AUTOMORPHIC FORMS OF NONNEGATIVE DIMENSION AND EXPONENTIAL SUMS

Marvin Isadore Knopp

I. INTRODUCTION

In this paper we extend the methods and results of the previous paper [2]. There we discuss the three groups $G(\sqrt{l})$ $(l = 1, 2, 3)$ of linear fractional transformations of the upper half-plane $\Im\tau > 0$ onto itself, where $G(\sqrt{l})$ is generated by the two transformations $S(\tau) = \tau + \sqrt{l}$ and $T(\tau) = -1/\tau$, and we construct automorphic forms of nonnegative *even* integral dimension, with multiplier system identically one, for these groups. Of course, $G(1)$ is the modular group.

Here the results of [2] are extended in the following way. The same groups are considered, but now we construct forms of *arbitrary* integral dimension $r \geq 0$, with arbitrary multiplier systems. Specifically, let Γ denote any one of the three groups in question and let $M \in \Gamma$, $M\tau = (\alpha\tau + \beta)/(\gamma\tau + \delta)$. Given any integer $r \geq 0$, we construct functions $F(\tau)$ that are regular in $\Im\tau > 0$ and satisfy throughout this half-plane and for all $M \in \Gamma$ the condition

$$F(M\tau) = \varepsilon(M) \cdot (-i(\gamma\tau + \delta))^{-r} \cdot F(\tau), \tag{1.1}$$

where $\varepsilon(M)$ does not depend on τ and $|\varepsilon(M)| = 1$ for all $M \in \Gamma$.

With each transformation $M \in \Gamma$ we associate the two matrices

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{and} \quad -M = \begin{pmatrix} -\alpha & -\beta \\ -\gamma & -\delta \end{pmatrix};$$

in this context we shall not distinguish between the two matrices. Therefore, applying (1.1) with M replaced by $-M$, we see that

$$\varepsilon(-M)(-i(-\gamma\tau - \delta))^{-r} = \varepsilon(M)(-i(\gamma\tau + \delta))^{r}. \tag{1.2}$$

Now, when there exists a function $F(\tau)$ satisfying (1.1) it follows in a simple fashion that if

$$M_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \in \Gamma, \qquad M_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} \in \Gamma,$$

then

$$\varepsilon(M_1 M_2)(-i(\gamma_3\tau + \delta_3))^{-r} = \varepsilon(M_1)\varepsilon(M_2)(-i(\gamma_1 M_2\tau + \delta_1))^{-r}(-i(\gamma_2\tau + \delta_2))^{-r}, \tag{1.3}$$

where $M_1 M_2 = \begin{pmatrix} * & * \\ \gamma_3 & \delta_3 \end{pmatrix}$. The multipliers $\varepsilon(M)$ are said to form a *multiplier system for* Γ *corresponding to the dimension* r, provided $\varepsilon(M)$ is a complex-valued function

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defined on Γ such that $|\varepsilon(M)| = 1$ for all $M \in \Gamma$ and $\varepsilon(M)$ satisfies (1.3).

Putting $M = S$ in (1.1), we obtain

$$F(S\tau) = \varepsilon(S)(-i)^{-r} \cdot F(\tau) = e^{2\pi i\alpha} F(\tau) \qquad (0 \leq \alpha < 1). \tag{1.4}$$

Applying (1.3) first with $M_1 = M$, $M_2 = S^q$ (q an integer) and then with $M_1 = S^q$, $M_2 = M$, and making use of (1.4), we have

$$\varepsilon(MS^q) = \varepsilon(S^q M) = e^{2\pi i q\alpha} \varepsilon(M). \tag{1.5}$$

Now, since S and T generate Γ it follows that the multiplier system $\varepsilon(M)$ is completely determined by α and $\varepsilon(T)$, and that

$$F(S\tau) = e^{2\pi i\alpha} F(\tau), \qquad F(T\tau) = \varepsilon(T)(-i\tau)^{-r} F(\tau) \tag{1.6}$$

together imply (1.1). Thus in order to show that a function $F(\tau)$ is a form of dimension r for Γ, it is necessary to prove only that $F(\tau)$ satisfies (1.6). (For a similar discussion of multiplier systems see [3, pp. 72-73].)

We outline our procedure for the modular group. Let r and ν be integers ($r \geq 0$, $\nu \geq 1$), and take any possible value of α as defined by (1.4), where now $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It is well known (see [6]) that in the case of the modular group each dimension r has six multiplier systems and hence six values of α connected with it. Now define

$$\begin{aligned} F(\tau; r, \nu) &= e^{2\pi i\alpha\tau} \left\{ e^{-2\pi i\nu\tau} + \sum_{m=0}^{\infty} a_m(r, \nu) e^{2\pi i m\tau} \right\}, \\ a_m(r, \nu) &= 2\pi \sum_{k=1}^{\infty} k^{-1} A_{k,\nu}(m) \left(\frac{\nu - \alpha}{m + \alpha} \right)^{(r+1)/2} \cdot I_{r+1}\left(\frac{4\pi}{k} \sqrt{(\nu - \alpha)(m + \alpha)} \right), \end{aligned} \tag{1.7}$$

with

$$A_{k,\nu}(m) = \sum_{0 \leq h < k}{}' \varepsilon^{-1}(M_{k,-h}) \exp\left[-\frac{2\pi i}{k} \{(\nu - \alpha)h' + (m + \alpha)h\} \right], \tag{1.8}$$

where h' is any integral solution of $hh' \equiv -1 \pmod k$,

$$M_{k,-h} = \begin{pmatrix} h' & -\frac{hh' + 1}{k} \\ k & -h \end{pmatrix} \in G(1),$$

and I_{r+1} is the modified Bessel function of the first kind. The dash (') over the summation sign indicates that the sum is taken over only those h for which $(h, k) = 1$.

Remark. It might seem, at first glance, that the terms of the sum $A_{k,\nu}(m)$ are not uniquely determined, since h' is only determined modulo k. In fact, h' can be replaced by $h' + qk$, where q is any integer. Then $M_{k,-h}$ is replaced by

$$N_{k,-h} = \begin{pmatrix} h' + qk & -\frac{hh'+1}{k} - qh \\ k & -h \end{pmatrix} = S^q M_{k,-h},$$

and, by (1.5), $\varepsilon(N_{k,-h}) = e^{2\pi iq\alpha} \cdot \varepsilon(M_{k,-h})$. The corresponding term in $A_{k,\nu}(m)$ is replaced by

$$\varepsilon^{-1}(N_{k,-h}) \exp\left[-\frac{2\pi i}{k}\{(\nu-\alpha)h' + (m+\alpha)h + (\nu-\alpha)qk\}\right]$$

$$= e^{-2\pi iq\alpha}\,\varepsilon(M_{k,-h})\,e^{-2\pi i(\nu-\alpha)q} \exp\left[-\frac{2\pi i}{k}\{(\nu-\alpha)h' + (m+\alpha)h\}\right]$$

$$= \varepsilon(M_{k,-h}) \exp\left[-\frac{2\pi i}{k}\{(\nu-\alpha)h' + (m+\alpha)h\}\right],$$

so that the terms of $A_{k,\nu}(m)$ are unaffected by the ambiguity in h'.

THEOREM (1.9). *Throughout* $\Im\tau > 0$, *the function* $F(\tau;\, r,\, \nu)$ *is regular and satisfies the condition*

$$\varepsilon^{-1}(T)(-i\tau)^r F(T\tau;\, r,\, \nu) = F(\tau;\, r,\, \nu) + p(\tau;\, r,\, \nu), \tag{1.10}$$

where $p(\tau;\, r,\, \nu)$ *is a polynomial in* τ *of degree at most* r.

This theorem, the proof of which will be given in Section IV, can be thought of as a weak converse, for integral dimension, to the main result of [6], where it is shown that every modular form of positive (integral or nonintegral) dimension, regular in $\Im\tau > 0$, has a Fourier expansion of the form (1.7). The actual converse to the result of [6], which would say that every function defined by (1.7) is a modular form (in other words, that $p(\tau;\, r,\, \nu) \equiv 0$), is not true.

On the basis of Theorem (1.9) we can then easily construct functions satisfying (1.6), where now $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; these functions will therefore be forms of dimension r for the modular group. The following theorem will enable us, in the same way, to construct forms of dimension r for the groups $G(\sqrt{2})$ and $G(\sqrt{3})$.

THEOREM (1.11). *Let* r *and* ν *be integers* $(r \geq 0,\ \nu \geq 1)$. *Choose any* α *consistent with* (1.4), *where now* $S = \begin{pmatrix} 1 & \sqrt{l} \\ 0 & 1 \end{pmatrix}$ $(l = 2, 3)$. *Put*

$$F_l(\tau;\, r,\, \nu) = e^{2\pi i\alpha\tau/\sqrt{l}} \left\{ e^{-2\pi i\nu\tau/\sqrt{l}} + \sum_{m=0}^{\infty} a_m(r,\, \nu,\, l)\, e^{2\pi im\tau/\sqrt{l}} \right\},$$

$$\begin{aligned} a_m(r,\, \nu,\, l) &= 2\pi \sum_{\substack{k=1 \\ k\equiv 0 \pmod{l}}}^{\infty} k^{-1} A_{k,\nu}(m) \left(\frac{\nu-\alpha}{m+\alpha}\right)^{(r+1)/2} I_{r+1}\left(\frac{4\pi}{k}\sqrt{(m+\alpha)(\nu-\alpha)}\right) \\ &\quad + \frac{2\pi}{\sqrt{l}} \sum_{\substack{k=1 \\ k\not\equiv 0 \pmod{l}}}^{\infty} k^{-1} A_{k,\nu}(m) \left(\frac{\nu-\alpha}{m+\alpha}\right)^{(r+1)/2} I_{r+1}\left(\frac{4\pi\sqrt{(m+\alpha)(\nu-\alpha)}}{\sqrt{l}\,k}\right) \\ &= a_{m,1}(r,\, \nu,\, l) + a_{m,2}(r,\, \nu,\, l). \end{aligned} \tag{1.12}$$

When $k \equiv 0 \pmod{l}$, $A_{k,\nu}(m)$ *is defined by*

$$(1.13) \qquad A_{k,\nu}(m) = \sum_{0 \le h < k}{}' \varepsilon^{-1}(M_{k/\sqrt{l},-h}) \exp\left[-\frac{2\pi i}{k} \{(\nu - \alpha)h' + (m + \alpha)h\} \right],$$

where h' *is again any integral solution of* $hh' \equiv -1 \pmod{k}$ *and*

$$M_{k/\sqrt{l},-h} = \begin{pmatrix} h' & -\frac{hh'+1}{k}\sqrt{l} \\ \frac{k}{l}\sqrt{l} & -h \end{pmatrix} \in G(\sqrt{l}).$$

When $k \not\equiv 0 \pmod{l}$, $A_{k,\nu}(m)$ *is defined by*

$$(1.14) \qquad A_{k,\nu}(m) = \sum_{0 \le h < k}{}' \varepsilon^{-1}(M_{k,-h\sqrt{l}}) \exp\left[-\frac{2\pi i}{k} \{(\nu - \alpha)h^* + (m + \alpha)h\} \right],$$

where h^* *is any integral solution of* $lhh^* \equiv -1 \pmod{k}$ *and*

$$M_{k,-h\sqrt{l}} = \begin{pmatrix} h^*\sqrt{l} & -\frac{lhh^*+1}{k} \\ k & -h\sqrt{l} \end{pmatrix} \in G(\sqrt{l}).$$

Then, throughout $\Im\tau > 0$, *the function* $F_1(\tau; r, \nu)$ *is regular and satisfies the condition*

$$(1.15) \qquad \varepsilon^{-1}(T)(-i\tau)^r F_1(T\tau; r, \nu) = F_1(\tau; r, \nu) + p_1(\tau; r, \nu),$$

where $p_1(\tau; r, \nu)$ *is a polynomial in* τ *of degree at most* r.

This theorem will be proved in Section V. The same remark that was made preceeding the statement of Theorem (1.9) can now be made in connection with the sums defined by (1.13) and (1.14). The proof is the same and will not be repeated here.

When $r > 0$, the changes that are needed in transferring the methods of [2] to arbitrary multiplier systems are almost all purely formal ones. The computations are more complicated, of course, but no essentially new problems arise. In particular, the trivial estimate $A_{k,\nu}(m) = O(k)$ for the exponential sums defined by (1.8), (1.13), and (1.14) will enable us to carry through the analysis.

However, when $r = 0$ we need an estimate of the form $A_{k,\nu}(m) = O(k^{1-\delta})$ $(\delta > 0)$, and it is shown, in fact, that for $r = 0$,

$$(1.16) \qquad A_{k,\nu}(m) = O(k^{2/3+\varepsilon}).$$

We accomplish this in Section II by showing that these exponential sums, which resemble the classical Kloosterman sum, can indeed be reduced (in a certain sense) to the Kloosterman sum when $r = 0$. We then apply the well-known estimate [7] for the Kloosterman sum to obtain (1.16).

II. ESTIMATION OF THE EXPONENTIAL SUMS

In order to reduce the exponential sums of (1.8), (1.13), and (1.14) to the classical Kloosterman sum we employ, among other things, a procedure used by Lehner [3, pp. 82-85]. Lehner estimates such sums connected with the modular group and the dimension $r = -2$, by making use of a parametrization (given by Rademacher and Zuckerman [6]) of the set of all modular forms. Now the sum defined by (1.8) is connected with the modular group, and we are interested in such a sum which occurs for the dimension $r = 0$. As might be suspected, it turns out that the sum can be handled by following Lehner's procedure almost exactly. Hence the reader is referred to Lehner's paper for the treatment of the sum (1.8).

The sums (1.13) and (1.14), on the other hand, are connected with the groups $G(\sqrt{2})$ or $G(\sqrt{3})$, depending on whether $l = 2$ or $l = 3$, and we proceed somewhat differently. Use is made here of the results of [1], in which all possible multiplier systems for these two groups corresponding to the dimension $r = 0$ are computed. The groups $G(\sqrt{2})$ and $G(\sqrt{3})$ will be treated separately below.

1. THE GROUP $G(\sqrt{2})$

As is shown in [1], this group has the four values $\alpha = 0, 1/4, 1/2, 3/4$ connected with the dimension $r = 0$. For each such value of α we can also choose $\varepsilon_0 \equiv \varepsilon(T) = \pm 1$. Since, as was previously mentioned, a multiplier system is completely determined by α and ε_0, it follows that there are eight multiplier systems for $G(\sqrt{2})$ corresponding to $r = 0$. The proof of (1.16) is divided into four cases according to the value of α, and each case into two subcases according to whether or not $k \equiv 0 \pmod 2$. In order to avoid a great deal of repetition, we discuss only the cases $\alpha = 1/2$ and $\alpha = 1/4$.

Case (i): $\alpha = 1/2$

(a) *Assume first that* k *is even;* we then need to consider the sum (1.13). According to [1], we have, for $\alpha = 1/2$,

$$\varepsilon(M_{k/\sqrt{2},-h}) = \exp\left[\pi i\left(\frac{hh'+1}{k}h - h'k/2\right)\right],$$

and (1.13) becomes

$$A_{k,\nu}(m) = \sum_{0\le h<k}{}' \exp\left[\pi i\left(-\frac{hh'+1}{k}h + h'k/2\right)\right]\cdot\exp\left[\frac{-2\pi i}{k}\left\{\left(\nu-\frac{1}{2}\right)h' + \left(m+\frac{1}{2}\right)h\right\}\right]$$

$$= \sum_{0\le h<k}{}' \exp\left[\frac{-2\pi i}{2k}\{h(hh'+1) - h'k^2/2 + 2\nu h' - h' + 2mh + h\}\right].$$

Now, $hh' \equiv -1 \pmod k$ implies that there exists an integer c such that $hh' + 1 = ck$, and furthermore, since k is even, h is odd, and we can choose c even. For if c is odd, replace c by $c + h$ and h' by $h' + k$. This choice does not affect the value of $A_{k,\nu}(m)$, as we have previously remarked. Then, $hh' \equiv -1 \pmod{2k}$ and

$$A_{k,\nu}(m) = \sum_{0\le h<k}{}' \exp\left[\frac{-2\pi i}{2k}(chk - h'k^2/2 + 2\nu h' - h' + 2mh + h)\right]$$

$$= \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{2k}\left\{h'\left(\frac{k^2}{2} + 1 - 2\nu\right) + h(-2m-1)\right\}\right].$$

From here we follow Lehner [3, p. 84], with some notational changes. Put $a = k^2/2 + 1 - 2\nu$, $b = -2m - 1$, and define

$$B_{k,\nu}(m) = \sum_{0 \leq j < 2k}{}' \exp\left[\frac{2\pi i}{2k}(aj' + bj)\right],$$

where j' is any integral solution of $jj' \equiv -1 \pmod{2k}$. In $B_{k,\nu}(m)$, put $j = qk + h$, where $0 \leq h < k$, $(h, k) = 1$, and q is 0 or 1. We can choose $j' = h'(1 + qkh')$, where $hh' \equiv -1 \pmod{2k}$, for

$$(qk + h)h'(1 + qkh') = qkh'(1 + hh') + q^2k^2h'^2 + hh' \equiv hh' \equiv -1 \pmod{2k}.$$

Then we have

$$\begin{aligned} B_{k,\nu}(m) &= \sum_{0 \leq h < k}{}' \sum_{q=0}^{1} \exp\left[\frac{2\pi i}{2k}\{ah'(1 + qkh') + b(h + qk)\}\right] \\ &= \sum_{0 \leq h < k}{}' \exp\left[\frac{2\pi i}{2k}(ah' + bh)\right] \cdot \sum_{q=0}^{1} \exp[\pi i q(ah'^2 + b)]. \end{aligned}$$

But $a + b = k^2/2 - 2\nu - 2m \equiv 0 \pmod 2$; also, since k is even, h' is odd. Therefore $ah'^2 + b \equiv 0 \pmod 2$, the inner sum equals 2, and $B_{k,\nu}(m) = 2A_{k,\nu}(m)$. But $B_{k,\nu}(m)$ is a classical Kloosterman sum and has, by [7], the estimate

$$B_{k,\nu}(m) = O[(2k)^{2/3+\varepsilon}(2k, a)^{1/3}].$$

Hence,

$$A_{k,\nu}(m) = O\left[(2k)^{2/3+\varepsilon}\left(2k, \frac{k^2}{2} + 1 - 2\nu\right)^{1/3}\right].$$

But

$$\left(2k, \frac{k^2}{2} + 1 - 2\nu\right) = \left(k, \frac{k^2}{2} + 1 - 2\nu\right) = (k, 1 - 2\nu) \leq 2\nu - 1,$$

since k is even. Therefore, $A_{k,\nu}(m) = O(k^{2/3+\varepsilon})$.

(b) *Now assume that* k *is odd;* then we must consider the sum (1.14). By [1], we have, for $\alpha = 1/2$,

$$\varepsilon(M_{k,-h\sqrt{2}}) = \varepsilon_0 \exp\left[\pi i\left(\frac{2hh^* + 1}{k} h - h^* k\right)\right],$$

and (1.14) becomes

$$\begin{aligned} A_{k,\nu}(m) &= \varepsilon_0 \sum_{0 \leq h < k}{}' \exp\left[\pi i\left(h^* k - \frac{2hh^* + 1}{k} h\right)\right] \exp\left[\frac{-2\pi i}{k}\left\{\left(\nu - \frac{1}{2}\right)h^* + \left(m + \frac{1}{2}\right)h\right\}\right] \\ &= \varepsilon_0 \sum_{0 \leq h < k}{}' \exp\left[\frac{2\pi i}{2k}\{h^* k^2 - h(2hh^* + 1) - 2\nu h^* + h^* - 2mh - h\}\right]. \end{aligned}$$

Since $2hh^* \equiv -1 \pmod{k}$, there exists an integer c such that $2hh^* + 1 = ck$, and it is clear that c is odd. We obtain

$$A_{k,\nu}(m) = \varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{2k}(h^*k^2 - ckh - 2\nu h^* + h^* - 2mh - h)\right]$$

$$= \varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{2k}\{h^*(k^2+1-2\nu) + h(-1-2m-k)\}\right].$$

But we can write $h' = 2h^*$ (where h' has its usual meaning), since $h\cdot 2h^* \equiv -1 \pmod{k}$. Therefore

$$A_{k,\nu}(m) = \varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{k}\left\{h'\left(\frac{k^2+1-2\nu}{4}\right) + h\left(\frac{-1-2m-k}{2}\right)\right\}\right].$$

Now $(-1-2m-k)/2$ is an integer, because k is odd. Also, $k^2 + 1 \equiv 2 \pmod{4}$.

Hence, *if* ν *is odd*, $4\,|\,k^2+1-2\nu$, $A_{k,\nu}(m)$ is a Kloosterman sum, and we get

$$A_{k,\nu}(m) = O\left[k^{2/3+\varepsilon}\left(k, \frac{k^2+1-2\nu}{4}\right)^{1/3}\right].$$

As before, $\left(k, \frac{k^2+1-2\nu}{4}\right) \le 2\nu - 1$, and we have $A_{k,\nu}(m) = O(k^{2/3+\varepsilon})$.

If ν *is even*, we can multiply each term of $A_{k,\nu}(m)$ by $e^{\pi i(2h^*k)} = 1$, and thus obtain

$$A_{k,\nu}(m) = \varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\pi i\left(3h^*k - \frac{2hh^*+1}{k}h\right)\right] \exp\left[\frac{-2\pi i}{k}\left\{\left(\nu - \tfrac{1}{2}\right)h^* + \left(m + \tfrac{1}{2}\right)h\right\}\right]$$

$$= \varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{k}\left\{h'\left(\frac{3k^2+1-2\nu}{4}\right) + h\left(\frac{-1-2m-k}{2}\right)\right\}\right].$$

Now $3k^2 + 1 \equiv 0 \pmod{4}$, $4\,|\,3k^2+1-2\nu$, and once more $A_{k,\nu}(m)$ is a Kloosterman sum. As before, $A_{k,\nu}(m) = O(k^{2/3+\varepsilon})$.

Case (ii): $\alpha = 1/4$

(a) *Let* k *be even.* By [1], we have, for $\alpha = 1/4$,

$$\varepsilon(M_{k/\sqrt{2},-h}) = \exp\left[\frac{\pi i}{2}\left\{-\frac{hh'+1}{k}h - \frac{h'k}{2} - (hh'+1)\right\}\right],$$

and (1.13) becomes

$$A_{k,\nu}(m) = \sum_{0\le h<k}{}' \exp\left[\frac{\pi i}{2}\left(-\frac{hh'+1}{k}h + \frac{h'k}{2} + hh' + 1\right)\right]\exp\left[\frac{-2\pi i}{k}\left\{\left(\nu - \tfrac{1}{4}\right)h' + \left(m + \tfrac{1}{4}\right)h\right\}\right]$$

$$= \sum_{0\le h<k}{}' \exp\left[\frac{\pi i}{2k}\left\{-h(hh'+1) + \frac{h'k^2}{2} + k(hh'+1) - 4\nu h' + h' - 4mh - h\right\}\right].$$

We have $hh' + 1 = ck$, where again c can be chosen even since k is even. Further, we can choose $c \equiv 0 \pmod 4$, for if $c \equiv 2 \pmod 4$, replace c by $c + 2h$ and h' by $h' + 2k$. Then $hh' \equiv -1 \pmod{4k}$ and

$$A_{k,\nu}(m) = \sum_{0\le h<k}{}' \exp\left[\frac{\pi i}{2k}\left(ckh + \frac{h'k^2}{2} + k^2c - 4\nu h' + h' - 4mh - h\right)\right]$$

$$= \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{4k}\left\{h'\left(\frac{k^2}{2} - 4\nu + 1\right) + h(-4m-1)\right\}\right].$$

Assume $k \equiv 0 \pmod 4$. Putting $a = \frac{k^2}{2} - 4\nu + 1$, $b = -4m - 1$, and

$$B_{k,\nu}(m) = \sum_{0\le j<4k}{}' \exp\left[\frac{2\pi i}{4k}(aj' + bj)\right] \qquad \text{with } jj' \equiv -1 \pmod{4k},$$

and noting that $a + b = k^2/2 - 4\nu - 4m \equiv 0 \pmod 4$, we may proceed as before to obtain $B_{k,\nu}(m) = 4A_{k,\nu}(m)$. Thus, since $B_{k,\nu}(m)$ is a Kloosterman sum, we have

$$A_{k,\nu}(m) = O\left[(4k)^{2/3+\varepsilon}\left(4k, \frac{k^2}{2} + 1 - 4\nu\right)^{1/3}\right] = O(k^{2/3+\varepsilon}).$$

Now let $k \equiv 2 \pmod 4$. We have chosen h' such that $hh' \equiv -1 \pmod{4k}$; hence $hh' \equiv -1 \pmod 8$. From this it follows immediately that $h + h' \equiv 0 \pmod 8$. Therefore, we can multiply each term of $A_{k,\nu}(m)$ by

$$\exp\left[\frac{\pi i}{2k}\cdot\frac{k}{2}(h + h')\right] = \exp\left[\frac{\pi i}{4}(h + h')\right] = 1,$$

and we get

$$A_{k,\nu}(m) = \sum_{0\le h<k}{}' \exp\left[\frac{\pi i}{2k}\left\{h'\left(\frac{k^2}{2} - 4\nu + 1 + \frac{k}{2}\right) + h\left(-4m - 1 + \frac{k}{2}\right)\right\}\right]$$

$$= \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{2k}\left\{h'\left(\frac{k^2/2 - 4\nu + 1 + k/2}{2}\right) + h\left(\frac{-4m - 1 + k/2}{2}\right)\right\}\right].$$

This time, let $a = (k^2/2 - 4\nu + 1 + k/2)/2$ and $b = (-4m - 1 + k/2)/2$. Since $k \equiv 2 \pmod 4$, a and b are integers with

$$a + b = \frac{1}{2}\left(\frac{k^2}{2} + k - 4\nu - 4m\right) \equiv 0 \pmod 2.$$

Hence we may put

$$B_{k,\nu}(m) = \sum_{0\le j<2k} \exp\left[\frac{2\pi i}{2k}(aj' + bj)\right],$$

with $jj' \equiv -1 \pmod{2k}$, and obtain $B_{k,\nu}(m) = 2A_{k,\nu}(m)$ and, once again, the estimate

$$A_{k,\nu}(m) = O\left[(2k)^{2/3+\varepsilon}\left(2k, \frac{k^2/2 - 4\nu + 1 + k/2}{2}\right)^{1/3}\right] = O(k^{2/3+\varepsilon}).$$

(b) *Let* k *be odd.* By [1], we have, for $\alpha = 1/4$,

$$\varepsilon(M_{k,-h\sqrt{2}}) = \varepsilon_0 \exp\left[\frac{\pi i}{2}\left(h^*k - \frac{2hh^*+1}{k}h - 2hh^*\right)\right],$$

and (1.14) becomes

$$A_{k,\nu}(m) = \varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{\pi i}{2}\left(-h^*k + \frac{2hh^*+1}{k}h + 2hh^*\right)\right]\exp\left[\frac{-2\pi i}{k}\left\{\left(\nu - \frac{1}{4}\right)h^* + \left(m + \frac{1}{4}\right)h\right\}\right]$$

$$= \varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{4k}\{-h^*k^2 + (2hh^*+1)h + 2hh^*k - 4\nu h^* + h^* - 4mh - h\}\right].$$

Now $2hh^* + 1 = ch$, and we can clearly choose h^* even, since k is odd. For if h^* is odd, we may replace h^* by $h^* + k$ and c by $c + 2h$. Thus $ck - 1 \equiv 0 \pmod 4$, and we have

$$A_{k,\nu}(m) = \varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{4k}\{-h^*k^2 + ckh + k(ck-1) - 4\nu h^* + h^* - 4mh - h\}\right]$$

$$= \varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{4k}\{h^*(1 - 4\nu - k^2) + h(ck - 4m - 1)\}\right].$$

As before, we may put $h' = 2h^*$, and $ck \equiv 1 \pmod 4$ implies $c \equiv k \pmod 4$. Thus, $A_{k,\nu}(m) = \varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{k}\left\{h'\left(\frac{1-4\nu-k^2}{8}\right) + h\left(\frac{k^2-4m-1}{4}\right)\right\}\right]$. But since k is odd, $k^2 \equiv 1 \pmod 8$, so that $4 \mid k^2 - 4m - 1$.

If ν *is even,* we also have $8 \mid 1 - 4\nu - k^2$, so that $A_{k,\nu}(m)$ is a Kloosterman sum and $A_{k,\nu}(m) = O(k^{2/3+\varepsilon})$, as before.

If ν *is odd,* multiply each term of $A_{k,\nu}(m)$ by $\exp\left[\frac{\pi i}{2}(-4h^*k)\right] = 1$ to obtain

$$A_{k,\nu}(m) = \varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{4k}\{-5h^*k^2 + (2hh^*+1)h + 2hh^*k - 4\nu h^* + h^* - 4mh - h\}\right]$$

$$= \varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{k}\left\{h'\left(\frac{1-5k^2-4\nu}{8}\right) + h\left(\frac{k^2-4m-1}{4}\right)\right\}\right].$$

Now $1 - 5k^2 - 4\nu \equiv 0 \pmod 8$, and once more $A_{k,\nu}(m) = O(k^{2/3+\varepsilon})$.

The case $\alpha = 0$ is similar to the case $\alpha = 1/2$, but somewhat simpler. The case $\alpha = 3/4$ can be handled in exactly the same way as $\alpha = 1/4$.

2. THE GROUP $G(\sqrt{3})$

It is shown in [1] that the group $G(\sqrt{3})$ has the six values $\alpha = 0, 1/6, 1/3, 1/2, 2/3, 5/6$ connected with the dimension $r = 0$. Again, for each value of α we can choose $\varepsilon_0 \equiv \varepsilon(T) = \pm 1$. As before, we divide the proof of (1.16) into six cases according to the value of α, and each case into two subcases according to whether or not $k \equiv 0 \pmod 3$. We shall treat only the cases $\alpha = 1/2$, $\alpha = 1/3$, and $\alpha = 1/6$. The case $\alpha = 0$ is quite simple, and the cases $\alpha = 2/3$, and $\alpha = 5/6$ can be handled in the same way as $\alpha = 1/3$ and $\alpha = 1/6$, respectively.

Case (i): $\alpha = 1/2$

(a) *Suppose* $k \equiv 0 \pmod 3$; we are then considering the sum (1.13). By [1], we have, for $\alpha = 1/2$,

$$\varepsilon(M_k/\sqrt{3}, -h) = \exp\left[\pi i\left(\frac{hh'+1}{k}h - \frac{h'k}{3} - \frac{hh'+1}{k}\cdot\frac{k}{3}\right)\right],$$

and

$$A_{k,\nu}(m) = \sum_{0\le h<k}{}' \exp\left[\pi i\left\{-\frac{h}{k}(hh'+1)+\frac{h'k}{3}+\frac{hh'+1}{3}\right\}\right]\exp\left[-\frac{2\pi i}{k}\left\{\left(\nu-\frac{1}{2}\right)h'+\left(m+\frac{1}{2}\right)h\right\}\right]$$

$$= \sum_{0\le h<k}{}' \exp\left[\frac{\pi i}{k}\left\{-h(hh'+1)+\frac{h'k^2}{3}+\frac{k^2}{3}(hh'+1) - 2\nu h' + h' - 2mh - h\right\}\right].$$

Let k *be even*, that is $k \equiv 0 \pmod 6$. Now $hh' + 1 = ck$, and since k is even, h is odd, so that we can choose c even, as before. Then,

$$A_{k,\nu}(m) = \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{2k}\{h'(1-2\nu) + h(-1-2m)\}\right],$$

with $hh' \equiv -1 \pmod{2k}$. Putting $a = 1 - 2\nu$ and $b = -1 - 2m$, we find that $a + b = -2\nu - 2m \equiv 0 \pmod 2$. Thus we can put

$$B_{k,\nu}(m) = \sum_{0\le h<2k}{}' \exp\left[\frac{2\pi i}{2k}(aj' + bj)\right],$$

with $jj' \equiv -1 \pmod{2k}$, obtain $B_{k,\nu}(m) = 2A_{k,\nu}(m)$, and finally $A_{k,\nu}(m) = O(k^{2/3+\varepsilon})$.

Let k *be odd*, that is, $k \equiv 3 \pmod 6$. We break up the sum defining $A_{k,\nu}(m)$ as follows:

$$A_{k,\nu}(m) = \sum_{\substack{0\le h<k\\ h \text{ odd}}}{}' + \sum_{\substack{0\le h<k\\ h \text{ even}}}{}' = A^{(1)}_{k,\nu}(m) + A^{(2)}_{k,\nu}(m).$$

In $A^{(1)}_{k,\nu}(m)$, h is odd, and we may choose c even, as before, where again $hh' + 1 = ck$. Then

$$A^{(1)}_{k,\nu}(m) = \sum_{\substack{0\le h<k\\ h \text{ odd}}}{}' \exp\left[\frac{\pi i}{k}\left(-hck + \frac{h'k^2}{3} + \frac{k}{3}ck - 2\nu h' + h' - 2mh - h\right)\right]$$

$$= -\sum_{\substack{0\le h<k\\ h \text{ odd}}} \exp\left[\frac{2\pi i}{k}\left\{h'\left(\frac{k^2/3 - 2\nu + 1}{2}\right) + h\left(\frac{-2m - 1 + k}{2}\right)\right\}\right],$$

where we have multiplied each term by $e^{\pi i h} = -1$. Since k is odd,

$$k^2/3 - 2\nu + 1 \equiv -2m - 1 + k \equiv 0 \pmod 2,$$

so that $A^{(1)}_{k,\nu}(m)$ is a generalized Kloosterman sum and by [7],

$$A^{(1)}_{k,\nu}(m) = O\left[k^{2/3+\varepsilon}\left(k, \frac{k^2/3 - 2\nu + 1}{2}\right)^{1/3}\right] = O(k^{2/3+\varepsilon}).$$

Now consider $A^{(2)}_{k,\nu}(m)$. In this sum, h is even, and therefore c is odd. Hence

$$A^{(2)}_{k,\nu}(m) = \sum_{\substack{0\leq h<k\\ h \text{ even}}}{}' \exp\left[\frac{\pi i}{k}\left(-hck + \frac{h'k^2}{3} + \frac{k}{3}ck - 2\nu h' + h' - 2mh - h\right)\right]$$

$$= -\sum_{\substack{0\leq h<k\\ h \text{ even}}}{}' \exp\left[\frac{2\pi i}{k}\left\{h'\left(\frac{k^2/3 - 2\nu + 1}{2}\right) + h\left(\frac{-2m - 1 + k}{2}\right)\right\}\right],$$

where we have used the fact that $e^{\pi i ck/3} = -1$, and multiplied each term by $e^{\pi i h} = 1$. Again, $A_{k,\nu}(m)$ is a generalized Kloosterman sum, so that $A^{(2)}_{k,\nu}(m) = O(k^{2/3+\varepsilon})$. Combining this with the previous result, we have $A_{k,\nu}(m) = O(k^{2/3+\varepsilon})$.

(b) *Assume* $k \not\equiv 0 \pmod 3$; we are then considering the sum (1.14). By [1], we have, for $\alpha = 1/2$,

$$\varepsilon(M_{k,-h\sqrt{3}}) = \varepsilon_0 \exp\left[\pi i\left(\frac{3hh^* + 1}{k}h - h^*k - hh^*\right)\right],$$

and thus

$$A_{k,\nu}(m) = \varepsilon_0 \sum_{0\leq h<k}{}' \exp\left[\frac{\pi i}{k}\{-h(3hh^* + 1) + h^*k^2 + hh^*k - 2\nu h^* + h^* - 2mh - h\}\right]$$

$$= \varepsilon_0 \sum_{0\leq h<k}{}' \exp\left[\frac{\pi i}{k}(-hck + h^*k^2 + 3hh^*k - 2\nu h^* + h^* - 2mh - h)\right],$$

where, as usual, $3hh^* + 1 = ck$.

Let k *be even.* Then h is odd, and we may assume that c is even (if c is odd, replace c by $c + 3h$ and h^* by $h^* + k$). Then, we have

$$A_{k,\nu}(m) = -\varepsilon_0 \sum_{0\leq h<k}{}' \exp\left[\frac{\pi i}{k}\left\{h'\left(\frac{1 - 2\nu}{3}\right) + h(-2m - 1)\right\}\right],$$

where we have put $h' = 3h^*$, so that $hh' \equiv -1 \pmod{2k}$. *If* $\nu \equiv 2 \pmod 3$, put $a = (1 - 2\nu)/3$, $b = -2m - 1$; *if* $\nu \equiv 1 \pmod 3$, write

$$A_{k,\nu}(m) = -\varepsilon_0 \sum_{0\leq h<k}{}' \exp\left[\frac{\pi i}{k}\left\{h'\left(\frac{k^2 + 1 - 2\nu}{3}\right) + h(-2m - 1)\right\}\right]$$

and put $a = (k^2 + 1 - 2\nu)/3$, $b = -2m - 1$; *if* $\nu \equiv 0 \pmod 3$, write

$$A_{k,\nu}(m) = -\varepsilon_0 \sum_{0 \le h < k}{}' \exp\left[\frac{\pi i}{k}\left\{h'\left(\frac{2k^2 + 1 - 2\nu}{3}\right) + h(-2m - 1)\right\}\right]$$

and put $a = (2k^2 + 1 - 2\nu)/3$, $b = -2m - 1$. It is easy to verify that in each case a and b are integers such that $a + b \equiv 0 \pmod 2$. Hence we let

$$B_{k,\nu}(m) = -\varepsilon_0 \sum_{0 \le j < 2k}{}' \exp\left[\frac{2\pi i}{2k}(aj' + bj)\right],$$

obtain $B_{k,\nu}(m) = 2A_{k,\nu}(m)$, and finally $A_{k,\nu}(m) = O(k^{2/3+\varepsilon})$.

Let k *be odd.* Let $3hh^* + 1 = ck$. If h is even, c is odd; on the other hand, if h is odd, we may choose c odd, by replacing c by $c + 3h$ if necessary. Then,

$$A_{k,\nu}(m) = \varepsilon_0 \sum_{0 \le h < k}{}' \exp\left[\frac{2\pi i}{k}\left\{h'\left(\frac{k^2 - 2\nu + 1}{6}\right) + h\left(\frac{-k - 2m - 1}{2}\right)\right\}\right],$$

where again we have put $h' = 3h^*$. Since k is odd,

$$k^2 - 2\nu + 1 \equiv -k - 2m - 1 \equiv 0 \pmod 2.$$

Also, since $k \not\equiv 0 \pmod 3$, $k^2 + 1 \equiv 2 \pmod 3$. Thus, *if* $\nu \equiv 1 \pmod 3$, then $3 \mid k^2 - 2\nu + 1$, so that $k^2 - 2\nu + 1 \equiv 0 \pmod 6$. *If* $\nu \equiv 0 \pmod 3$, write

$$A_{k,\nu}(m) = \varepsilon_0 \sum_{0 \le h < k}{}' \exp\left[\frac{2\pi i}{k}\left\{h'\left(\frac{-k^2 - 2\nu + 1}{6}\right) + h\left(\frac{-k - 2m - 1}{2}\right)\right\}\right];$$

if $\nu \equiv 2 \pmod 3$, write

$$A_{k,\nu}(m) = \varepsilon_0 \sum_{0 \le h < k}{}' \exp\left[\frac{2\pi i}{k}\left\{h'\left(\frac{3k^2 - 2\nu + 1}{6}\right) + h\left(\frac{-k - 2m - 1}{2}\right)\right\}\right].$$

In each case, h' is multiplied by an integer, $A_{k,\nu}(m)$ is a Kloosterman sum, and $A_{k,\nu}(m) = O(k^{2/3+\varepsilon})$.

Case (ii): $\alpha = 1/3$

(a) *Assume* that $k \equiv 0 \pmod 3$; we are then considering (1.13). By [1], we have, for $\alpha = 1/3$,

$$\varepsilon(M_{k/\sqrt{3},-h}) = \exp\left[\frac{2\pi i}{3}\left(\frac{hh' + 1}{k}h - h'k/3\right)\right],$$

and thus

$$A_{k,\nu}(m) = \sum_{0 \le h < k}{}' \exp\left[\frac{2\pi i}{3k}\left\{\frac{h'k^2}{3} - h(hh' + 1) - 3\nu h' + h' - 3mh - h\right\}\right].$$

As before, write $hh' + 1 = ck$, and choose $c \equiv 0 \pmod 3$. This is possible, since $k \equiv 0 \pmod 3$ and $(h, k) = 1$ together imply that $(h, 3k) = 1$. With this choice of c (and h'), $hh' \equiv -1 \pmod{3k}$. Then

$$A_{k,\nu}(m) = \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{3k}\left\{h'\left(\frac{k^2}{3} - 3\nu + 1\right) + h(-3m-1)\right\}\right].$$

If we put $a = k^2/3 - 3\nu + 1$, $b = -3m - 1$, we find that $a + b \equiv 0 \pmod 3$. Hence, as before, we may put

$$B_{k,\nu}(m) = \sum_{0\le j<3k}{}' \exp\left[\frac{2\pi i}{3k}(aj' + bj)\right],$$

and we find that $B_{k,\nu}(m) = 3A_{k,\nu}(m)$. Thus $A_{k,\nu}(m) = O(k^{2/3+\varepsilon})$.

(b) *Let* $k \not\equiv 0 \pmod 3$. We have from [1], for $\alpha = 1/3$,

$$\varepsilon(M_{k,-h\sqrt{3}}) = \varepsilon_0 \exp\left[\frac{2\pi i}{3}\left(-\frac{3hh^*+1}{k}h + h^*k\right)\right],$$

and therefore

$$A_{k,\nu}(m) = \varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{3k}\{h(3hh^*+1) - h^*k^2 - 3\nu h^* + h^* - 3mh - h\}\right].$$

Putting $3hh^* + 1 = ck$, we find that $c \equiv k \pmod 3$, since $ck \equiv 1 \pmod 3$. Hence

$$A_{k,\nu}(m) = \varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{3k}\{h^*(1 - k^2 - 3\nu) + h(k^2 - 3m - 1)\}\right].$$

Now, $k \not\equiv 0 \pmod 3$ implies that $1 - k^2 - 3\nu \equiv k^2 - 3m - 1 \equiv 0 \pmod 3$. *If* $1 - k^2 - 3\nu \equiv 0 \pmod 9$, we are done, for we write $h' = 3h^*$, and

$$A_{k,\nu}(m) = \varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{k}\left\{h'\left(\frac{1-k^2-3\nu}{9}\right) + h\left(\frac{k^2-3m-1}{3}\right)\right\}\right]$$

is a Kloosterman sum. *If* $1 - k^2 - 3\nu \equiv 3 \pmod 9$, write

$$A_{k,\nu}(m) = \varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{3k}\{h^*(1 - 4k^2 - 3\nu) + h(k^2 - 3m - 1)\}\right],$$

and proceed as before. *If* $1 - k^2 - 3\nu \equiv 6 \pmod 9$, write

$$A_{k,\nu}(m) = \varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{3k}\{h^*(1 + 2k^2 - 3\nu) + h(k^2 - 3m - 1)\}\right],$$

and again follow the same procedure.

Case (iii): $\alpha = 1/6$

Suppose $k \equiv 0 \pmod 3$. By [1] we have, for $\alpha = 1/6$,

$$\varepsilon(M_{k/\sqrt{3},-h}) = \exp\left[\frac{\pi i}{3}\left\{\frac{hh'+1}{k}h - \frac{h'k}{3} - (hh'+1)\right\}\right],$$

so that

$$A_{k,\nu}(m) = \sum_{0 \leq h < k}{}' \exp\left[\frac{2\pi i}{6k}\left\{- h(hh' + 1) + \frac{h'k^2}{3} + (hh' + 1)k - 6\nu h' + h' - 6mh - h\right\}\right].$$

Assume that k *is even,* that is, $k \equiv 0 \pmod 6$. Since $(h, k) = 1$, it follows that $(h, 6k) = 1$; therefore we can choose h' such that $hh' \equiv -1 \pmod{6k}$, and we have

$$A_{k,\nu}(m) = \sum_{0 \leq h < k}{}' \exp\left[\frac{2\pi i}{6k}\left\{h'\left(\frac{k^2}{3} - 6\nu + 1\right) + h(- 6m - 1)\right\}\right].$$

Let $a = k^2/3 - 6\nu + 1$, $b = - 6m - 1$, and

$$B_{k,\nu}(m) = \sum_{0 \leq h < 6k}{}' \exp\left[\frac{2\pi i}{6k}(aj' + bj)\right] \qquad (jj' \equiv -1 \pmod{6k}).$$

We find that $a + b \equiv 0 \pmod 6$ $B_{k,\nu}(m) = 6A_{k,\nu}(m)$, and $A_{k,\nu}(m) = O(k^{2/3+\varepsilon})$, as before.

Suppose that k *is odd,* that is, $k \equiv 3 \pmod 6$. We write $hh' + 1 = ck$ and notice that $3 | k$ and $(h, k) = 1$ together imply that $(h, 3k) = 1$, whence we can choose h' such that $hh' \equiv -1 \pmod{3k}$. With such a choice of h', $c \equiv 0 \pmod 3$. Now, if h is even, c is odd, and if h is odd, we can choose c odd by replacing c by $c + 3h$ and h' by $h' + 3k$, if necessary. After such a replacement, we still have $c \equiv 0 \pmod 3$. In any case, we have $c \equiv 3 \pmod 6$. Therefore

$$A_{k,\nu}(m) = - \sum_{0 \leq h < k}{}' \exp\left[\frac{2\pi i}{6k}\left(- 3hk + \frac{h'k^2}{3} - 6\nu h' + h' - 6mh - h\right)\right]$$

$$= - \sum_{0 \leq h < k}{}' \exp\left[\frac{2\pi i}{3k}\left\{h'\left(\frac{k^2/3 - 6\nu + 1}{2}\right) + h\left(\frac{- 3k - 6m - 1}{2}\right)\right\}\right].$$

If we put $a = \dfrac{k^2/3 - 6\nu + 1}{2}$ and $b = \dfrac{- 3k - 6m - 1}{2}$, we find that a and b are integers with $a + b \equiv 0 \pmod 3$. Thus we put

$$B_{k,\nu}(m) = - \sum_{0 \leq h < 3k}{}' \exp\left[\frac{2\pi i}{3k}(aj' + bj)\right] \qquad (jj' \equiv -1 \pmod{3k}),$$

obtain $B_{k,\nu}(m) = 3A_{k,\nu}(m)$, and finally $A_{k,\nu}(m) = O(k^{2/3+\varepsilon})$.

(b) *Suppose that* $k \not\equiv 0 \pmod 3$. By [1] we have, for $\alpha = 1/6$,

$$\varepsilon(M_{k, -h\sqrt{3}}) = \varepsilon_0 \exp\left[\frac{\pi i}{6}\left(h^*k - \frac{3hh^* + 1}{k}h - 3hh^*\right)\right],$$

so that

$$A_{k,\nu}(m) = \varepsilon_0 \sum_{0 \leq h < k}{}' \exp\left[\frac{2\pi i}{6k}\{- h^*k^2 + h(3hh^* + 1) + 3hh^*k - 6\nu h^* + h^* - 6mh - h\}\right].$$

Suppose that k *is even.* Write $3hh^* + 1 = ck$; since k is even we can choose h^* so that c is even, that is, $3hh^* \equiv -1 \pmod{2k}$. Now $ck \equiv 1 \pmod 3$ implies that

$c \equiv k \pmod 3$; but since c and k are both even, we actually have $c \equiv k \pmod 6$. Also, hh^* is odd, and we obtain

$$A_{k,\nu}(m) = -\varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{6k}(-h^*k^2 + hk^2 + 6\nu h^* + h^* - 6mh - h)\right]$$

$$= -\varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{2k}\left\{h'\left(\frac{-k^2 - 6\nu + 1}{9}\right) + h\left(\frac{k^2 - 6m - 1}{3}\right)\right\}\right],$$

where again we have put $h' = 3h^*$, so that $hh' \equiv -1 \pmod{2k}$. Now, since $k \not\equiv 0 \pmod 3$,

$$-k^2 - 6\nu + 1 \equiv k^2 - 6m - 1 \equiv 0 \pmod 3.$$

Put $b = \dfrac{k^2 - 6m - 1}{3}$. *If* $-k^2 - 6\nu + 1 \equiv 0 \pmod 9$, put $a = \dfrac{-k^2 - 6\nu + 1}{9}$. *If* $-k^2 - 6\nu + 1 \equiv 3 \pmod 9$, write

$$A_{k,\nu}(m) = -\varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{2k}\left\{h'\left(\frac{-4k^2 - 6\nu + 1}{9}\right) + h\left(\frac{k^2 - 6m - 1}{3}\right)\right\}\right]$$

and put $a = (-4k^2 - 6\nu + 1)/9$. *If* $-k^2 - 6\nu + 1 \equiv 6 \pmod 9$, write

$$A_{k,\nu}(m) = -\varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{2k}\left\{h'\left(\frac{2k^2 - 6\nu + 1}{9}\right) + h(k^2 - 6m - 1)\right\}\right]$$

and put $a = \dfrac{2k^2 - 6\nu + 1}{9}$. In every case, a and b are integers such that $a + b \equiv 0 \pmod 2$. Hence we put

$$B_{k,\nu}(m) = -\varepsilon_0 \sum_{0\le h<2k} \exp\left[\frac{2\pi i}{2k}(aj' + bj)\right] \qquad (jj' \equiv -1 \pmod{2k}))$$

and obtain $B_{k,\nu}(m) = 2A_{k,\nu}(m)$. Thus $A_{k,\nu}(m) = O(k^{2/3+\varepsilon})$.

Suppose that k *is odd.* As before, $c \equiv k \pmod 3$; furthermore, whether h is even or odd, we can choose h^* so that c is odd. Since k is odd, it follows that $c \equiv k \pmod 6$, and $3hh^* = ck - 1 \equiv 0 \pmod 6$. Then,

$$A_{k,\nu}(m) = \varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{6k}(-h^*k^2 + hk^2 - 6\nu h^* + h^* - 6mh - h)\right]$$

$$= \varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{k}\left\{h'\left(\frac{-k^2 - 6\nu + 1}{18}\right) + h\left(\frac{k^2 - 6m - 1}{6}\right)\right\}\right],$$

where again $3h^* = h'$. Now, $-k^2 - 6\nu + 1 \equiv k^2 - 6m - 1 \equiv 0 \pmod 6$. *If* $-k^2 - 6\nu + 1 \equiv 0 \pmod{18}$, $A_{k,\nu}(m)$ is a Kloosterman sum, and we have our estimate. *If* $-k^2 - 6\nu + 1 \equiv 6 \pmod{18}$, write

$$A_{k,\nu}(m) = \varepsilon_0 \sum_{0\le h<k}{}' \exp\left[\frac{2\pi i}{2}\left\{h'\left(\frac{-7k^2 - 6\nu + 1}{18}\right) + h\left(\frac{k^2 - 6m - 1}{6}\right)\right\}\right].$$

If $-k^2 - 6\nu + 1 \equiv 12 \pmod{18}$ write

$$A_{k,\nu}(m) = \varepsilon_0 \sum_{0 \le h < k}{}' \exp\left[\frac{2\pi i}{k}\left\{h'\left(\frac{5k^2 - 6\nu + 1}{18}\right) + h\left(\frac{k^2 - 6m - 1}{6}\right)\right\}\right].$$

Since $6k^2 \equiv 6 \pmod{18}$, we find that in each case $A_{k,\nu}(m)$ is a Kloosterman sum, so that $A_{k,\nu}(m) = O(k^{2/3+\varepsilon})$.

III. THE RADEMACHER LEMMA

The method of this paper goes back, basically, to a paper by Rademacher [5] in which the functional equation $J(-1/\tau) = J(\tau)$ for the modular invariant $J(\tau)$ is derived directly from the Fourier expansion of $J(\tau)$. The principal analytic tool of [5] is a lemma in which the terms of a certain conditionally convergent double sum are rearranged. Several variations of this lemma can be found in [2] and [3]. Here we derive several more variations which will be needed in Sections IV and V.

We make essential use of the asymptotic estimate of the sums $A_{k,\nu}(m)$ that was obtained for the case $r = 0$, in Section II. Define

$$\mathscr{A}_{k,\nu}(m) = k^{-r} A_{k,\nu}(m). \tag{3.1}$$

When $r = 0$, $\mathscr{A}_{k,\nu}(m) = A_{k,\nu}(m) = O(k^{2/3+\varepsilon})$, by the results of Section II. When $r > 0$, $\mathscr{A}_{k,\nu}(m) = O(k^{1-r})$, since $A_{k,\nu}(m) = O(k)$, a trivial estimate, in every case. For the sake of simplicity, we shall assume that r is a nonnegative integer, although the lemmas that we shall state can actually be proved for any real $r \geq 0$. Of course we apply these lemmas only when r is an integer. Therefore, when $r > 0$, we again have the estimate $\mathscr{A}_{k,\nu}(m) = O(k^{2/3+\varepsilon})$, so that, in all cases considered.

$$\mathscr{A}_{k,\nu}(m) = O(k^{2/3+\varepsilon}). \tag{3.2}$$

In the remainder of this paper we shall assume that $\alpha \neq 0$. This is done since the inclusion of $\alpha = 0$ would complicate the use of the Lipschitz summation formula, to be stated below, and the proofs of the lemmas, and since the case $\alpha = 0$ was treated in [2].

LEMMA (3.3). *Let* $\tau = iy$ $(y > 0)$, *let* r *and* ν *be integers* $(r \geq 0,\ \nu > 0)$, *and let* $l = 1$, $l = 2$, *or* $l = 3$. *Then*

$$\sum_{\substack{k=1\\ l|k}}^{\infty} \sum_{\substack{m=-\infty\\ (m,k)=1}}^{\infty} \frac{\varepsilon^{-1}(M_{k/\sqrt{l},-m}) \exp\left[\frac{-2\pi i}{k}(\nu-\alpha)m'\right]}{k^{r+1}(-i(k\tau - m))}$$
$$= \lim_{K\to\infty} \sum_{\substack{k=1\\ l|k}}^{K} \sum_{\substack{|m|\le K\\ (m,k)=1}} \frac{\varepsilon^{-1}(M_{k/\sqrt{l},-m}) \exp\left[\frac{-2\pi i}{k}(\nu-\alpha)m'\right]}{k^{r+1}(-i(k\tau - m))}. \tag{3.4}$$

LEMMA (3.5). *Let* τ, y, r, *and* ν *be as before, and let* $l = 2$ *or* $l = 3$. *Then*

(3.6)
$$\sum_{\substack{k=1\\ l\nmid k}}^{\infty} \sum_{\substack{m=-\infty\\ l\mid m}}^{\infty} \frac{\varepsilon^{-1}(M_{k,-m/\sqrt{l}}) \exp\left[\frac{-2\pi i}{k}(\nu-\alpha)m'\right]}{k^{r+1}(-i(k\tau - m)}$$
$$= \lim_{K\to\infty} \sum_{\substack{k=1\\ l\nmid k}}^{K} \sum_{\substack{|m|\le K\\ l\mid k\\ (m,k)=1}} \frac{\varepsilon^{-1}(M_{k,-m/\sqrt{l}}) \exp\left[\frac{-2\pi i}{k}(\nu-\alpha)m'\right]}{k^{r+1}(-i(k\tau - m))}.$$

Remark. Lemmas (3.3) and (3.5) are used in an important way in the proofs of Theorems (1.9) and (1.11), and since the lemmas are proved only for purely imaginary τ, these theorems follow only for such τ. We can, however, extend the results to $\Im\,\tau > 0$ by analytic continuation.

Proof of (3.3). The Lipschitz summation formula (see [4]) is

$$\sum_{n=0}^{\infty} (n+\alpha)^p e^{-2\pi t(n+\alpha)} = \frac{\Gamma(p+1)}{(2\pi)^{p+1}} \sum_{q=-\infty}^{\infty} e^{2\pi i q\alpha}(t+qi)^{-p-1}, \tag{3.7}$$

where $\Re\, t > 0$, $0 < \alpha < 1$, and $p > -1$. First, we show the convergence of the left-hand side of (3.4). Put $m = qk + h$, with $0 \le h < k$ and $(h, k) = 1$. Then $M_{k/\sqrt{l},-h} = M_{k/\sqrt{l},-m}S^q$, so that, according to (1.5),

$$\varepsilon(M_{k/\sqrt{l},-h}) = e^{2\pi i q\alpha}\cdot\varepsilon(M_{k/\sqrt{l},-m}),$$

and we can choose $h' = m'$. Therefore,

$$\sum_{\substack{m=-\infty\\ (m,k)=1}}^{\infty} \frac{\varepsilon^{-1}(M_{k/\sqrt{l},-m})\, e^{-2\pi i(\nu-\alpha)m'/k}}{k^{r+1}(-i(k\tau - m))}$$
$$= \frac{1}{k^{r+2}} \sum_{0\le h<k}{}' \varepsilon^{-1}(M_{k/\sqrt{l},-h})\, e^{-2\pi i(\nu-\alpha)h'/k} \sum_{q=-\infty}^{\infty} \frac{e^{2\pi i q\alpha}}{\left(-i\cdot\left(\tau - \frac{h}{k} - q\right)\right)}.$$

We now apply (3.7), with $p = 0$ and $t = -i(\tau - h/k)$, and the above becomes

$$\frac{1}{k^{r+2}} \sum_{0\le h<k}{}' \varepsilon^{-1}(M_{k/\sqrt{l},-h})\, e^{-2\pi i(\nu-\alpha)h'/k}\cdot 2\pi \sum_{n=0}^{\infty} e^{2\pi i(\tau-h/k)(n+\alpha)}$$
$$= \frac{2\pi}{k^2} \sum_{n=0}^{\infty} e^{-2\pi(n+\alpha)y}\, k^{-r} \sum_{0\le h<k}{}' \varepsilon^{-1}(M_{k/\sqrt{l},-h}) \exp\left[\frac{-2\pi i}{k}\{(\nu-\alpha)h' + (n+\alpha)h\}\right]$$
$$= \frac{2\pi}{k^2} \sum_{n=0}^{\infty} e^{-2\pi(n+\alpha)y}\cdot \mathscr{A}_{k,\nu}(n) = O\left(k^{-4/3+\varepsilon}\frac{e^{-2\pi\alpha y}}{1 - e^{-2\pi\alpha y}}\right),$$

where we have made use of (3.1) and (3.2). This proves that the left-hand side of (3.4) converges.

Now we can state the lemma in the form

$$\lim_{K\to\infty} \sum_{\substack{k=1\\ 1|k}}^{K} \sum_{\substack{|m|>K\\ (m,k)=1}} \frac{\varepsilon^{-1}(M_{k/\sqrt{1},-m})\, e^{-2\pi i(\nu-\alpha)m'/k}}{k^{r+1}(-i(k\tau - m))} = 0. \tag{3.8}$$

Put

$$T_k(K) = \sum_{\substack{|m|>K\\ (m,k)=1}} \frac{\varepsilon^{-1}(M_{k/\sqrt{1},-m})\, e^{-2\pi i(\nu-\alpha)m'/k}}{k^{r+1}(-i(k\tau - m))},$$

and define the function

$$g(m) = \begin{cases} \varepsilon^{-1}(M_{k/\sqrt{1},-m})\, e^{-2\pi i(\nu-\alpha)m'/k} & \text{if } (m, k) = 1,\\ 0 & \text{otherwise.}\end{cases}$$

If we replace m by $m + k$, we have $M_{k/\sqrt{1},-m}$ replaced by $M_{k/\sqrt{1},-m}\cdot S^{-1}$ and, by (1.5), $\varepsilon^{-1}(M_{k/\sqrt{1},-m})$ goes over into $e^{2\pi i\alpha}\,\varepsilon^{-1}(M_{k/\sqrt{1},-m})$. Hence $e^{-2\pi i\alpha m/k}g(m)$ is periodic in m with period k, and it can be expressed as a finite exponential sum,

$$e^{-2\pi i\alpha m/k}g(m) = \sum_{j=0}^{k=1} B_j\, e^{2\pi i jm/k},$$

in other words,

$$g(m) = \sum_{j=0}^{k=1} B_j \exp\left[\frac{2\pi i m}{k}(j+\alpha)\right].$$

A little computation shows that

$$B_j = \frac{1}{k} \sum_{0\le m<k}{}' \varepsilon^{-1}(M_{k/\sqrt{1},-m}) \exp\left[\frac{-2\pi i}{k}\{(\nu-\alpha)m' + (j+\alpha)m\}\right]. \tag{3.9}$$

Now,

$$T_k(K) = \sum_{|m|>K} \frac{g(m)}{k^{r+1}(-i(k\tau - m))} = \sum_{|m|>K} \sum_{j=0}^{k-1} B_j \frac{\exp\left[\frac{2\pi i m}{k}(j+\alpha)\right]}{k^{r+1}(-i(k\tau - m))}$$

$$= \sum_{j=0}^{k=1} B_j \sum_{|m|>K} \exp\left[\frac{2\pi i m}{k}(j+\alpha)\right] \cdot k^{-r-1}(-i(k\tau - m))^{-1},$$

and we notice from (3.9) that $k^{-r+1}B_j = \mathscr{A}_{k,\nu}(j)$. Therefore

(3.10)
$$T_k(K) = \frac{1}{k^2}\sum_{j=0}^{k-1} \mathscr{A}_{k,\nu}(j) \sum_{m=K+1}^{\infty} \frac{\exp\left[\frac{2\pi i m}{k}(j+\alpha)\right]}{(-i(k\tau - m))}$$
$$+\frac{1}{k^2}\sum_{j=0}^{k-1} \mathscr{A}_{k,\nu}(j) \sum_{m=K+1}^{\infty} \frac{\exp\left[\frac{-2\pi i m}{k}(j+\alpha)\right]}{(-i(k\tau + m))} = S_1 + S_2.$$

In order to handle S_1, put

$$E_m = \sum_{p=K+1}^{\infty} e^{2\pi i(j+\alpha)p/k} = \frac{e^{2\pi i(j+\alpha)(m+1/2)/k} - e^{2\pi i(j+\alpha)(K+1/2)/k}}{e^{\pi i(j+\alpha)/k} - e^{-\pi i(j+\alpha)/k}}.$$

Since $0 < \alpha \le j + \alpha \le k - 1 + \alpha < k$, we have

$$|E_m| \le \frac{2}{|e^{\pi i(j+\alpha)/k} - e^{-\pi i(j+\alpha)/k}|} = \frac{1}{\sin \pi(j+\alpha)/k}$$
$$\le \max\left\{\frac{k}{2(j+\alpha)}, \frac{k}{2(k-j-\alpha)}\right\} \le \frac{k}{2}\left(\frac{1}{j+\alpha} + \frac{1}{k-j-\alpha}\right).$$

But

$$\sum_{m=K+1}^{\infty} \frac{\exp\left[\frac{2\pi i m}{k}(j+\alpha)\right]}{-i(k\tau - m)} = \sum_{m=K+1}^{\infty} \frac{E_m - E_{m-1}}{-i(k\tau - m)} = \sum_{m=K+1}^{\infty} iE_m\left(\frac{1}{k\tau - m} - \frac{1}{k\tau - m - 1}\right),$$

and therefore

$$\left|\sum_{m=K+1}^{\infty} \frac{\exp\left[\frac{2\pi i m}{k}(j+\alpha)\right]}{-i(k\tau - m)}\right| \le \frac{k}{2}\left(\frac{1}{j+\alpha} + \frac{1}{k-j-\alpha}\right) \sum_{m=K+1}^{\infty} \frac{1}{(k^2y^2 + m^2)^{1/2}\{k^2y^2 + (m+1)^2\}^{1/2}}$$
$$< \frac{k}{2}\left(\frac{1}{j+\alpha} + \frac{1}{k-j-\alpha}\right)\int_K^{\infty} \frac{dx}{x^2} = \frac{k}{2}\left(\frac{1}{j+\alpha} + \frac{1}{k-j-\alpha}\right)K^{-1}.$$

From this it follows that

(3.11)
$$S_1 = \frac{1}{k^2}\sum_{j=0}^{k-1} \mathscr{A}_{k,\nu}(j) \sum_{m=K+1}^{\infty} \frac{\exp\left[\frac{2\pi i m}{k}(j+\alpha)\right]}{-i(k\tau - m)}$$
$$= O\left\{k^{-2}\sum_{j=0}^{k-1} k^{2/3+\varepsilon} \cdot \frac{k}{2}\left(\frac{1}{j+\alpha} + \frac{1}{k-j-\alpha}\right)\right\} = O(k^{-1/3+\varepsilon}K^{-1}\log k),$$

where we have used (3.2) and replaced the summation on j by an integral.

We can proceed in exactly the same way to obtain

(3.12) $$S_2 = O(k^{-1/3+\varepsilon} K^{-1} \log k) .$$

Using (3.10), (3.11), and (3.12), we have

$$T_k(K) = O(k^{-1/3+\varepsilon} K^{-1} \log k) ,$$

and finally

$$\sum_{\substack{k=1 \\ l|k}}^{K} \frac{\varepsilon^{-1}(M_{k/\sqrt{l},-m}) e^{-2\pi i(\nu-\alpha)m'/k}}{k^{r+1}(-i(k\tau - m))} = \sum_{\substack{k=1 \\ l|k}}^{K} T_k(K) = O\left(K^{-1} \sum_{\substack{k=1 \\ l|k}}^{K} k^{-1/3+\varepsilon} \log k\right)$$

$$= O(K^{-1/3+\varepsilon} \log K) .$$

Therefore, (3.8) follows and the lemma is proved.

Proof of (3.5). We show the convergence of the left-hand side of (3.6). If we put $m = lqk + lh$, it is easy to see that as q runs through all the integers and h through the integers modulo k with the condition $(h, k) = 1$, the expression $lqk + lh$ takes on, exactly once, each integer value m such that $m \equiv 0 \pmod{l}$ and $(m, k) = 1$. Now we can choose $(m/l)^* = h^*$, since then

$$l\left(\frac{m}{l}\right)\left(\frac{m}{l}\right)^* = l(qk + h)h^* \equiv -1 \pmod{k} ;$$

likewise we may choose $m' = h^*$, since $mm' = (lqk + lh)h^* \equiv -1 \pmod{k}$. Then $M_{k,-m/\sqrt{l}} \equiv M_{k,-\frac{m}{l}\sqrt{l}} = M_{k,-h\sqrt{l}} \cdot S^{-q}$, so that by (1.5),

$$\varepsilon(M_{k,-m/\sqrt{l}}) = e^{-2\pi i\alpha q} \varepsilon(M_{k,-h\sqrt{l}}) .$$

Therefore,

$$\sum_{\substack{m=-\infty \\ l|m \\ (m,k)=1}}^{\infty} \frac{\varepsilon^{-1}(M_{k,-m/\sqrt{l}}) e^{-2\pi i(\nu-\alpha)m'/k}}{k^{r+1}(-i(k\tau - m))}$$

$$= \frac{1}{lk^{r+2}} \sum_{0 \le h < k}{}' \varepsilon^{-1}(M_{k,-h\sqrt{l}}) e^{-2\pi i(\nu-\alpha)h^*/k} \sum_{q=-\infty}^{\infty} \frac{e^{2\pi i\alpha q}}{\left(-i\left(\frac{\tau}{l} - \frac{h}{k} - q\right)\right)} .$$

Now we again apply (3.7) with $p = 0$ and $t = -i(\tau - h/k)$, and proceed as in the proof of Lemma (3.3) to show the convergence of the left-hand side of (3.6).

We can state the lemma in the form

(3.13) $$\lim_{K\to\infty} \sum_{\substack{k=1 \\ l\nmid k}}^{K} \sum_{\substack{|m|>K \\ l|m \\ (m,k)=1}} \frac{\varepsilon^{-1}(M_{k,-m/\sqrt{l}}) e^{-2\pi i(\nu-\alpha)m'/k}}{k^{r+1}(-i(k\tau - m))} = 0 .$$

Let

$$T_k(K) = \sum_{\substack{|m|>K \\ l|m \\ (m,k)=1}} \frac{\varepsilon^{-1}(M_{k,-m/\sqrt{l}})\, e^{-2\pi i(\nu-\alpha)m'/k}}{k^{r+1}(-i(k\tau - m))}$$

$$= \frac{1}{l} \sum_{\substack{|m|>K \\ l|m \\ (m,k)=1}} \frac{\varepsilon^{-1}(M_{k,-m/\sqrt{l}})\, e^{-2\pi i(\nu-\alpha)m'/k}}{k^{r+1}(-i(k\tau - m))}.$$

If we replace m by m/l, then m' is replaced by m*, and we obtain

$$T_k(K) = l^{-1} \sum_{\substack{|m|>K/l \\ (m,k)=1}} \frac{\varepsilon^{-1}(M_{k,-m\sqrt{l}})\, e^{-2\pi i(\nu-\alpha)m^*/k}}{k^{r+1}(-i(k\tau/l - m))}.$$

This time, define

$$g(m) = \begin{cases} \varepsilon^{-1}(M_{k,-m\sqrt{l}})\, e^{-2\pi i(\nu-\alpha)m^*/k} & \text{if } (m, k) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Replacing m by m + k, we find that $M_{k,-m\sqrt{l}}$ is replaced by $M_{k,-m\sqrt{l}} \cdot S^{-1}$, so that, by (1.5), $\varepsilon(M_{k,-m\sqrt{l}})$ goes into $e^{2\pi i\alpha}\varepsilon^{-1}(M_{k,-m\sqrt{l}})$. Thus $e^{-2\pi i\alpha m/k} g(m)$ is periodic in m with period k, and, as before,

$$g(m) = \sum_{j=0}^{k-1} B_j \exp\left[\frac{2\pi i m}{k}(j + \alpha)\right],$$

with

$$B_j = \frac{1}{k} \sum_{0\leq m<k}{}' \varepsilon^{-1}(M_{k,-m\sqrt{l}}) \exp\left[\frac{-2\pi i}{k}\{(\nu - \alpha)m^* + (j + \alpha)m\}\right]. \tag{3.14}$$

As in the proof of Lemma (3.3), we find that

$$\begin{aligned} T_k(K) = {} & \frac{1}{lk^2} \sum_{j=0}^{k-1} \mathscr{A}_{k,\nu}(j) \sum_{m=\left[\frac{K}{l}\right]+1}^{\infty} \frac{\exp\left[\frac{2\pi i m}{k}(j + \alpha)\right]}{k^{r+1}(-i(k\tau/l - m))} \\ & + \frac{1}{lk^2} \sum_{j=0}^{k-1} \mathscr{A}_{k,\nu}(j) \sum_{m=\left[\frac{K}{l}\right]+1}^{\infty} \frac{\exp\left[\frac{-2\pi i m}{k}(j + \alpha)\right]}{k^{r+1}(-i(k\tau/l + m))} = S_1 + S_2. \end{aligned} \tag{3.15}$$

For the remainder of the proof we can proceed exactly as in the proof of Lemma (3.3).

IV. THE MODULAR GROUP

1. This section is devoted primarily to the proof of Theorem (1.9). We begin with

LEMMA (4.1). *Define* $c_m(l)$ *by*

$$(4.2)\quad c_m(l) = 2\pi \sum_{\substack{k=1 \\ k\equiv 0 (\mathrm{mod}\, l)}}^{\infty} k^{-1} A_{k,\nu}(m) \left(\frac{\nu - \alpha}{m + \alpha}\right)^{(r+1)/2} I_{r+1}\left(\frac{4\pi}{k}\sqrt{(m + \alpha)(\nu - \alpha)}\right).$$

Then $c_m(l) \sim \dfrac{A_{l,\nu}(m)(\nu - \alpha)^{r/2+1/4} \exp\left[\frac{4\pi}{l}(\nu - \alpha)^{1/2}(m + \alpha)^{1/2}\right]}{\sqrt{2l}\,(m + \alpha)^{r/2+3/4}}$.

Proof. The first term that occurs in the sum defining $c_m(l)$ is for $k = l$. This term is equal to

$$2\pi l^{-1} A_{l,\nu}(m)\left(\frac{\nu - \alpha}{m + \alpha}\right)^{(r+1)/2} I_{r+1}\left(\frac{4\pi}{l}\sqrt{(m + \alpha)(\nu - \alpha)}\right).$$

It is a simple consequence of the power series definition of I_{r+1},

$$(4.3)\qquad I_{r+1}(x) = \sum_{p=0}^{\infty} \frac{(x/2)^{2p+r+1}}{p!\,(p + r + 1)!} \qquad (x \text{ real}),$$

that

$$(4.4)\qquad I_{r+1}(x) \leq x^r \sinh x,$$

for every nonnegative integer r. Also, it is easy to see that $\sinh x \leq (x/B) \sinh B$ for $0 \leq x \leq B$. Hence

$$\left| c_m(l) - 2\pi l^{-1} A_{l,\nu}(m)\left(\frac{\nu - \alpha}{m + \alpha}\right)^{(r+1)/2} I_{r+1}\left(\frac{4\pi}{l}\sqrt{(m + \alpha)(\nu - \alpha)}\right)\right|$$

$$= \left| 2\pi\left(\frac{\nu - \alpha}{m + \alpha}\right)^{(r+1)/2} \sum_{\substack{k=2l \\ k\equiv 0 (\mathrm{mod}\, l)}}^{\infty} k^{-1} A_{k,\nu}(m)\, I_{r+1}\left(\frac{4\pi}{k}\sqrt{(m + \alpha)(\nu - \alpha)}\right)\right|$$

$$\leq C_1\left(\frac{\nu - \alpha}{m + \alpha}\right)^{(r+1)/2} \sum_{\substack{k=2l \\ k\equiv 0 (\mathrm{mod}\, l)}}^{\infty} k^{-1}|A_{k,\nu}(m)|\left(\frac{4\pi}{k}\sqrt{(m+\alpha)(\nu-\alpha)}\right)^{r+1} \frac{\sinh\left(\frac{4\pi}{2l}\sqrt{(m+\alpha)(\nu-\alpha)}\right)}{\left(\frac{4\pi}{2l}\sqrt{(m+\alpha)(\nu-\alpha)}\right)}$$

$$= C_1 \frac{2l(4\pi)^r (\nu - \alpha)^{r+1/2}}{(m + \alpha)^{1/2}} \sinh\left(\frac{4\pi}{2l}\sqrt{(m + \alpha)(\nu - \alpha)}\right) \sum_{\substack{k=2l \\ k\equiv 0 (\mathrm{mod}\, l)}}^{\infty} \frac{1}{k^2}|\mathscr{A}_{k,\nu}(m)|$$

$$\leq C_2 (m+\alpha)^{-1/2} \sinh\left(\frac{4\pi}{2l}\sqrt{(m+\alpha)(\nu-\alpha)}\right) \sum_{\substack{k=2l \\ k\equiv 0 \pmod l}}^{\infty} k^{-4/3+\varepsilon}$$

$$\leq C_3 (m+\alpha)^{-1/2} \exp\left[\frac{2\pi}{l}\sqrt{(m+\alpha)(\nu-\alpha)}\right],$$

where we have used (3.1) and (3.2). Now in [8, p. 203] it is shown that

$$I_{r+1}(x) \sim e^x/\sqrt{2\pi x},$$

so that

$$I_{r+1}\left(\frac{4\pi}{l}\sqrt{(m+\alpha)(\nu-\alpha)}\right) \sim \frac{\exp[4\pi l^{-1}(\nu-\alpha)^{1/2}(m+\alpha)^{1/2}]}{2\pi\sqrt{2/l}\,(\nu-\alpha)^{1/4}(m+\alpha)^{1/4}},$$

and the result follows.

Lemma (4.1) with $l = 1$ shows that the Fourier coefficients $a_m(r, \nu)$ of $F(\tau; r, \nu)$, defined by (1.7), satisfy

$$a_m(r,\nu) \sim \frac{\varepsilon^{-1}(T)(\nu-\alpha)^{r/2+1/4}\exp[4\pi(\nu-\alpha)^{1/2}(m+\alpha)^{1/2}]}{\sqrt{2}\,(m+\alpha)}. \tag{4.5}$$

In particular, therefore, the series defining $F(\tau; r, \nu)$ converges absolutely for $\Im\tau > 0$, and $F(\tau; r, \nu)$ is regular there.

In order to derive the transformation properties of $F(\tau; r, \nu)$, we transform (1.7) into a certain double series. The computations are repetitions of those found in [5, pp. 244-5] and in [2] and [3], and we omit them here. Briefly, the series definition of $a_m(r, \nu)$ is inserted into the series for $F(\tau; r, \nu)$, I_{r+1} is replaced by the power series (4.3); the inequality (4.4) and (4.5) are used to justify several interchanges of summation, and use is made of the Lipschitz formula (3.7). We obtain

$$\begin{aligned} F(\tau; r,\nu) &= e^{-2\pi i(\nu-\alpha)\tau} \\ &+ \sum_{k=1}^{\infty}\ \sideset{}{'}\sum_{0\leq h<k} \varepsilon^{-1}(M_{k,-h})\, e^{-2\pi i(\nu-\alpha)h'/k} \sum_{q=-\infty}^{\infty} e^{2\pi i\alpha q}[-i(k\tau-h)+ikq]^r \\ &\times\left(\exp\left[\frac{2\pi i(\nu-\alpha)}{k(k\tau-h-kq)}\right] - \sum_{p=0}^{r}\frac{1}{p!}\left\{\frac{2\pi i(\nu-\alpha)}{k(k\tau-h-kq)}\right\}^p\right). \end{aligned} \tag{4.6}$$

2. In (4.6), put $m = h + qk$. Then $(m, k) = 1$ follows from $(h, k) = 1$, $m' = h'$, and $M_{k,-h} = M_{k,-m}S^q$. Therefore, if we apply (1.5), $\varepsilon(M_{k,-h}) = e^{2\pi i\alpha q}\cdot\varepsilon(M_{k,-m})$, and (4.6) becomes

$$\begin{aligned} F(\tau; r,\nu) &= e^{-2\pi i(\nu-\alpha)\tau} \\ &+ \sum_{k=1}^{\infty}\ \sum_{\substack{m=-\infty \\ (m,k)=1}}^{\infty} \varepsilon^{-1}(M_{k,-m})\, e^{-\frac{2\pi i}{k}(\nu-\alpha)m'}[-i(k\tau-m)]^r\left\{\exp\left[\frac{2\pi i(\nu-\alpha)}{k(k\tau-m)}\right] - \sum_{p=0}^{r}\frac{1}{p!}\left(\frac{2\pi i(\nu-\alpha)}{k(k\tau-m)}\right)^p\right\}. \end{aligned} \tag{4.7}$$

Expand the expression in the braces in (4.7) into a power series to obtain

$$F(\tau;\, r,\, \nu) - e^{-2\pi i(\nu-\alpha)\tau}$$

$$= \sum_{k=1}^{\infty} \sum_{\substack{m=-\infty \\ (m,k)=1}}^{\infty} \varepsilon^{-1}(M_{k,-m})\, e^{-\frac{2\pi i}{k}(\nu-\alpha)m'} [-i(k\tau - m)]^r \sum_{p=r+1}^{\infty} \frac{1}{p!}\left(\frac{2\pi i(\nu-\alpha)}{k(k\tau - m)}\right)^p$$

(4.8) $$= \sum_{k=1}^{\infty} \sum_{\substack{m=-\infty \\ (m,k)=1}}^{\infty} \varepsilon^{-1}(M_{k,-m})\, e^{-\frac{2\pi i}{k}(\nu-\alpha)m'} \frac{2\pi(\nu-\alpha)}{(r+1)!\,k^{r+1}(-i(k\tau - m))}$$

$$+ \sum_{k=1}^{\infty} \sum_{\substack{m=-\infty \\ (m,k)=1}}^{\infty} \varepsilon^{-1}(M_{k,-m})\, e^{-\frac{2\pi i}{k}(\nu-\alpha)m'} [-i(k\tau - m)]^r \sum_{p=r+2}^{\infty} \frac{1}{p!}\left(\frac{2\pi i(\nu-\alpha)}{k(k\tau - m)}\right)^p .$$

This separation into two sums is justified, since the first converges by Lemma (3.3), with $l = 1$, and the second is an absolutely convergent triple sum. The second sum can therefore be rearranged in any fashion. Make use of this fact and apply Lemma (3.3) (with $l = 1$) to the first sum, and (4.8) becomes

$$F(\tau;\, r,\, \nu) - e^{-2\pi i(\nu-\alpha)\tau}$$

(4.9) $$= \lim_{K\to\infty} \sum_{k=1}^{K} \sum_{\substack{|m|\le K \\ (m,k)=1}} \varepsilon^{-1}(M_{k,-m})\, e^{-\frac{2\pi i}{k}(\nu-\alpha)m'} \frac{2\pi(\nu-\alpha)}{(r+1)!\,k^{r+1}(-i(k\tau - m))}$$

$$+ \lim_{K\to\infty} \sum_{k=1}^{K} \sum_{\substack{|m|\le K \\ (m,k)=1}} \varepsilon^{-1}(M_{k,-m})\, e^{-\frac{2\pi i}{k}(\nu-\alpha)m'} [-i(k\tau - m)]^r \sum_{p=r+2}^{\infty} \frac{1}{p!}\left(\frac{2\pi i(\nu-\alpha)}{k(k\tau - m)}\right)^p$$

Therefore

$$F(\tau;\, r,\, \nu) = e^{-2\pi i(\nu-\alpha)\tau} + \varepsilon^{-1}(T)(-i\tau)^r \left\{ \exp \frac{2\pi i(\nu-\alpha)}{\tau} - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i(\nu-\alpha)}{\tau}\right)^p \right\}$$

(4.10) $$+ \lim_{K\to\infty} \sum_{k=1}^{K} \sum_{\substack{|m|=1 \\ (m,k)=1}}^{K} \varepsilon^{-1}(M_{k,-m})\, e^{-\frac{2\pi i}{k}(\nu-\alpha)m'} [-i(k\tau - m)]^r$$

$$\times \left\{ \exp\left[\frac{2\pi i(\nu-\alpha)}{k(k\tau - m)}\right] - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i(\nu-\alpha)}{k(k\tau - m)}\right)^p \right\},$$

where we have combined the two sums of (4.9) and separated out the term corresponding to $m = 0$ and $k = 1$.

Now let

$$S_K(\tau) = \sum_{k=1}^{K} \sum_{\substack{|m|=1 \\ (m,k)=1}}^{K} \varepsilon^{-1}(M_{k,-m})(-i(k\tau - m))^r e^{-\frac{2\pi i}{k}(\nu-\alpha)m'} \exp\left[\frac{2\pi i(\nu - \alpha)}{k(k\tau - m)}\right],$$

and put $-k' = (mm' + 1)/k$, so that $kk' + mm' + 1 = 0$ and thus $kk' \equiv -1 \pmod m$. A little computation shows that

$$(4.11)\qquad \begin{aligned} S_K(\tau) = &\sum_{\substack{k=1 \\ (m,k)=1}}^{K} \sum_{m=1}^{K} \varepsilon^{-1}(M_{k,-m})(-i(k\tau - m))^r \exp\left[2\pi i(\nu - \alpha)\frac{-k' - m'\tau}{k\tau - m}\right] \\ &+ \sum_{\substack{k=1 \\ (m,k)=1}}^{K} \sum_{m=1}^{K} \varepsilon^{-1}(M_{k,m})(-i(k\tau + m))^r \exp\left[2\pi i(\nu - \alpha)\frac{-k' + m'\tau}{k\tau + m}\right], \end{aligned}$$

where we have separated the terms for $m > 0$ and $m < 0$, and used the fact that if m is replaced by $-m$, then m' is replaced by $-m'$. Therefore,

$$\begin{aligned} &\varepsilon^{-1}(T)(-i\tau)^r S_K(T\tau) \\ &= \sum_{\substack{k=1 \\ (m,k)=1}}^{K} \sum_{m=1}^{K} \varepsilon^{-1}(T)\varepsilon^{-1}(M_{k,-m})(-i\tau)^r(-i(kT\tau - m))^r \exp\left[2\pi i(\nu - \alpha)\frac{-k'\tau + m'}{-k - m\tau}\right] \\ &+ \sum_{\substack{k=1 \\ (m,k)=1}}^{K} \sum_{m=1}^{K} \varepsilon^{-1}(T)\varepsilon^{-1}(M_{k,m})(-i\tau)^r(-i(kT\tau + m))^r \exp\left[2\pi i(\nu - \alpha)\frac{-k'\tau - m'}{-k + m\tau}\right]. \end{aligned}$$

Now apply (1.3), and this becomes

$$(4.12)\qquad \begin{aligned} &\varepsilon^{-1}(T)(-i\tau)^r S_K(T\tau) \\ &= \sum_{\substack{k=1 \\ (m,k)=1}}^{K} \sum_{m=1}^{K} \varepsilon^{-1}(M_{-m,-k})(-i(-m\tau - k))^r \exp\left[2\pi i(\nu - \alpha)\frac{-m' + k'\tau}{m\tau + k}\right] \\ &+ \sum_{\substack{k=1 \\ (m,k)=1}}^{K} \sum_{m=1}^{K} \varepsilon^{-1}(M_{m,-k})(-i(m\tau - k))^r \exp\left[2\pi i(\nu - \alpha)\frac{-m' - k'\tau}{m\tau - k}\right]. \end{aligned}$$

By (1.2), $\varepsilon^{-1}(M_{-m,-k})(-i(-m\tau - k))^r = \varepsilon^{-1}(M_{m,k})(-i(m\tau + k))^r$. Inserting this into the first double sum of (4.12), interchanging the roles of m and k, and comparing the result with (4.11), we find that

$$(4.13)\qquad \varepsilon^{-1}(T)(-i\tau)^r S_K(T\tau) = S_K(\tau).$$

From (4.10), it follows that

$$\varepsilon^{-1}(T)(-i\tau)^r F(T\tau; r, \nu) = \varepsilon^{-1}(T)(-i\tau)^r e^{2\pi i(\nu-\alpha)/\tau} + e^{-2\pi i(\nu-\alpha)\tau} - \sum_{p=0}^{r} \frac{1}{p!}\{-2\pi i(\nu - \alpha)\tau\}^p$$

$$+ \lim_{K\to\infty} \Bigg\{ \varepsilon^{-1}(T)(-i\tau)^r S_K(T\tau) - \sum_{k=1}^{K} \sum_{\substack{|m|=1\\(m,k)=1}}^{K} \varepsilon^{-1}(M_{-m,-k})\, e^{-2\pi i(\nu-\alpha)m'/k}$$

$$\times (-i(-m\tau - k))^r \sum_{p=0}^{r} \frac{1}{p!}\left[\frac{-2\pi i(\nu - \alpha)\tau}{k(m\tau + k)}\right] \Bigg\},$$

where we have used (1.3) and the fact that $\varepsilon^2(T) = 1$. We now compare this with (4.10) and apply (4.13) to obtain, finally

$$F(\tau; r, \nu) - \varepsilon^{-1}(T)(-i\tau)^r F(T\tau; r, \nu)$$

$$= \lim_{K\to\infty} \sum_{k=1}^{\infty} \sum_{\substack{|m|\le K\\(m,k)=1}} e^{-2\pi i(\nu-\alpha)m'/k} \Bigg\{ \varepsilon^{-1}(M_{-m,-k})(-i(-m\tau - k))^r \sum_{p=0}^{r} \frac{1}{p!}\left[\frac{-2\pi i(\nu - \alpha)\tau}{k(m\tau + k)}\right]^p$$

(4.14)

$$- \varepsilon^{-1}(M_{k,-m})(-i(k\tau - m))^r \sum_{p=0}^{r} \frac{1}{p!}\left[\frac{2\pi i(\nu - \alpha)}{k(k\tau - m)}\right]^p \Bigg\}.$$

But the right-hand side of (4.14) is a polynomial in τ of degree at most r, which we denote by $-p(\tau; r, \nu)$, and the proof of Theorem (1.9) is complete.

3. CONSTRUCTION OF MODULAR FORMS

Let α and r be fixed, and let $1 \le \nu_1 < \nu_2 < \cdots < \nu_\mu$ be integers. Consider the function defined by

$$F(\tau) = \sum_{j=1}^{\mu} a_{-j} F(\tau; r, \nu_j). \tag{4.15}$$

From Theorem (1.9), we have

$$\varepsilon^{-1}(T)(-i\tau)^r F(T\tau) = F(\tau) + \sum_{j=1}^{\mu} a_{-\mu}\, p(\tau; r, \nu_j). \tag{4.16}$$

Putting $p(\tau) = \Sigma_{j=1}^{\mu} a_{-j}\, p(\tau; r, \nu_j)$ and replacing τ by $T\tau$ in (4.16), we see that $\varepsilon^{-1}(T)(-i\tau)^r p(-1/\tau) = -p(\tau)$. Therefore the zeros of $p(\tau)$ occur in pairs (with some obvious exceptions). Let [x] denote the greatest integer less than or equal to x. Let τ_n ($n = 1, \cdots, [r/2] + 1$) be nonexceptional values of τ, so that $p(\tau) = 0$ implies $p(-1/\tau_n) = 0$ and $\tau_n \neq -1/\tau_{n'}$. If we can guarantee that $p(\tau_n) = 0$ for $n = 1, \cdots, [r/2] + 1$, then $p(\tau)$ has $2([r/2] + 1) > r$ roots and therefore, since $p(\tau)$ is of degree at most r, $p(\tau) \equiv 0$. Then, by (4.16) we have $\varepsilon^{-1}(T)(-i\tau)^r F(T\tau) = F(\tau)$. Also, it follows directly from the definition (1.7) of $F(\tau; r, \nu)$ that $F(S\tau) = e^{2\pi i\alpha} F(\tau)$, so that, by a previous remark, $F(\tau)$ is a modular form of dimension r.

We consider therefore the homogeneous linear system

$$\sum_{j=1}^{\mu} a_{-j} p(\tau_n; r, \nu_j) = 0 \qquad (n = 1, \cdots, [r/2] + 1) \tag{4.17}$$

of $[r/2] + 1$ equations in the μ unknowns $a_{-1}, \cdots, a_{-\mu}$. This of course has $\mu - ([r/2] + 1)$ linearly independent solutions $(a_{-1}, \cdots, a_{-\mu})$. Hence we may state

THEOREM (4.18). *Let* μ *be an integer greater than* $[r/2] + 1$. *If we define* $F(\tau)$ *by* (4.15) *with* $(a_{-1}, \cdots, a_{-\mu})$ *chosen to satisfy* (4.17), *then* $F(\tau)$ *is a modular form of dimension* r. *The principal part of* $F(\tau)$ *at* $i\infty$ *is*

$$a_{-\mu} e^{-2\pi i(\nu_\mu - \alpha)\tau} + \cdots + a_{-1} e^{-2\pi i(\nu_1 - \alpha)\tau}.$$

V. THE GROUPS $G(\sqrt{2})$ AND $G(\sqrt{3})$

1. In this section we prove Theorem (1.11). The proof of Lemma (4.1) can be modified to derive

LEMMA (5.1). *Let* $l = 2$ *or* $l = 3$, *and let* $a_{m,2}(r, \nu, l)$ *be defined as in* (1.12). *Then*

$$a_{m,2}(r, \nu, l) \sim \frac{\varepsilon^{-1}(T)(\nu - \alpha)^{r/2+1/4} \exp[4\pi(\nu - \alpha)^{1/2}(m + \alpha)^{1/2}]}{\sqrt{2l}\,(m + \alpha)^{r/2+3/4}}.$$

Now when $l = 2$ or $l = 3$, $c_m(l)$ defined by (4.2) is the same as $a_{m,1}(r, \nu, l)$ defined in (1.12). Thus, Lemmas (4.1) and (5.1) together show that the Fourier coefficients $a_m(r, \nu, l)$ of $F_l(\tau; r, \nu)$, defined by (1.12), satisfy

$$a_m(r, \nu, l) \sim \frac{\varepsilon^{-1}(T)(\nu - \alpha)^{r/2+1/4} \exp[4\pi(\nu - \alpha)^{1/2}(m + \alpha)^{1/2}]}{\sqrt{2l}\,(m + \alpha)^{r/2+3/4}}. \tag{5.2}$$

Thus the series defining $F_l(\tau; r, \nu)$ converges absolutely for $\Im\tau > 0$, so that $F_l(\tau; r, \nu)$ is regular in that half-plane.

Let

$$f_1(\tau) = \sum_{m=0}^{\infty} a_{m,1}(r, \nu, l)\, e^{2\pi i(m+\alpha)\tau/\sqrt{l}},$$

$$f_2(\tau) = e^{-2\pi i(\nu-\alpha)\tau/\sqrt{l}} + \sum_{m=0}^{\infty} a_{m,2}(r, \nu, l)\, e^{2\pi i(m+\alpha)\tau/\sqrt{l}},$$

so that $F_l(\tau; r, \nu) = f_1(\tau) + f_2(\tau)$. We apply the argument used in the case of the modular group to derive (4.10), use Lemma (3.3), with $l = 2$ or $l = 3$, and obtain

$$f_1(\tau) = \lim_{K\to\infty} \sum_{\substack{k=1 \\ k\equiv 0 (\mathrm{mod}\, l)}}^{K} \sum_{\substack{|m|=1 \\ (m,k)=1}}^{K} \varepsilon^{-1}(M_{k/\sqrt{l},-m})\, e^{-2\pi i(\nu-\alpha)m'/k}$$

(5.3)

$$\times [-i(k\tau/\sqrt{l} - m)]^r \left\{ \exp\left[\frac{2\pi i(\nu-\alpha)}{k(k\tau/\sqrt{l} - m)}\right] - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i(\nu-\alpha)}{k(k\tau/\sqrt{l} - m)}\right)^p \right\}.$$

Notice that there are no terms with $m = 0$, since $k \equiv 0 \pmod{l}$ and $(m, k) = 1$. We can again apply essentially the same argument to $f_2(\tau)$, this time using Lemma (3.5), to obtain

$$f_2(\tau) = e^{-2\pi i(\nu-\alpha)\tau/\sqrt{l}} + \varepsilon^{-1}(T)(-i\tau)^r \left\{ e^{\frac{2\pi i(\nu-\alpha)}{\sqrt{l}\,\tau}} - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i(\nu-\alpha)}{\sqrt{l}\,\tau}\right)^p \right\}$$

(5.4)
$$+ \lim_{K\to\infty} \sum_{\substack{k=1 \\ k\not\equiv 0 (\mathrm{mod}\, l)}}^{K} \sum_{\substack{|m|=1 \\ m\equiv 0 (\mathrm{mod}\, l) \\ (m,k)=1}}^{K} \varepsilon^{-1}(M_{k,-\frac{m}{l}\sqrt{l}})\, e^{-2\pi i(\nu-\alpha)m'/k} [-i(k\tau - m)]^r$$

$$\times \left\{ \exp\left[\frac{2\pi i(\nu-\alpha)}{k(\sqrt{l}k\tau - m)}\right] - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i(\nu-\alpha)}{k(\sqrt{l}k\tau - m)}\right)^p \right\}.$$

The term for $k = 1$ and $m = 0$ has been written separately.

2. Now put

$S_{K,1}(\tau)$

$$= \sum_{\substack{k=1 \\ k\equiv 0 (\mathrm{mod}\, l)}}^{K} \sum_{\substack{|m|=1 \\ (m,k)=1}}^{K} \varepsilon^{-1}(M_{k/\sqrt{l},-m})\, e^{-\frac{2\pi i}{k}(\nu-\alpha)m'} (-i(k\tau/\sqrt{l} - m))^r \exp\left[\frac{2\pi i(\nu-\alpha)}{k(k\tau/\sqrt{l} - m)}\right].$$

Again letting $-k' = (mm' + 1)/k$, we find, as before, that

$S_{K,1}(\tau)$

$$= \sum_{\substack{k=1 \\ k\equiv 0 (\mathrm{mod}\, l)}}^{K} \sum_{\substack{m=1 \\ (m,k)=1}}^{K} \varepsilon^{-1}(M_{k/\sqrt{l},-m}) (-i(k\tau/\sqrt{l} - m))^r \exp\left[2\pi i(\nu-\alpha)\frac{-k' - m'\tau/\sqrt{l}}{k\tau/\sqrt{l} - m}\right]$$

(5.5)

$$+ \sum_{\substack{k=1 \\ k\equiv 0 (\mathrm{mod}\, l)}}^{K} \sum_{\substack{m=1 \\ (m,k)=1}}^{K} \varepsilon^{-1}(M_{k/\sqrt{l},m}) (-i(k\tau/\sqrt{l} + m))^r \exp\left[2\pi i(\nu-\alpha)\frac{-k' + m'\tau/\sqrt{l}}{k\tau/\sqrt{l} + m}\right].$$

Let

$$S_{K,2}(\tau) = \sum_{\substack{k=1 \\ k\not\equiv 0 \pmod{l}}}^{K} \sum_{\substack{|m|=1 \\ m\equiv 0 \pmod{l} \\ (m,k)=1}}^{K} \varepsilon^{-1}(M_{k,-\frac{m}{l}\sqrt{l}})\, e^{-\frac{2\pi i}{k}(\nu-\alpha)m'} (-i(k\tau - m/\sqrt{l}))^{r} \exp\left[\frac{2\pi i(\nu-\alpha)}{k(\sqrt{l}k\tau - m)}\right],$$

and again

$$S_{K,2}(\tau) = \sum_{\substack{k=1 \\ k\not\equiv 0 \pmod{l}}}^{K} \sum_{\substack{m=1 \\ m\equiv 0 \pmod{l} \\ (m,k)=1}}^{K} \varepsilon^{-1}(M_{k,-\frac{m}{l}\sqrt{l}}) (-i(k\tau - m/\sqrt{l}))^{r} \exp\left[2\pi i(\nu-\alpha)\frac{-k' - \sqrt{l}m'\tau}{\sqrt{l}k\tau - m}\right]$$

$$(5.6) \qquad + \sum_{\substack{k=1 \\ k\not\equiv 0 \pmod{l}}}^{K} \sum_{\substack{m=1 \\ m\equiv 0 \pmod{l} \\ (m,k)=1}}^{K} \varepsilon^{-1}(M_{k,\frac{m}{l}\sqrt{l}}) (-i(k\tau + m/\sqrt{l}))^{r} \exp\left[2\pi i(\nu-\alpha)\frac{-k' + \sqrt{l}m'\tau}{\sqrt{l}k\tau + m}\right].$$

Now

$$\varepsilon^{-1}(T)(-i\tau)^{r} S_{K,1}(T\tau) = \sum_{\substack{k=1 \\ k\equiv 0 \pmod{l}}}^{K} \sum_{\substack{m=1 \\ (m,k)=1}}^{K} \varepsilon^{-1}(T)\varepsilon^{-1}(M_{\frac{k}{\sqrt{l}},-m})(-i\tau)^{r}\left(-i\left(\frac{k}{\sqrt{l}}T\tau - m\right)\right)^{r} \exp\left[2\pi i(\nu-\alpha)\frac{-\sqrt{l}k'\tau + m'}{-k - \sqrt{l}m\tau}\right]$$

$$(5.7) \qquad + \sum_{\substack{k=1 \\ k\equiv 0 \pmod{l}}}^{K} \sum_{\substack{m=1 \\ (m,k)=1}}^{K} \varepsilon^{-1}(T)\varepsilon^{-1}(M_{\frac{k}{\sqrt{l}},m})(-i\tau)^{r}\left(-i\left(\frac{k}{\sqrt{l}}T\tau + m\right)\right)^{r} \exp\left[2\pi i(\nu-\alpha)\frac{-\sqrt{l}k'\tau - m'}{-k + \sqrt{l}m\tau}\right].$$

But $M_{k/\sqrt{l},-m} \cdot T = M_{-m,-k/\sqrt{l}}$ and $M_{k/\sqrt{l},m} \cdot T = M_{m,-k/\sqrt{l}}$, so that, by (1.3),

$$\varepsilon^{-1}(T)\,\varepsilon^{-1}(M_{k/\sqrt{l},-m})(-i\tau)^{r}\left(-i\left(\frac{k}{\sqrt{l}}T\tau - m\right)\right)^{r} = \varepsilon^{-1}(M_{-m,-k/\sqrt{l}})(-i(-m\tau - k/\sqrt{l}))^{r}$$

and

$$\varepsilon^{-1}(T)\,\varepsilon^{-1}(M_{k/\sqrt{l},m})(-i\tau)^{r}\left(-i\left(\frac{k}{\sqrt{l}}T\tau + m\right)\right)^{r} = \varepsilon^{-1}(M_{m,-k/\sqrt{l}})(-i(m\tau - k/\sqrt{l}))^{r}.$$

Also, by (1.2),

$$\varepsilon^{-1}(M_{-m,-k/\sqrt{l}})(-i(-m\tau - k/\sqrt{l}))^{r} = \varepsilon^{-1}(M_{m,k/\sqrt{l}})(-i(m\tau + k/\sqrt{l}))^{r},$$

so that (5.7) becomes

$$\varepsilon^{-1}(T)(-i\tau)^r S_{K,1}(T\tau)$$

$$(5.8)\quad \begin{aligned} &= \sum_{\substack{k=1\\ k\equiv 0 \pmod l}}^{K} \sum_{\substack{m=1\\ (m,k)=1}}^{K} \varepsilon^{-1}(M_{m,k/\sqrt{l}})(-i(m\tau + k/\sqrt{l}))^r \exp\left[2\pi i(\nu-\alpha)\frac{-m' + \sqrt{l}k'\tau}{\sqrt{l}m\tau + k}\right] \\ &+ \sum_{\substack{k=1\\ k\equiv 0 \pmod l}}^{K} \sum_{\substack{m=1\\ (m,k)=1}}^{K} \varepsilon^{-1}(M_{m,-k/\sqrt{l}})(-i(m\tau - k/\sqrt{l}))^r \exp\left[2\pi i(\nu-\alpha)\frac{-m' - \sqrt{l}k'\tau}{\sqrt{l}m\tau - k}\right]. \end{aligned}$$

Interchanging the roles of m and k, noticing that $(m, k) = 1$ and $m \equiv 0 \pmod l$ imply $k \not\equiv 0 \pmod l$, and comparing the result with (5.6), we find that

$$(5.9)\qquad \varepsilon^{-1}(T)(-i\tau)^r S_{K,1}(T\tau) = S_{K,2}(\tau).$$

It follows directly that

$$(5.10)\qquad \varepsilon^{-1}(T)(-i\tau)^r S_{K,2}(T\tau) = S_{K,1}(\tau).$$

Finally, (5.3), (5.4), (5.9), (5.10), and the fact that $F_1(\tau; r, \nu) = f_1(\tau) + f_2(\tau)$ together imply

$$\varepsilon^{-1}(T)(-i\tau)^r F_1(T\tau; r, \nu) - F_1(\tau; r, \nu)$$

$$(5.11)\quad \begin{aligned} &= \lim_{K\to\infty} \sum_{\substack{k=1\\ k\equiv 0 \pmod l}}^{K} \sum_{\substack{|m|=1\\ (m,k)=1}}^{K} e^{-\frac{2\pi i}{k}(\nu-\alpha)m'} \left\{\varepsilon^{-1}(M_{k/\sqrt{l},-m})(-i(k\tau/\sqrt{l} - m))^r \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i(\nu-\alpha)}{k(k\tau/\sqrt{l} - m)}\right)^p \right. \\ &\qquad \left. - \varepsilon^{-1}(M_{-m,-k/\sqrt{l}})(-i(-m\tau - k/\sqrt{l}))^r \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{-2\pi i(\nu-\alpha)\tau}{k(m\tau + k/\sqrt{l})}\right)^p \right\} \\ &+ \lim_{K\to\infty} \sum_{\substack{k=1\\ k\not\equiv 0 \pmod l}}^{K} \sum_{\substack{|m|\le K\\ m\equiv 0 \pmod l\\ (m,k)=1}} e^{-\frac{2\pi i}{k}(\nu-\alpha)m'} \\ &\times \left\{\varepsilon^{-1}(M_{k,-m/\sqrt{l}})(-i(k\tau - m/\sqrt{l}))^r \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i(\nu-\alpha)}{k(\sqrt{l}k\tau - m)}\right)^p \right. \\ &\qquad \left. - \varepsilon^{-1}(M_{-m/\sqrt{l},-k})(-i(-m\tau/\sqrt{l} - k))^r \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{-2\pi i(\nu-\alpha)\tau}{k(m\tau + \sqrt{l}k)}\right)^p \right\}, \end{aligned}$$

where we have also made use of (1.3) and the fact that $\varepsilon^2(T) = 1$ (the term for $k = 1$ and $m = 0$ has now been absorbed into the second double sum). The right-hand side of (5.11) is a polynomial in τ of degree at most r, which we denote by $p_1(\tau; r, \nu)$, and the proof of Theorem (1.11) is complete.

We can now proceed as in Section IV to construct forms for the groups $G(\sqrt{2})$ and $G(\sqrt{3})$, by making use of Theorem (1.11). That is, we fix α and r and form a linear combination of the $F_1(\tau; r, \nu)$ (with different ν) in such a way that the resulting linear combination of polynomials arising from Theorem (1.11) vanishes identically.

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