0)$, $(0, 1, 0)$, $(L, 1, 0)$ as vertices determines a quadrangle [3; p. 16]. We denote the module of this quadrangle for the class of curves joining the sides on $Y = 0$, $Y = 1$ by $m(L, K, H, g, h)$.

LEMMA 1. *With preassigned* K, H *and given* $n > 1$, *we can determine* g, h *such that, for* L *large enough,* $m(L, K, H, g, h) < L/n$.

We choose $g(Y)$ so that

$$\int_0^1 [1 + (g'(Y))^2]^{1/2} \, dY = 2n$$

(as may be done in many ways), and h accordingly. By $\sigma|dw|$ we denote the conformal metric induced on $S(L, K, H, g, h)$ by the Euclidean metric. Let

$$G = \max_{0 \leq Y \leq 1} [1 + (g'(Y))^2]^{1/2}.$$

Let the area of $S(L, K, H, g, h)$ lying over the rectangles

$$0 \leq X \leq K + H, \ 0 \leq Y \leq 1 \quad \text{and} \quad L - (K + H) \leq X \leq L, \ 0 \leq Y \leq 1$$

be A. On the surface we now define the following metric, admissible in the module problem:

$$\rho(P)\,|dw(P)| = \frac{1}{2n}\,\sigma(P)\,|dw(P)| \quad (P \text{ over } K + H \leq X \leq L - (K + H), \ 0 \leq Y \leq 1),$$

$$\rho(P)\,|dw(P)| = \frac{1}{2n}\,G\,\sigma(P)\,|dw(P)| \quad (P \text{ over } 0 \leq X < K + H, \ 0 \leq Y \leq 1, \ L - (K + H) < X \leq L, \ 0 \leq Y \leq 1).$$

Evidently every curve in the given class has length at least one in this metric, since an arc joining the lines $Y = Y_1$, $Y = Y_2$ $(Y_1 < Y_2)$ has length at least

$$\frac{1}{2n}\int_{Y_1}^{Y_2} [1 + (g'(Y))^2]^{1/2} \, dY,$$

while the area of the surface is at most

$$\frac{1}{2n} L + \frac{1}{4n^2} G^2 A .$$

As soon as L is large enough, this is less than L/n.

The surface S(L, K, H, g, h) admits a conformal mapping p as a quadrangle, with the obvious correspondence of vertices onto a rectangle $0 < u < L'$, $0 < v < 1$, $(w = u + iv)$. In $0 < X < K$, $0 < Y < 1$ we can use Z as local uniformizing parameter, and by the reflection principle the preceding mapping can be extended to a mapping of $-K < X < K$, $-1 < Y < 2$. On this domain we denote the mapping function by p(Z).

LEMMA 2. *There exist positive constants* $\lambda(K)$, $\mu(K)$, *depending only on* K, *not on the other parameters, such that*

$$\lambda(K) \leq |p'(Z)| \leq \mu(K) \qquad (X = 0,\ 0 \leq Y \leq 1).$$

The function p(Z) maps the rectangle $-K < X < K$, $-1 < Y < 2$ into the strip $-1 < v < 2$, with the segment $X = 0$, $0 \leq Y \leq 1$ going into the segment $u = 0$, $0 \leq v \leq 1$. In many ways, for example by treating a suitable module problem, we see that $|p'(Z)|$ is bounded, say $|p'(Z)| \leq \mu(K)$, for $0 \leq Y \leq 1$. Similarly, we see that the image of $-K < X < K$, $-1 < Y < 2$ must cover a strip $-\nu(K) < u < \nu(K)$, $-1 < v < 2$, where $\nu > 0$ depends only on K. Applying the preceding argument to the mapping p^{-1} inverse to p, we obtain the bound

$$\lambda(K) \leq |p'(Z)| \qquad (X = 0,\ 0 \leq Y \leq 1).$$

LEMMA 3. *The surface* S(L, K, H, g, h) *admits a quasiconformal mapping* q *onto a rectangle* $0 < u < L'$, $0 < v < 1$, *with the obvious correspondence of vertices, with maximal dilation less than a quantity depending only on* K, *and such that*

(1) $$q(0, Y) = iY \qquad (0 \leq Y \leq 1),$$

(2) $$q(X, 1) = q(X, 0) + i \qquad (0 \leq X \leq L),$$

(3) $$q(L, Y) = iY \qquad (0 \leq Y \leq 1).$$

We start with the mapping p introduced above, and follow it with the self-mapping s of the rectangle $0 \leq u \leq L'$, $0 \leq v \leq 1$ defined by

$$s(u, v) = u + p^{-1}(iv).$$

Then evidently the mapping sp satisfies the condition (1). Conditions (2) and (3) follow from the symmetry of the surface S(L, K, H, g, h). The maximal dilation of the mapping q is not greater than $\max(\mu(K), 1/\lambda(K))$, where the quantities are those of Lemma 2. For a similar argument in a slightly different context, see [2].

4. Construction: *We preassign the quantities* K, H, *then determine a sequence of values* L_j *and associated functions* g_j, h_j $(j = 0, 1, 2, \cdots)$ *such that, by Lemma* 1,

$$m(L_j, K, H, g_j, h_j) < \frac{1}{n_j} L_j ,$$

where $\sum_{j=0}^{\infty} n_j^{-1} < \infty$. *Then we define the function* $f(x, y)$ *by*

$$f(x, y) = 0 \qquad (x \leq 0),$$

$$f(x, y) = L_j^{-1} F(L_j(x - j), L_j y - [L_j y], L_j, K, H, g_j, h_j) \qquad (j < x \leq j + 1;\ j = 0, 1, 2, \cdots).$$

THEOREM. *The surface defined by* $t = f(x, y)$ *is of hyperbolic type.*

To prove this result, it is enough to show that the surface admits a quasiconformal mapping onto a plane domain of hyperbolic type. Such a mapping is constructed so as to be the identity on the points for which $x \leq 0$. By Lemma 3, the portion of the surface over $j \leq x \leq j + 1$ admits a quasiconformal mapping onto a vertical strip, of width less than n_j^{-1}, such that the boundary points of the surface correspond to boundary points of the strip with the same value of y. When these mappings are combined to give a continuous mapping of the surface, its image will be a domain in the half-plane $x < \Sigma_{j=0}^{\infty} n_j^{-1}$. Since we have used a fixed value of K, and since the mapping is quasiconformal, apart from isolated vertical lines, the entire mapping will be quasiconformal.

It is clear that the construction can be modified in numerous ways.

Further the existence of a hyperbolic surface with $f(x, y)$ real analytic in x, y can be shown. This remark must be regarded as well known, since at the Bombay Conference on Function Theory (January, 1960), four or five people remarked that it could be proved by the Whitney approximation theorem [6], no further details being given. Actually the full force of this result is not required for the proof. Indeed, if two parametric surfaces, sufficiently differentiable, with metrics

$$E\,dx^2 + 2F\,dx\,dy + G\,dy^2, \qquad E'\,dx^2 + 2F'\,dx\,dy + G'\,dy^2$$

are put into correspondence by relating points with the same parameter values (x, y), the mapping will be quasiconformal if $(E'G - 2F'F + EG')/JJ'$ is bounded, where J, J' are the Jacobians $(EG - F^2)^{1/2}$, $(E'G' - F'^2)^{1/2}$. Now let $f(x, y)$ be infinitely differentiable, with the corresponding surface hyperbolic, and let $\tilde{f}(x, y)$ be real analytic, with

$$|f_x(x, y) - \tilde{f}_x(x, y)| < \frac{1}{2}, \qquad |f_y(x, y) - \tilde{f}_y(x, y)| < \frac{1}{2}$$

for all points. The natural metrics on the corresponding surfaces are given as above with

$$E = 1 + f_x^2, \qquad F = f_x f_y, \qquad G = 1 + f_y^2,$$

$$E' = 1 + \tilde{f}_x^2, \qquad F' = \tilde{f}_x \tilde{f}_y, \qquad G' = 1 + \tilde{f}_y^2,$$

$$J = (1 + f_x^2 + f_y^2)^{1/2}, \qquad J' = (1 + \tilde{f}_x^2 + \tilde{f}_y^2)^{1/2}.$$

The above criterion for quasiconformality then requires the boundedness of

$$\frac{2 + f_x^2 + f_y^2 + \tilde{f}_x^2 + \tilde{f}_y^2 + (\tilde{f}_x f_y - \tilde{f}_y f_x)^2}{(1 + f_x^2 + f_y^2)^{1/2} (1 + \tilde{f}_x^2 + \tilde{f}_y^2)^{1/2}},$$

which is not larger than

$$\frac{2 + f_x^2 + f_y^2 + \tilde{f}_x^2 + \tilde{f}_y^2 + 2f_y^2(\tilde{f}_x - f_x)^2 + 2f_x^2(\tilde{f}_y - f_y)^2}{(1 + f_x^2 + f_y^2)^{1/2}(1 + \tilde{f}_x^2 + \tilde{f}_y^2)^{1/2}}.$$

If we write $f_x^2 + f_y^2 = M^2$, this last quantity is not larger than

$$\frac{2\frac{1}{16} + \frac{1}{2}M + 2\frac{1}{2}M^2}{(1 + M^2)^{1/2}\left(\max\left(1, M^2 - \frac{1}{2}M + 1\right)\right)^{1/2}},$$

a quantity which clearly has a bound independent of M. Thus the surface corresponding to $\tilde{f}(x, y)$ is also hyperbolic.

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