SUBRINGS OF SIMPLE ALGEBRAS

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This paper is essentially an appendix to the work [1] of R. A. Beaumont and the author. Its purpose is to clarify the concept introduced there of the smallest field of definition for a subring of a simple rational algebra. However, the main results can be formulated for subrings of quite general algebras, and the proofs do not depend on the developments in [1]. We are indebted to the referee for this observation and for substantial simplification of the paper generally.

Let Λ be an integral domain, and suppose that Q is the quotient field of Λ. Throughout the paper, S is to be a finite-dimensional Q-algebra containing the subring A such that A is a Λ-module and QA = S. Let C be the center of A. A field F is called a *field of definition* of A if $\Lambda \subset F \subset C$ and there exist an F-basis $a_1, \cdots, a_k$ of S in A and a nonzero element $\lambda \in \Lambda$ such that

(1) $$\lambda A \subset (A \cap F)a_1 + \cdots + (A \cap F)a_k.$$

It is routine to show that this property does not depend on the choice of $a_1, \cdots, a_k$. In case Λ is the ring of integers, the last condition is equivalent to this, that the group $(A \cap F)a_1 + \cdots + (A \cap F)a_k$ is of finite index in A. If also A contains the identity 1 of S, then $F \subset C$ is a field of definition of A if and only if A is a finitely generated $A \cap F$-module.

For any Λ-submodule B of a Q-space T, define

(2) $$QE(B) = \{h \in \mathrm{Hom}_Q(T, T) \mid \lambda h(B) \subset B \text{ for some } \lambda \neq 0 \text{ in } \Lambda\} .$$

If B satisfies QB = T and $h \in \mathrm{Hom}_\Lambda(B, B)$, then h can be extended to T by defining $h(t) = \lambda^{-1} h(\lambda t)$, where $\lambda \neq 0$ in Λ is such that $\lambda t \in B$. Thus, $\mathrm{Hom}_\Lambda(B, B)$ can be identified with $E(B) = \{h \in \mathrm{Hom}_Q(T, T) \mid h(B) \subset B\}$. Consequently, by (2), $Q \otimes_\Lambda \mathrm{Hom}_\Lambda(B, B)$ can be identified with Q(E(B)) = QE(B). In particular, QE(B) does not depend on the manner in which B is imbedded in T.

If F is any field between Q and C, then $\mathrm{Hom}_F(S, S)$ can be identified with the subring of $\mathrm{Hom}_Q(S, S)$ consisting of all Q-endomorphisms which commute with multiplication by elements of F. Henceforth this identification will be made.

LEMMA 1. *The field* F *is a field of definition of* A *if and only if* $\Lambda \subset F \subset C$ *and* $\mathrm{Hom}_F(S, S) \subset QE(A)$.

Proof. Let $a_1, \cdots, a_k$ be an F-basis of S in A. Put

$$B = (A \cap F)a_1 + \cdots + (A \cap F)a_k.$$

Assume that F is a field of definition of A, and let $\lambda \neq 0$ in Λ be such that $\lambda A \subset B \subset A$. Let $h \in \mathrm{Hom}_F(S, S)$. Since QA = S, there exists $\mu \neq 0$ in Λ such that $\mu h(a_i) \in A$ for $i = 1, \cdots, k$. Then

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$$\mu\lambda h(A) \subset \mu h(B) = (A \cap F)\mu h(a_1) + \cdots + (A \cap F)\mu h(a_k) \subset A.$$

Thus, $h \in QE(A)$. Conversely, suppose that $\Lambda \subset F \subset C$ and $\mathrm{Hom}_F(S, S) \subset QE(A)$. Define $p_i: S \to F \subset S$ by $p_i(\Sigma f_i a_i) = f_i$. Then $p_i \in \mathrm{Hom}_F(S, S) \subset QE(A)$, so that there exists $\lambda \neq 0$ satisfying $\lambda p_i(A) \subset A$ for $i = 1, \cdots, k$. Thus, if $a \in A$, then $\lambda a = (\lambda p_1(a))a_1 + \cdots + (\lambda p_k(a))a_k \in B$. Hence, F is a field of definition of A.

The next lemma—a slight generalization of part of the Jacobson-Bourbaki Theorem (see [2, p. 159])—was suggested by the referee.

LEMMA 2. *Let* S_R *and* S_L *be the subrings of* $\mathrm{Hom}_Q(S, S)$ *respectively consisting of right and left multiplications by elements of* S. *If* E *is a subring of* $\mathrm{Hom}_Q(S, S)$ *containing* S_R *and* S_L, *then there is a field* F *between* Q *and* C *such that* $E = \mathrm{Hom}_F(S, S)$.

Proof. Since S is a simple algebra and $S_R, S_L \subset E$, it follows that S is an irreducible E-module. Moreover, the elements of the centralizer of E in $\mathrm{Hom}_Q(S, S)$ commute with the elements of S_R and S_L. Hence this centralizer is a subfield F of C. By the Density Theorem [2, p. 28] and the finite-dimensionality of S over Q, $E = \mathrm{Hom}_F(S, S)$.

THEOREM. *Let* Λ *be an integral domain with quotient field* Q. *Suppose that* S *is a finite-dimensional simple* Q-*algebra and that* A *is a* Λ-*subalgebra of* S *such that* QA = S. *Let* C *be the center of* S. *Then there exists a smallest field of definition* F *of* A *between* Λ *and* C, *and this field satisfies*

$$QE(A) = \mathrm{Hom}_F(S, S).$$

Proof. Clearly QE(A) contains S_R and S_L; therefore, by Lemma 2, there exists a field F between Q and C such that $QE(A) = \mathrm{Hom}_F(S, S)$. By Lemma 1, F is a field of definition of A. Suppose that G is any other field of definition of A. Then $Q \subset G \subset C$ and $\mathrm{Hom}_G(S, S) \subset \mathrm{Hom}_F(S, S)$, by Lemma 1. Thus G, the centralizer of $\mathrm{Hom}_G(S, S)$ (in $\mathrm{Hom}_Q(S, S)$), contains the centralizer of $\mathrm{Hom}_F(S, S)$, which is F.

The hypotheses for the following corollaries are uniform: S is a finite-dimensional, simple Q-algebra; A is a Λ-subalgebra of S such that QA = S; F is the smallest field of definition of A; the dimension of S over F is k; 1 is the identity element of S.

COROLLARY 1. *The center of* QE(A) *consists of all scalar multiplications by elements of* F. *If* $1 \in A$, *then the center of* $\mathrm{Hom}_\Lambda(A, A)$ *consists of all scalar multiplications by elements of* $A \cap F$.

Proof. The first statement follows from the theorem and the fact that F is the center of $\mathrm{Hom}_F(S, S)$. If $1 \in A$, then for any $\alpha \in S$, $\alpha A \subset A$ implies $\alpha \in A$. This proves the last statement.

COROLLARY 2. *Suppose that* S *is a field,* F = S, *and* $1 \in A$. *Then* A *is isomorphic (as a* Λ-*algebra) to* $\mathrm{Hom}_\Lambda(A, A)$.

Proof. By the Theorem, $QE(A) = \mathrm{Hom}_F(F, F) = F$ is commutative. Thus, Corollary 2 follows from Corollary 1.

COROLLARY 3. *Suppose that* B *and* C *are independent submodules of* A *such that* $\lambda A \subset B + C$ *for some* $\lambda \neq 0$. *Then* QB *and* QC *are* F-*subspaces of* S.

Proof. Let p be the projection of S onto QB corresponding to the decomposition $S = QB \oplus QC$. Then $\lambda p(A) = p(\lambda A) \subset p(B + C) = B \subset A$. Thus, $p \in QE(A) = \mathrm{Hom}_F(S, S)$. Consequently, QB and QC are F-subspaces of S.

Definition. Let A be a torsion-free Λ-module. Then A is called *strongly indecomposable* if there exists no decomposable submodule B of A for which there is a $\lambda \neq 0$ in Λ such that $\lambda A \subset B$.

COROLLARY 4. *Suppose that* S *is a field. Then* $F = S$ *if and only if* A *is strongly indecomposable.*

Proof. Let $a_1, \cdots, a_k$ be an F-basis of S in A. Then

$$B = (A \cap F)a_1 + \cdots + (A \cap F)a_k$$

satisfies $\lambda A \subset B \subset A$ for some $\lambda \neq 0$. Hence if A is strongly indecomposable, then $k = 1$ and $F = S$. The converse follows from Corollary 3.

If G is a field of definition of A and $H \subset G$ is a field of definition of $A \cap G$, then clearly H is also a field of definition of A.

COROLLARY 5. *There exists a* Λ*-subalgebra* C *of* A *such that*

(i) $\lambda A \subset C$ *for some* $\lambda \neq 0$ *in* Λ;

(ii) $C = B_1 \oplus \cdots \oplus B_k$, *where each* B_i *is a* Λ*-submodule of* C *isomorphic to* $A \cap F$ *and each* B_i *is strongly indecomposable.*

Proof. Let $a_1, \cdots, a_k$ be an F-basis of S in A. Suppose $a_u a_v = \Sigma_w f_{uvw} a_w$ ($f_{uvw} \in F$). Choose $\mu \neq 0$ in Λ so that $\mu f_{uvw} \in A$ for all u, v, w. Let $b_i = \mu a_i$. Then $b_1, \cdots, b_k$ is an F-basis of S such that $C = (A \cap F)b_1 + \cdots + (A \cap F)b_k$ is a Λ-subalgebra of A. Since F is a field of definition of A, there exists $\lambda \neq 0$ such that $\lambda A \subset C$. By the remark above, F is the smallest field of definition of $A \cap F$. Therefore $A \cap F$ is strongly indecomposable, by Corollary 4.

COROLLARY 6. *Suppose that* $1 \in A$. *Then there exist a* Λ*-subalgebra* E *of* $\mathrm{Hom}_\Lambda(A, A)$ *and a* $\lambda \neq 0$ *in* Λ *such that*

(i) $\lambda^2 \mathrm{Hom}_\Lambda(A, A) \subset E \subset \mathrm{Hom}_\Lambda(A, A)$, *and*

(ii) E *is isomorphic to the ring of all matrices of order* k *with elements in the ring* $\lambda(A \cap F)$.

Proof. Let $C = B_1 \oplus \cdots \oplus B_k$ be as in Corollary 5, so that $\lambda A \subset C \subset A$. Then the ring $\lambda \mathrm{Hom}_\Lambda(C, C) = \mathrm{Hom}_\Lambda(C, \lambda C)$ is mapped isomorphically onto a subring E of $\mathrm{Hom}_\Lambda(A, A)$ by the correspondence $h \to h^\lambda$, where $h^\lambda(a) = \lambda^{-1} h(\lambda a)$ for $a \in A$. If $g \in \mathrm{Hom}_\Lambda(A, A)$, then $\lambda^2 g(C) \subset \lambda C$ and $(\lambda^2 g)^\lambda = \lambda^2 g$. Consequently, E satisfies (i). It is well known that $\mathrm{Hom}_\Lambda(C, C)$ is isomorphic to the ring of matrices of order k in $\mathrm{Hom}_\Lambda(A \cap F, A \cap F)$. By Corollary 2 and the remark preceding Corollary 5, $\mathrm{Hom}_\Lambda(A \cap F, A \cap F)$ is isomorphic to $A \cap F$. Thus, E satisfies (ii).

REFERENCES

1. R. A. Beaumont and R. S. Pierce, *Torsion free rings*, Illinois J. Math. (to appear).

2. N. Jacobson, *Structure of Rings*, Amer. Math. Soc. Colloquium Publication, 37 (1956).

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