p-ADIC TRANSFORMATION GROUPS

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1. INTRODUCTION

The present paper is motivated by considerations of the question whether a p-adic group can act effectively as a topological transformation group on a manifold. Our purpose is to study the topological transformation groups (G, X) in which G is a p-adic group and X is a locally compact Hausdorff space. We prove that if X is of homology dimension not greater than n (with respect to reals modulo 1), the homology dimension of the orbit space X/G is not greater than n + 3. If in particular X is an n-dimensional manifold and G acts effectively on X, then the homology dimension of X/G is actually equal to n + 2.

From our result it is easy to verify the following known theorem. If G is a p-adic group (respectively, a p-adic solenoid group) acting *freely* on an n-dimensional manifold X, then the orbit space X/G is of dimension either n + 2 (respectively, n + 1) or ∞. It remains to be seen whether our results can be used to prove the well-known conjecture that a p-adic group can not act effectively on a manifold.

In proving our results, we make extensive use of a modified special homology theory of Smith in which reals modulo 1 are used as coefficients. For any compact Hausdorff space X on which a prime-power order cyclic group or a p-adic group acts, special homology groups are defined and several exact sequences are established.

2. COVERINGS

Let X be a space, and A a subset of X. An *open covering* of A in X is a collection λ of open subsets of X such that every $U \in \lambda$ intersects A and such that the union $\bigcup\{U \mid U \in \lambda\}$ contains A. A *closed covering* of A in X is a collection of closed subsets of X the interiors of which form an open covering of A in X. By a *covering* we mean either an open covering or a closed covering.

Let λ and μ be coverings of A in X. If every member of μ is contained in some member of λ, we say that μ *refines* λ. For every $V \in \mu$, the *star* of V in μ, denoted by (V, μ), is defined to be the union of the members of μ which meet V. If for every $V \in \mu$, (V, μ) is contained in some member of λ, we say that μ *star-refines* λ.

(2.1) Let X be a compact Hausdorff space, and A a closed subset of X. Then, for every open covering λ of A in X, there exists an open covering μ of A in X star-refining λ.

(2.2) Let X be a normal space, and A a closed subset of X. Let α be an open covering of A in X. Then for every finite closed covering $\lambda = \{F_1, \cdots, F_m\}$ of A in X refining α there exists an open covering $\mu = \{U_1, \cdots, U_m\}$ of A in X refining α such that every F_i is contained in U_i and such that $F_{i_1} \cap \cdots \cap F_{i_j} \neq \emptyset$ if and only if $U_{i_1} \cap \cdots \cap U_{i_j} \neq \emptyset$.

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Let X be a space, and let T be a homeomorphism of X onto itself. A subset A of X is T-*invariant* if $T(A) = A$. Let A be a T-invariant subset of X. A covering λ of A in X is T-*invariant* if for every $U \in \lambda$, $T(U) \in \lambda$.

(2.3) In (2.1) and (2.2), if both A and λ are invariant under a periodic map T of X, then we can claim the existence of a T-invariant μ.

Throughout this paper we let p be an arbitrary but fixed prime number, and for any nonnegative integer i, we let p^i be abbreviated by [i].

Let X be a compact Hausdorff space, and let T be a periodic map of X such that for some integer $r \geq 0$, $T^{[r]}$ is the identity map. Then the fixed point set of $T^{[i]}$, denoted by F_i, is compact and T-invariant. Moreover,

$$F_0 \subset \cdots \subset F_r = X.$$

Notice that we do not exclude the possibility that $T^{[i]}$ is the identity map for some integer i less than r.

Let A be a T-invariant closed subset of X. A covering λ of A in X is called *special* if the following conditions are satisfied.

1) λ is finite and T-invariant.
2) For every $U \in \lambda$ there exists an integer $t(U) \geq 0$ such that $T^{[t(U)]}(U) = U$ and the $T^i(U)^-$ (that is, the closures of the $T^i(U)$) for $i = 0, \cdots, [t(U)] - 1$ are mutually disjoint.
3) Let $U \in \lambda$ and let s be a positive integer. If there exists a $V \in \lambda$ such that $V \cap U \neq \emptyset$, $T^s(V) \cap U \neq \emptyset$, and $V \cap T^s(V) = \emptyset$, then $T^s(U) = U$.
4) For any $T^{[i]}$-invariant members $U_0, \cdots, U_q$ of λ with $U_0 \cap \cdots \cap U_q \neq \emptyset$, $U_0 \cap \cdots \cap U_q \cap A \cap F_i \neq \emptyset$.

Notice that if $r = 1$, this definition of special covering is the one used by Smith in his special homology theory.

(2.4) LEMMA. *Let* X *be a compact Hausdorff space: let* T *be a periodic map of* X *such that for some nonnegative integer* r, $T^{[r]}$ *is the identity map; and let* A *be a* T-*invariant closed subset of* X. *Then every covering of* A *in* X *is refined by a special open covering of* A *in* X *and is refined by a special closed covering of* A *in itself.*

Proof. Let λ be a covering of A in X. We first show that λ is refined by a special closed covering μ of A in itself. Since our assertion is the existence of a special closed covering μ of A in itself which refines the covering $\{U \cap A \mid U \in \lambda\}$ of A in itself, we may assume that $A = X$.

Let F_i be the fixed point set of $T^{[i]}$ ($i = 0, \cdots, r$), and let $F_{-1} = \emptyset$. We construct by induction a finite sequence

$$\mu_{-1} = \emptyset \subset \mu_0 \subset \cdots \subset \mu_r$$

in which every μ_i is a special closed covering of F_i in X refining λ.

Suppose that for some integer t $(0 \leq t \leq r)$ $\mu_{-1} = \emptyset \subset \mu_0 \subset \cdots \subset \mu_{t-1}$ has been constructed. Let

$$X_t = F_t - \bigcup \{U^\circ \mid U \in \mu_{t-1}\},$$

where U° means the interior of U in X. Clearly, X_t is compact and T-invariant.

For any $x \in X_t$, there is a neighborhood B(x) of x contained in some member of λ such that

(i) $T^{[t]}(B(x)) = B(x)$,

(ii) the $T^i(B(x))^-$ for $i = 0, \cdots, [t] - 1$ are mutually disjoint, and

(iii) any $U \in \mu_{t-1}$ intersects B(x) if and only if $x \in U$.

Then $\beta = \{B(x) \mid x \in X_t\}$ is an open covering of X_t in X. By (2.1) there exists an open covering γ of X_t in X which star-refines a star-refinement of β.

We take for every $x \in X_t$ a $T^{[t]}$-invariant closed neighborhood D(x) contained in some member of γ such that $D(T(x)) = T(D(x))$ for $x \in X_t$. Let S be a finite T-invariant subset of X_t such that $\{D(x) \mid x \in S\}$ is a closed covering of X_t in X. For every $x \in S$, we let

$$E(x) = D(x) \cup \{y \mid y \in S,\ D(y) \cap D(x) \neq \emptyset\}.$$

It is not hard to show that

$$\mu_t = \mu_{t-1} \cup \{E(x) \mid x \in S\}$$

is a special closed covering of F_t in X. Hence the sequence $\mu_{-1} = \emptyset \subset \mu_0 \subset \cdots \subset \mu_r$ can be constructed by induction. The covering μ_r is clearly as desired.

Now we show that every covering λ of A in X is refined by a special open covering of A in X, where A is an arbitrary T-invariant closed subset of X. From our result above, the open covering $\alpha = \{U^{\circ} \mid U \in \lambda\}$ is refined by a special closed covering $\mu = \{F_1, \cdots, F_m\}$ of A in itself. By (2.2) and (2.3), there exists a T-invariant open covering $\nu = \{V_1, \cdots, V_m\}$ of A in X refining α such that every V_i contains F_i and such that $V_{i_1} \cap \cdots \cap V_{i_j} \neq \emptyset$ if and only if $F_{i_1} \cap \cdots \cap F_{i_j} \neq \emptyset$. It is easily seen that ν is a special open covering of A in X refining λ.

3. HOMOLOGY DIMENSION

Let X be a locally compact Hausdorff space. If there exists a least integer $n \geq -1$ such that the Lebesgue covering dimension of every compact subset of X is not greater than n, we say that X is of *dimension* n. Otherwise, we say that X is of dimension ∞. The dimension of X is written dim X.

Whenever (M, N) is a compact pair, $H_k(M, N)$ denotes the k^{th} Čech homology group with reals modulo 1 as coefficients. If there exists a least integer $n \geq -1$ such that whenever (M, N) is a compact pair with $M \subset X$, $H_k(M, N) = 0$ for all $k > n$, we say that X is of *homology dimension* n. Otherwise, we say that X is of homology dimension ∞. The homology dimension of X is written hd X.

(3.1) Whenever X is a locally compact Hausdorff space, hd X $\leq$ dim X, and the equality holds if dim X is finite [1].

As a consequence of (3.1) we have

(3.2) A locally compact Hausdorff space of homology dimension n is of dimension either n or ∞.

The following results can be found in [2].

(3.3) Let X be a locally compact Hausdorff space, and A a closed subset of X. Then

$$\text{hd } X = \max(\text{hd } A, \text{hd}(X - A)) .$$

(3.4) In a locally compact Hausdorff space of homology dimension n, there exists a point x such that every neighborhood of x is of homology dimension n.

(3.5) If X is a locally compact Hausdorff space of homology dimension n and R is the real line, then

$$\text{hd } (X \times R) = 1 + \text{hd } X .$$

As a consequence of (3.3) and (3.4) we have

(3.6) If X is a locally compact Hausdorff space of finite homology dimension, and G is a finite group acting on X, then the orbit space X/G is of the same homology dimension as X.

Proof. Let G_x, for every $x \in X$, be the isotropy subgroup of G at x. For every subgroup H of G we denote by X_H the subspace of X consisting of all the points x with G_x conjugate to H. Then X_H is locally compact and G-invariant (that is, T-invariant for every $T \in G$). Since the projection of X onto X/G defines a local homeomorphism of X_H onto X_H/G, it follows from (3.4) that $\text{hd } X_H = \text{hd } X_H/G$. Hence we infer from (3.3) that

$$\text{hd } X/G = \max \text{hd } X_H/G = \max \text{hd } X_H = \text{hd } X .$$

4. SPECIAL HOMOLOGY GROUPS

Unless the contrary is stated, we use the group $\mathfrak{P}$ of reals modulo 1 as the coefficient group.

Let K be a finite simplicial complex, and let T be a periodic simplicial map of K such that, for some nonnegative integer r, $T^{[r]}$ is the identity map. Again we do not exclude the possibility that $T^{[i]}$ is the identity map for some nonnegative integer i less than r.

Let $C_k(K)$ be the group of k-chains of K based on ordered simplexes. Let

$$\sigma = \sum_{i=0}^{[r]-1} T^i, \qquad \tau = 1 - T$$

be endomorphisms of $C_k(K)$, and let ρ and ρ' stand for σ and τ, respectively, or vice versa. Denote by $C_k^\rho(K)$ the kernel of ρ: $C_k(K) \to C_k(K)$. Since $\partial\rho = \rho\partial$, $\partial C_k^\rho(K) \subset C_{k-1}^\rho(K)$, and therefore we may define groups

$$Z_k^\rho(K) = Z_k(K) \cap C_k^\rho(K) ,$$

$$B_k^\rho(K) = \partial C_{k+1}^\rho(K) ,$$

$$H_k^\rho(K) = Z_k^\rho(K)/B_k^\rho(K) .$$

With $\overline{C}_k^\rho(K) = \rho' C_k(K)$ in place of $C_k^\rho(K)$, we similarly define groups $\overline{Z}_k^\rho(K)$, $\overline{B}_k^\rho(K)$, $\overline{H}_k^\rho(K)$. Both $H_k^\rho(K)$ and $\overline{H}_k^\rho(K)$ are called *special* homology groups of K with respect to T.

Since $\rho\rho' = 0$, $\overline{C}_k^\rho(K) \subset C_k^\rho(K)$. The inclusion homomorphism $\iota: \overline{C}_k^\rho(K) \to C_k^\rho(K)$ induces a homomorphism

$$\iota_*: \overline{H}_k^\rho(K) \to H_k^\rho(K).$$

Remark. If T is of prime order p (that is, r = 1) and the group of integers modulo p is taken as the coefficient group, then $H_k^\rho(K)$ is the special homology group in the sense of Smith. If moreover the fixed point set L of T is a sub-complex of K, then $\overline{H}_k^\rho(K)$ is the relative special homology group $H_k(K, L)$ in the sense of Smith, and ι_* maps $\overline{H}_k^\rho(K)$ isomorphically onto a direct summand of $H_k^\rho(K)$.

Let $\omega: C_k^\rho(K) \to C_k(K)$ be the inclusion homomorphism. Then the sequence

$$0 \to C_k^{\rho'}(K) \xrightarrow{\omega} C_k(K) \xrightarrow{\rho'} \overline{C}_k^\rho(K) \to 0$$

is exact. Hence a standard argument yields

(4.1) The sequence

$$\cdots \leftarrow H_{k-1}^{\rho'}(K) \leftarrow \overline{H}_k^\rho(K) \xleftarrow{\rho'_*} H_k(K) \xleftarrow{\omega_*} H_k^{\rho'}(K) \leftarrow \cdots$$

is exact, where ω_* and ρ'_* are induced by ω and ρ', respectively, and

$$\overline{H}_k^\rho(K) \to H_{k-1}^{\rho'}(K)$$

is the appropriate boundary homomorphism.

(4.2) $\overline{C}_k^\tau(K) = C_k^\tau(K)$, and hence $\iota_*: \overline{H}_k^\tau(K) \to H_k^\tau(K)$ is an isomorphism onto.

Proof. Every k-chain of K can be uniquely written

$$a = \sum_{i=1}^{m} \sum_{j=0}^{[t(u_i)]-1} \alpha_{ij} T^j u_i,$$

where $u_1, \cdots, u_m$ are ordered k-simplexes of K such that whenever $i \neq i'$ and j is any integer, $u_{i'} \neq T^j u_i$, where every $t(u_i)$ is the smallest nonnegative integer such that $T^{[t(u_i)]}(u_i) = u_i$, and where $\alpha_{ij} \in \mathfrak{P}$. If $a \in C_k^\tau(K)$, then the α_{ij}, for

$$j = 0, \cdots, [t(u_i)] - 1,$$

are all equal. Let β_i be an element of $\mathfrak{P}$ such that

$$[r - t(u_i)]\beta_i = \alpha_{i0}.$$

Let

$$b = \sum_{i=1}^{m} \beta_i u_i .$$

Then $\sigma b = a$, and hence $a \in \bar{C}_k^{\tau}(K)$. This proves that $C_k^{\tau}(K) \subset \bar{C}_k^{\tau}(K)$. But $\bar{C}_k^{\tau}(K)$ is contained in $C_k^{\tau}(K)$. Hence $\bar{C}_k^{\tau}(K) = C_k^{\tau}(K)$.

(4.3) If e is an element of $H_k(K)$ which is left fixed by the induced homomorphism $T_*: H_k(K) \to H_k(K)$, then

$$\omega_* \iota_* \sigma_* e = [r]e .$$

Proof. Let c by a cycle in e. Since $T_* e = e$, the cycles $c, Tc, \cdots, T^{[r]-1}c$ are homologous to one another. Hence $\omega \iota \sigma c = \sigma c$ is homologous to $[r]c$.

Now we let the simplicial complex K satisfy the following condition. If u and v are vertices of K and s is an integer such that $T^s u \neq u$ but (u, v) and $(T^s u, v)$ are ordered 1-simplexes of K, then $T^s v = v$. Notice that if X is a compact Hausdorff space and T is a periodic map of X such that $T^{[r]}$ is the identity map, then for every special open covering λ the nerve K_λ of λ together with the periodic simplicial map T_λ of K_λ defined by T satisfies this condition.

From this condition it is easily seen that the fixed point set of $T^{[i]}$, for $i = 0, \cdots, [r]$, is a subcomplex L_i of K. Moreover

$$L_0 \subset L_1 \subset \cdots \subset L_r = K.$$

Let G be the group generated by T. Under the condition above, the orbit space K/G is a simplicial complex and the projection π of K onto K/G is a simplicial map. Moreover, every L_i is G-invariant, and $\pi(L_i) = L_i/G$ is a subcomplex of K/G.

Whenever t is a positive integer we denote by $\mathfrak{C}_t$ the cyclic subgroup of $\mathfrak{P}$ of order t. Let $C_k(L_i/G; \mathfrak{C}_{[r-i]})$ be the subgroup of k-chains of L_i/G with coefficients in $\mathfrak{C}_{[r-i]}$ $(i = 0, \cdots, r - 1)$. Then

$$D_k(K) = \sum_{i=0}^{r-1} C_k(L_i/G; \mathfrak{C}_{[r-i]})$$

is a subgroup of $C_k(K/G)$ with $\partial D_k(K) \subset D_{k-1}(K)$. Let $I_k(K)$ be the quotient group of the kernel of $\partial: D_k(K) \to D_{k-1}(K)$ by the image of $\partial: D_{k+1}(K) \to K_k(K)$. The following is immediate.

(4.4) If $r = 1$, then $I_k(K) = H_k(L; \mathfrak{C}_p)$, where $L = L_0$ is the fixed point set of T and $H_k(L; \mathfrak{C}_p)$ is the k^{th} homology group of L with coefficients in $\mathfrak{C}_p$.

Since the sequence

$$0 \to \bar{C}_k^{\sigma}(K) \xrightarrow{\iota} C_k^{\sigma}(K) \xrightarrow{\pi} D_k(K) \to 0$$

is exact, it follows that

(4.5) The sequence

$$\cdots \leftarrow \bar{H}_{k-1}^{\sigma}(K) \leftarrow I_k(K) \xleftarrow{\pi_*} H_k^{\sigma}(K) \xleftarrow{\iota_*} H_k(K) \leftarrow \cdots$$

is exact, where ι_* and π_* are induced by the homomorphisms ι and π, respectively, and where $I_k(K) \to \bar{H}^{\sigma}_{k-1}(K)$ is the appropriate boundary homomorphism.

Whenever u is an ordered k-simplex of K/G, we let $\tilde{u}$ be an ordered k-simplex of K with $\pi\tilde{u} = u$. Then there exists a homomorphism $\kappa: C_k(K/G) \to \bar{C}^{\tau}_k(K)$ defined by

$$\kappa u = \sigma\tilde{u}.$$

Since the diagram

$$\begin{array}{ccc} \bar{C}^{\tau}_k(K) & \xleftarrow{\ \sigma\ } & C_k(K) \\ {\scriptstyle\kappa}\nwarrow & & \swarrow{\scriptstyle\pi} \\ & C_k(K/G) & \end{array}$$

is commutative, it follows that

(4.6) The diagram

$$\begin{array}{ccc} \bar{H}^{\tau}_k(K) & \xleftarrow{\ \sigma_*\ } & H_k(K) \\ {\scriptstyle\kappa_*}\nwarrow & & \swarrow{\scriptstyle\pi_*} \\ & H_k(K/G) & \end{array}$$

is commutative, where σ_*, π_*, κ_* are induced by σ, π, κ, respectively.

(4.7) Whenever $e \in H_k(K/G)$, $\pi_*\omega_*\iota_*\kappa_* e = [r]e$.

Proof. It follows from the definition of κ that if u is an ordered k-simplex of K/G and $\tilde{u}$ is an ordered k-simplex of K with $\pi\tilde{u} = u$, then

$$\pi\omega\iota\kappa u = \pi\omega\iota\sigma\tilde{u} = \pi\sigma\tilde{u} = [r]u.$$

Hence our assertion follows.

Let θ: $D_k(K) \to C_k(K/G)$ be the inclusion homomorphism. Since the sequence

$$0 \to D_k(K) \xrightarrow{\theta} C_k(K/G) \xrightarrow{\kappa} \bar{C}^{\tau}_k(K) \to 0$$

is exact, it follows that

(4.8) The sequence

$$\cdots \leftarrow I_{k-1}(K) \leftarrow \bar{H}^{\tau}_k(K) \xleftarrow{\kappa_*} H_k(K/G) \xleftarrow{\theta_*} I_k(K) \leftarrow \cdots$$

is exact, where θ_* and κ_* are induced by θ and κ, respectively, and where $\bar{H}^{\tau}_k(K) \to I_{k-1}(K)$ is the appropriate boundary homomorphism.

Now let us establish these results for compact Hausdorff spaces. Let X be a compact Hausdorff space, and let T be a periodic map of X such that for some nonnegative integer r, $T^{[r]}$ is the identity. As before we do not exclude the possibility that $T^{[i]}$ is the identity map for some nonnegative integer i less than r. Let G be the group generated by T, and let π be the projection of X onto the orbit space X/G.

Whenever λ is a special open covering of X, we denote by K_λ the nerve of λ, by T_λ the periodic simplicial map of K_λ defined by T, and by G_λ the group generated by T_λ. Then $T_\lambda^{[r]}$ is the identity map. Hence for K_λ and T_λ we can construct groups and homomorphisms as appeared in (4.1) to (4.8). We shall let A_λ be one of these groups, and $h_\lambda: A_\lambda \to B_\lambda$ one of these homomorphisms.

Let μ be a special open covering of X refining λ, and let $\pi_{\lambda\mu}$ be a projection of K_μ into K_λ such that $\pi_{\lambda\mu} T_\mu = T_\lambda \pi_{\lambda\mu}$. If A_μ is the corresponding group of A_λ for μ, then $\pi_{\lambda\mu}$ induces a homomorphism $\pi_{\lambda\mu *}: A_\mu \to A_\lambda$ independent of the choice of $\pi_{\lambda\mu}$. Hence $\{A_\lambda, \pi_{\lambda\mu *}\}$ is an inverse system, and therefore $\underleftarrow{\lim}\, A_\lambda$ is defined. Let

$$H_k^\rho(X) = \underleftarrow{\lim}\, H_k^\rho(K_\lambda), \qquad \bar{H}_k^\rho(X) = \underleftarrow{\lim}\, \bar{H}_k^\rho(K_\lambda),$$

$$I_k(X) = \underleftarrow{\lim}\, I_k(K_\lambda).$$

$H_k^\rho(X)$ and $\bar{H}_k^\rho(X)$ are called *special* homology groups of X. Notice that it follows from (2.4) that

$$H_k(X) = \underleftarrow{\lim}\, H_k(K_\lambda), \qquad H_k(X/G)\ \underleftarrow{\lim}\, H_k(K_\lambda/G_\lambda).$$

If $h_\mu: A_\mu \to B_\mu$ is the corresponding homomorphism of $h_\lambda: A_\lambda \to B_\lambda$ for μ, then $h_\lambda \pi_{\lambda\mu *} = \pi_{\lambda\mu *} h_\mu$. Hence $\{h_\lambda\}$ gives a homomorphism of $\underleftarrow{\lim}\, A_\lambda$ into $\underleftarrow{\lim}\, B_\lambda$. Let

$$\rho_* = \{\rho_{\lambda *}\}, \qquad \iota_* = \{\iota_{\lambda *}\}, \qquad \text{and so forth.}$$

From (4.1) to (4.8) we can easily prove

(4.9) LEMMA. *The sequence*

$$\cdots \leftarrow H_{k-1}^{\rho'}(X) \leftarrow \bar{H}_k^\rho(X) \overset{\rho'_*}{\leftarrow} H_k(X) \overset{\omega_*}{\leftarrow} H_k^{\rho'}(X) \leftarrow \cdots$$

is exact.

(4.10) LEMMA. *The homomorphism* $\iota_*: \bar{H}_k^\tau(X) \to H_k^\tau(X)$ *is an isomorphism onto.*

(4.11) LEMMA. *If* e *is an element of* $H_k(X)$ *such that* $T_* e = e$, *then*

$$\omega_* \iota_* \sigma_* e = [r]e.$$

(4.12) LEMMA. *The sequence*

$$\cdots \leftarrow \bar{H}_{k-1}^\sigma(X) \leftarrow I_k(X) \overset{\pi_*}{\leftarrow} H_k^\sigma(X) \overset{\iota_*}{\leftarrow} \bar{H}_k^\sigma(X) \leftarrow \cdots$$

is exact.

(4.13) LEMMA. *If* $r = 1$, *then* $I_k(X) = H_k(F; \mathfrak{C}_p)$, *where* F *is the fixed point set of* T *and* $H_k(F; \mathfrak{C}_p)$ *is the* k^{th} *Čech homology group of* F *with coefficients in* $\mathfrak{C}_p$.

(4.14) LEMMA. *The diagram*

$$\begin{array}{ccc} \overline{H}_k^{\tau}(X) & \xleftarrow{\sigma_*} & H_k(X) \\ & {}_{\kappa_*}\nwarrow \quad \swarrow_{\pi_*} & \\ & H_k(X/G) & \end{array}$$

is commutative.

(4.15) LEMMA. *Whenever* $e \in H_k(X/G)$, $\pi_* \omega_* \iota_* \kappa_* e = [r]e$. *Hence*

$$[r]\, H_k(X/G) \subset \pi_* H_k(X)\,,$$

and consequently the quotient group $H_k(X/G)/\pi_* H_k(X)$ *is isomorphic to the limit-group of an inverse system of finite abelian groups with elements of order* p^s $(s \leq r)$.

(4.16) LEMMA. *The sequence*

$$\cdots \leftarrow I_{k-1}(X) \leftarrow \overline{H}_k^{\tau}(X) \overset{\kappa_*}{\leftarrow} H_k(X/G) \overset{\theta_*}{\leftarrow} I_k(X) \leftarrow \cdots$$

is exact.

(4.9), (4.12) and (4.16) follow from the fact that the limit-sequence of an inverse system of exact sequences of compact abelian groups is exact.

(4.17) LEMMA. *If the fixed point set of* $T^{[r-1]}$ *is of homology dimension* $\leq n$, *then* $I_k(X) = 0$ *whenever* $k > n$. *Hence*

$$\iota_*\colon \overline{H}_k^{\sigma}(X) \to H_k^{\sigma}(X), \qquad \kappa_*\colon H_{k+1}(X/G) \to \overline{H}_{k+1}^{\tau}(X)$$

are isomorphisms onto, when $k > n$, *and they are isomorphisms into when* $k = n$.

Proof. We shall prove that for any special open covering μ of X there exists a special open covering ν of X refining μ such that the homomorphism

$$\pi_{\mu\nu *}\colon I_k(K_\nu) \to I_k(K_\mu)$$

induced by a projection $\pi_{\mu\nu}$ is trivial.

Let F_i $(i = 0, \cdots, r)$ be the fixed point set of $T^{[i]}$. Then $F_0 \subset \cdots \subset F_r = X$, and every F_i is a closed subset invariant under T. Since, by hypothesis, F_{r-1} is of homology dimension $\leq n$, it follows from (3.3) and (3.6) that F_i/G, for $i = 0, \cdots, r-1$, is of homology dimension $\leq n$.

Whenever λ is a special open covering of X, we denote by K_λ the nerve of λ, by T_λ the periodic simplicial map of K_λ defined by T, by G_λ the cyclic group generated by T_λ, and by $L_{i\lambda}$ the fixed point set of $T_\lambda^{[i]}$. Then $L_{i\lambda}$ is a T_λ-invariant subcomplex of K_λ, and $L_{i\lambda}/G_\lambda$ is a subcomplex of the simplicial complex K_λ/G_λ.

From (2.4) it is easily seen that for each open covering α of F_i/G $(0 \leq i < r)$ there exists a special open covering λ of X such that $\{(U \cap F_i)/G \mid U \in \lambda\}$ refines α. We infer that

$$\varprojlim H_k(L_{i\lambda}/G;\ \mathfrak{C}_p) = H_k(F_i/G;\ \mathfrak{C}_p)\,,$$

where $H_k(\ ;\mathfrak{C}_p)$ means the k^{th} (Čech) homology group with $\mathfrak{C}_p$ as the coefficient group.

Let μ be a given special open covering of X. Since F_i/G is of homology dimension $\leq n$, there is a sequence of special open coverings of X,

$$\lambda_r = \mu,\ \lambda_{r-1},\ \cdots,\ \lambda_1,\ \lambda_0 = \nu,$$

such that every λ_{i+1} is refined by λ_i, and such that the homomorphism of $H_k(L_{i\lambda_i}/G_{\lambda_i};\ \mathfrak{C}_p)$ into $H_k(L_{i\lambda_{i+1}}/G_{\lambda_{i+1}};\ \mathfrak{C}_p)$ induced by a projection $\pi_{\lambda_{i+1}\lambda_i}$ of $K_{\lambda_i}/G_{\lambda_i}$ onto $K_{\lambda_{i+1}}/G_{\lambda_{i+1}}$ is trivial for $k > n$, where $i = 0, \cdots, r-1$. Whenever $0 \leq i < j \leq r-1$, we let $\pi_{ij} = \pi_{\lambda_j\lambda_{j-1}} \cdots \pi_{\lambda_{i+1}\lambda_i}$.

Let

$$c^{(0)} = c_0^{(0)} + \cdots + c_{r-1}^{(0)}$$

be an arbitrary element of $D_k(K_{\lambda_0})$, with

$$\partial c^{(0)} = 0,$$

$$c_i^{(0)} \in C_k(L_{i\lambda_0}/G_{\lambda_0};\ \mathfrak{C}_{[r-i]}) \qquad (i = 0, \cdots, r-1).$$

Then $[r-1]c_0^{(0)} = [r-1]c^{(0)} \in Z_k(L_{0\lambda_0}/G_{\lambda_0};\ \mathfrak{C}_p)$, and it follows that

$$\pi_{10}[r-1]c_0^{(0)} = \partial[r-1]a_0$$

for some $a_0 \in C_{k+1}(L_{0\lambda_1}/G_{\lambda_1};\ \mathfrak{C}_{[r]})$. Let

$$c_1^{(1)} = \pi_{10}c_0^{(0)} - \partial a_0 + \pi_{10}c_1^{(0)},$$

$$c_i^{(1)} = \pi_{10}c_i^{(0)} \qquad (i = 2, \cdots, r-1),$$

$$c^{(1)} = \pi_{10}c^{(0)} - \partial a_0.$$

Then

$$c^{(1)} = c_1^{(1)} + \cdots + c_{r-1}^{(1)},$$

$$c^{(1)} - \pi_{10}c^{(0)} \in \partial D_{k+1}(K_{\lambda_1}),$$

and

$$c_i^{(1)} \in C_k(L_{i\lambda_1}/G_{\lambda_1};\ \mathfrak{C}_{[r-i]}) \qquad (i = 1, \cdots, r-1).$$

Repeating this process, we can construct, for every $j = 1, \cdots, r$,

$$c^{(j)} = c_j^{(j)} + \cdots + c_{r-1}^{(j)} \in D_k(K_{\lambda_j})$$

such that

$$c^{(j)} - \pi_{j0}\, c^{(0)} \in \partial D_{k+1}(K_{\lambda_j}),$$

$$c_i^{(j)} \in C_k(L_{i\lambda_j}/G_{\lambda_j};\ C_{[r-i]}) \quad (i = j, \cdots, r-1).$$

Since $c^{(r)} = 0$, it follows that

$$\pi_{\mu\nu}\, c^{(0)} \in \partial D_{k+1}(K_\mu).$$

Hence the homomorphism of $I_k(K_\nu)$ into $I_k(K_\mu)$ induced by a projection

$$\pi_{\mu\nu}\colon K_\nu/G_\nu \to K_\mu/G_\mu$$

is trivial. This completes the proof that $I_k(X) = 0$ for all $k > n$.

From this result the rest of our lemma is a direct consequence of (4.12) and (4.16).

(4.18) LEMMA. *Assume that* X *is of homology dimension* $\leq n$. *Then for* $k > n$, $\bar{H}_k^\rho(X) = H_k^\rho(X) = 0$ *and hence* $\omega_*\colon H_n^\rho(X) \to H_n(X)$ *is an isomorphism into.*

Proof. By (3.6), X/G is of homology dimension $\leq n$, and hence $H_k(X\ G) = 0$ for $k > n$. Since the fixed point set of $T^{[r-1]}$ is a closed subset of X and is consequently of homology dimension $\leq n$ by (3.3), it follows from (4.17) and (4.10) that for $k > n+1$, $\bar{H}_k^\tau(X) = H_k^\tau(X) = 0$. Making use of (4.9) and the fact that for $k > n$, $H_k(X)$ and $\bar{H}_{k+1}^\tau(X)$ are both trivial, we infer that for $k > n$, $H_k^\sigma(X) = 0$. Hence, by (4.17), $\bar{H}_k^\sigma(X) = 0$ for $k > n$. Similarly $\bar{H}_{n+1}^\tau(X) = H_{n+1}^\tau(X) = 0$. Since $\bar{H}_{n+1}^\rho(X) = 0$, it follows from (4.9) that $\omega_*\colon H_n^\rho(X) \to H_n(X)$ is an isomorphism into.

5. p-ADIC TRANSFORMATION GROUPS

Let X be a compact Hausdorff space, and let G be a p-adic group acting as a topological transformation group on X, where p is an arbitrary prime number.

Let $G = G_0 \supset G_1 \supset \cdots$ be the sequence of open subgroups of G such that whenever $j \geq i$, G_i/G_j is a cyclic group of order $[j-i]$ $(= p^{j-i})$. Let

$$h_{ij}\colon G/G_j \to G/G_i, \qquad h_i\colon G \to G/G_i$$

be homomorphisms induced by the identity homomorphism of G. Then $\{G/G_i;\ h_{ij}\}$ is an inverse system, and $\{h_i\}$ gives an isomorphism of G onto the limit-group $\varprojlim G/G_i$.

Similarly we let

$$\pi_{ij}\colon X/G_j \to X/G_i, \qquad \pi_i\colon X \to X/G_i$$

be maps induced by the identity map of X. Then $\{X/G_i;\ \pi_{ij}\}$ is an inverse system, and $\{\pi_i\}$ gives a homeomorphism of X onto the limit-space $\varprojlim X/G_i$.

Let T be an element of G not in G_1, and for every nonnegative integer i, let T_i be the coset TG_i in G/G_i. Then T_i is a periodic map of X/G_i, with $T_i^{[i]}$ being the identity map. Hence we can apply all results of the last section to X/G_i with respect to T_i. Notice that the replacement of T by another element of $G - G_1$ only results in a replacement of T_i by one of the generators of the group G/G_i.

Since $X = \varprojlim X/G_i$, $H_k(X) = \varprojlim H_k(X/G_i)$. With $H_k^\rho(X/G_i)$ and $\bar{H}_k^\rho(X/G_i)$ in place of $H_k(X/G_i)$, we define

$$H_k^\rho(X) = \varprojlim H_k^\rho(X/G_i), \qquad \bar{H}_k^\rho(X) = \varprojlim \bar{H}_k^\rho(X/G_i),$$

and we call them *special* homology groups of X with respect to the p-adic group G. Notice that since π_{ij} does not induce a homomorphism of $I_k(X/G_j)$ into $I_k(X/G_i)$, we are not able to define a group $I_k(X)$ with respect to G.

Let ω_{i*}, ι_{i*}, τ_{i*}, σ_{i*}, κ_{i*}, π_{i*}, θ_{i*} be the analogues of the homomorphisms ω_*, ι_*, τ_*, σ_*, κ_*, π_*, θ_* in Section 4 for X/G_i with respect to T_i. Since

$$\omega_{i*}\pi_{ij*} = \pi_{ij*}\omega_{j*},$$

$\{\omega_{i*}\}$ gives a homomorphism

$$\omega_*: H_k^\rho(X) \to H_k(X).$$

Similarly, we have homomorphisms

$$\tau_*: H_k(X) \to \bar{H}_k^\sigma(X), \qquad \iota_*: \bar{H}_k^\rho(X) \to H_k^\rho(X).$$

Since $\sigma_{i*}\pi_{ij*} \neq \pi_{ij*}\sigma_{j*}$, $\{\sigma_{j*}\}$ does not give a homomorphism of $H_k(X)$ into $\bar{H}_k^\tau(X)$. Also, none of $\{\kappa_{i*}\}$, $\{\pi_{i*}\}$, $\{\theta_{i*}\}$ gives a homomorphism.

From (4.9) and (4.10) we can easily prove

(5.1) LEMMA. *The sequence*

$$\cdots \leftarrow H_{k-1}^\tau(X) \leftarrow \bar{H}_k^\sigma(X) \overset{\tau_*}{\leftarrow} H_k(X) \overset{\omega_*}{\leftarrow} H_k^\tau(X) \leftarrow \cdots$$

is exact, where $\bar{H}_k^\sigma(X) \to H_{k-1}^\tau(X)$ *is the appropriate boundary homomorphism.*

(5.2) LEMMA. *The homomorphism* $\iota_*: \bar{H}_k^\tau(X) \to H_k^\tau(X)$ *is an isomorphism onto.*

(5.1) establishes just one case of (4.9) for the p-adic transformation group G. Since a corresponding homomorphism σ_* is not defined for the p-adic group G, we can not have the second case of (4.9) here. However, we are able to prove a weaker exact sequence in (5.4) below.

(5.3) LEMMA. *If for every nonnegative integer* i *the stationary point set of* G_i *is of homology dimension* $\leq n$, *then whenever* $k > n$, $I_k(X/G_i) = 0$ *for all* i. *Hence*

$$\iota_*: \bar{H}_k^\sigma(X) \to H_k^\sigma(X) \quad \textit{and} \quad \kappa_{i*}: H_{k+1}(X/G) \to \bar{H}_{k+1}^\tau(X/G_i) \quad (i = 0, 1, \cdots),$$

are isomorphisms onto when $k > n$, *and isomorphisms into when* $k = n$.

Proof. The projection, $\pi_i: X \to X/G_i$ maps the stationary point set of G_{i-1} homeomorphically onto the fixed point set of $T_i^{[r-1]}$. Hence our result follows from (4.17).

(5.4) COROLLARY. *If for every nonnegative integer* i *the stationary point set of* G_i *is of homology dimension* $\leq n$, *then the sequence*

$$H_{n+1}(X/G) \xleftarrow{\pi_*} H_{n+1}(X) \xleftarrow{\omega_*\iota_*} \bar{\bar{H}}^{\sigma}_{n+1}(X) \longleftarrow H_{n+2}(X/G) \longleftarrow \cdots$$

is exact, where $H_{k+1}(X/G) \to \bar{\bar{H}}^{\sigma}_k(X)$ *is the appropriate boundary homomorphism.*

Proof. By (4.9), there exists, for every nonnegative integer i, an exact sequence

$$\bar{\bar{H}}^{\tau}_{n+1}(X/G_i) \xleftarrow{\sigma_{i*}} H_{n+1}(X/G_i) \xleftarrow{\omega_{i*}} H^{\sigma}_{n+1}(X/G_i) \longleftarrow \bar{\bar{H}}^{\tau}_{n+2}(X/G_i) \longleftarrow \cdots .$$

Since the stationary point set of G_{i-1} is of homology dimension $\leq$ n, it follows from (5.3) and (4.14) that the sequence

$$H_{n+1}(X/G) \xleftarrow{\pi_{0i*}} H_{n+1}(X/G_i) \xleftarrow{\omega_{i*}\iota_{i*}} \bar{\bar{H}}^{\sigma}_{n+1}(X/G_i) \longleftarrow H_{n+2}(X/G) \longleftarrow \cdots$$

is exact, where $H_{k+1}(X/G) \to \bar{\bar{H}}^{\sigma}_k(X/G_i)$ is the composition

$$H_{k+1}(X/G) \xrightarrow{\kappa_{i*}} \bar{\bar{H}}^{\tau}_{k+1}(X/G_i) \longrightarrow H^{\sigma}_k(X/G_i) \xrightarrow{\iota^{-1}_{i*}} \bar{\bar{H}}^{\sigma}_k(X/G_i) .$$

Hence the limit-sequence, namely our desired sequence, is exact.

6. MAIN THEOREMS

Making use of (4.15) and the fact that every compact totally disconnected abelian group may be regarded as the limit-group of an inverse system of finite abelian groups, one can easily show

(6.1) PROPOSITION. *Let* G *be a compact totally disconnected abelian group acting on a compact Hausdorff space* X, *and let* π *be the projection of* X *onto the orbit space* X/G. *Then the induced homomorphism* π_*: $H_k(X) \to H_k(X/G)$ *maps the identity component of* $H_k(X)$ *onto that of* $H_k(X/G)$. *If moreover* G *is isomorphic to the limit-group of an inverse system of finite abelian groups whose orders are powers of a fixed prime number* p, *then so is* $H_k(X/G)/\pi_* H_k(X)$, *with the same* p.

A compact additive abelian group A is called *elementary* if its identity component A^0 is a finite-dimensional toral group and the quotient group A/A^0 is finite.

(6.2) PROPOSITION. *Let* G *be a compact totally disconnected abelian group acting on a compact Hausdorff space* X. *If* $H_k(X)$ *is elementary, then there exists an open subgroup* H *of* G *such that whenever* G' *is an open subgroup of* G *contained in* H, *the projection of* X *onto* X/G' *induces an isomorphism of* $H_k(X)$ *into* $H_k(X/G')$.

(6.3) COROLLARY. *Let* G *be a* p*-adic group acting on a compact Hausdorff space* X. *If* $H_k(X) = 0$, *then* $H_k(X/G)$ *is isomorphic to the limit-group of an inverse system of finite abelian groups whose orders are powers of* p.

(6.4) THEOREM. *Let* X *be a compact Hausdorff space, and let* G *be a* p*-adic group acting as topological transformation group on* X. *Let* n *be an integer* $(n \geq 0)$ *such that the stationary point set of every open subgroup of* G *is of homology dimension* $\leq n - 1$ *and such that* $H_n(X)$, $H_{n+1}(X)$, *and* $H_{n+2}(X)$ *are elementary. Let* d *be the dimension of* $H_n(X)$, *and let* F *be the maximal subgroup of* $H_{n+1}(X)/H_{n+1}(X)^0$ *with its order being a power of* p, *where* $H_{n+1}(X)^0$ *is the identity component of* $H_{n+1}(X)$. *Let* G' *be an open subgroup of* G *such that the projection* π: $X \to X/G'$ *induces an isomorphism* π_* *of* $H_k(X)$ *into* $H_k(X/G')$ *for* $k = n, n+1, n+2$, *and such that every element of* G' *induces the identity homomorphism of* $H_k(X)$ *into itself*

($k = n, n+1$). *Then there exists an exact sequence*

$$0 \leftarrow K \leftarrow H_{n+2}(X/G')/\pi_* H_{n+2}(X) \leftarrow F \leftarrow 0,$$

where K *is a group having a subgroup isomorphic to* G^d. *If, moreover, the stationary point set of every open subgroup of* G *is of homology dimension less than* $\max(0, n-1)$, *then* K *is isomorphic to* G^d.

Proof. Let us first observe the existence of an open subgroup G' of G satisfying our hypothesis. By (6.2), there exists an open subgroup H of G such that for every open subgroup G of H the projection of X onto X/G' induces an isomorphism of $H_k(X)$ into $H_k(X/G')$, for $k = n, n+1, n+2$. Let α be a finite open covering of X such that for $k = n, n+1$, the projection of $H_k(X)$ into $H_k(K_\alpha)$, where K_α is the nerve of α, is an isomorphism into. Then there exists an open subgroup G' of H such that α is refined by a finite open covering β of X which has the property that every $V \in \beta$ is G'-invariant. This open subgroup G' of G satisfies our hypothesis. Notice that if G satisfies our hypothesis, so does every open subgroup of G'.

For the sake of convenience, we assume that $G' = G$.

By (5.4), there exists an exact sequence

$$H_{n+1}(X/G) \overset{\pi_*}{\leftarrow} H_{n+1}(X) \leftarrow \overline{H}^\sigma_{n+1}(X) \leftarrow H_{n+2}(X/G) \overset{\pi_*}{\leftarrow} H_{n+2}(X).$$

Since, by assumption, $\pi_*: H_k(X) \to H_k(X/G)$ is an isomorphism into for $k = n+1$, $n+2$, the sequence

$$0 \leftarrow \overline{H}^\sigma_{n+1}(X) \leftarrow H_{n+2}(X/G) \overset{\pi_*}{\leftarrow} H_{n+2}(X) \leftarrow 0$$

is exact. Hence there exists an isomorphism of $\overline{H}^\sigma_{n+1}(X)$ onto $H_{n+2}(X/G)/\pi_* H_{n+2}(X)$.

The symbols G_i, π_i, and so forth used below are the same as in the preceding section.

Since the stationary point set of every G_i is of homology dimension $\leq n-1$, it follows from (5.3) that

$$\kappa_{i*}: H_{n+1}(X/G) \to \overline{H}^\tau_{n+1}(X/G_i)$$

is an isomorphism onto. If e is an element of $H_{n+1}(X)$ which for every positive integer i is divisible by [i], then there exists a sequence $e_0, e_1, \cdots$ in $H_{n+1}(X)$ such that $e_0 = e$ and such that $e_i = [j-i]e_j$, for every $i \leq j$. Let

$$e_i' = \kappa_{i*}\pi_* e_i \in \overline{H}^\tau_{n+1}(X/G_i) \qquad (i = 0, i, \cdots).$$

Then

$$\begin{aligned} \pi_{ij*} e_j' &= \pi_{ij*}\kappa_{j*}\pi_* e_j = \pi_{ij*}\sigma_{j*}\pi_{j*} e_j \\ &= [j-i]\sigma_{i*}\pi_{ij*}\pi_{j*} e_j = \sigma_{i*}\pi_{i*} e_i = e_i'. \end{aligned} \tag{4.14}$$

Hence $e' = \{e_i'\}$ is an element of $\overline{H}^\tau_{n+1}(X)$ with $\omega_* \iota_* e' = e$. Conversely, it can be seen that every element of $\omega_* \iota_* \overline{H}^\tau_{n+1}(X)$ is divisible by [i], for every positive integer i. Hence $H_{n+1}(X)/\omega_* H^\tau_{n+1}(X)$ is isomorphic to the maximal subgroup F of $H_{n+1}(X)/H_{n+1}(X)^0$ with its order being a power of p.

Let K be the kernel of $\omega_*: H_n^\tau(X) \to H_n(X)$. By (5.1), the sequence

$$0 \leftarrow K \leftarrow \bar{H}_{n+1}^\sigma(X) \leftarrow H_{n+1}(X) \leftarrow H_{n+1}^\tau(X)$$

is exact. Using the result just proved, we obtain an exact sequence

$$0 \leftarrow K \leftarrow \bar{H}_{n+1}^\sigma(X) \leftarrow F \leftarrow 0.$$

Hence the theorem is proved if we are able to show that there exists an isomorphism of G^d into K or an isomorphism of G^d onto K, according as every open subgroup of G has a stationary point set of dimension $\leq n - 1$ or $< \max(n - 1, 0)$.

Let E be the limit-group of the inverse system $\{E_i; f_{ij}\}$ indexed by nonnegative integers, where $E_i = H_n(X)$ for all i, and where, for all $j \geq i$, f_{ij} maps every element e into $[j - i]e$.

Since

$$\pi_{ij*}\sigma_{j*}\pi_{j*}e = [j - i]\sigma_{i*}\pi_{ij*}\pi_{j*}e = \sigma_{i*}\pi_{i*}f_{ij}e,$$

it follows that $\phi = \{\sigma_{i*}\pi_{i*}\}$ is a homomorphism of E into $\bar{H}_n^\tau(X)$. By hypothesis, the stationary point set of G_{i-1} is of homology dimension $\leq n - 1$. We infer from (5.3) that $\kappa_{i*}: H_n(X/G) \to \bar{H}_n^\tau(X/G_i)$ is an isomorphism into. Since

$$\pi_*: H_n(X) \to H_n(X/G)$$

is assumed to be an isomorphism into, it follows from (4.14) that $\sigma_{i*}\pi_{i*} = \kappa_{i*}\pi_*$ is an isomorphism into. Hence ϕ is one-to-one.

It is easily seen that the homomorphism $\omega_*\iota_*\phi: E \to H_n(X)$ maps every element $\{e_i\}$ into e_0, and that the kernel of $\omega_*\iota_*\phi$ is isomorphic to G^d. Hence the kernel K of ω_* has a subgroup isomorphic to G^d, because ϕ is an isomorphism into and ι_* is an isomorphism onto by (5.3).

Suppose that the stationary point set of every open subgroup of G is of homology dimension $< \max(0, n - 1)$. Then for every i, κ_{i*} is an isomorphism onto. Therefore we can show that ϕ is an isomorphism onto. Hence K is isomorphic to G^d.

Let X be a locally compact Hausdorff space, and let A and B be closed subsets of X with $A \supset B$. Let $X \cup \infty$ be the one-point-compactification of X. Then $H_k(A, B)$ denotes the k^{th} Čech homology group of the compact pair $(A \cup \infty, B \cup \infty)$.

(6.5) THEOREM. *Let* X *be a locally compact Hausdorff space of homology dimension* $\leq n$, *and let* G *be a* p*-adic group acting as a topological transformation group on* X. *Then the homology dimension of the orbit space* X/G *is at most* n + 3. *If, moreover, the stationary point set of every open subgroup of* G *is of homology dimension* $\leq n - 1$, *then* X/G *is of homology dimension* $\leq n + 2$. *If, furthermore,* $H_n(M, N)$ *is an elementary group of dimension* > 0 *for some closed subsets* M *and* N *of* X, *with* $M = G(M) \supset N = G(N)$, *then the homology dimension of* X/G *is exactly equal to* n + 2.

Proof. The first part means that for each compact pair (M^*, N^*) with M^* contained in X/G, $H_{n+4}(M^*, N^*) = 0$. Let π be the projection of X onto X/G. Let Y be the one-point-compactification of $\pi^{-1}(M^*) - \pi^{-1}(N^*)$. Then the action of G on $\pi^{-1}(M^*)$ defines an action of G on Y, and Y/G may be regarded as the one-point-compactification of $M^* - N^*$. Hence $H_k(Y/G)$ is isomorphic to $H_k(M^*, N^*)$ for $k > 0$.

Using (3.3), we can easily see that Y is of homology dimension $\leq$ n. It follows from the last part of (6.4) that $H_{n+4}(Y/G) = 0$. Hence the first part of (6.5) is proved.

The second part of (6.5) can be proved by the same argument. In order to prove the last part, we let Y be the one-point-compactification of M - N. Then the action of G on M defines an action of G on Y, and $H_n(Y)$ is isomorphic to $H_n(M, N)$, which by hypothesis is an elementary group of dimension $d > 0$. By the first part of (6.4), $H_{n+2}(Y/G')$ contains a subgroup isomorphic to G^d, for some open subgroup G' of G. Hence $\text{hd } X/G \geq \text{hd } Y/G = \text{hd } Y/G' \geq n + 2$.

(6.6) COROLLARY. *If* G *is a* p-*adic group acting freely on an* n-*dimensional manifold* X, *then the orbit space* X/G *is of dimension either* n + 2 or ∞.

Proof. Let y be a point of X, and let U be a neighborhood of y homeomorphic to euclidean n-space. Then there exists an open subgroup G" of G such that $G''y \subset U$. It is easily seen that $V = \bigcap \{gU \mid g \in G''\}$ is open in X. Therefore the component X' of V containing y is an orientable n-dimensional manifold. Let G' be an open subgroup of G' with $G'y \subset X'$. Then G' is a p-adic group acting freely on X'. By (6.5), X'/G' is of homology dimension n + 2. Hence our assertion follows if we apply results of Section 3.

As a consequence of (3.5) and (6.6), we have

(6.7) COROLLARY. *If* G *is a* p-*adic solenoid group acting freely on an* n-*dimensional manifold* X, *then the orbit space* X/G *is of dimension either* n + 1 *or* ∞.

7. p-ADIC TRANSFORMATION GROUPS ON A HOMOLOGY MANIFOLD

The purpose of this section is to improve results (6.6) and (6.7).

A locally compact Hausdorff space X is said to have the *property* $P^n(\mathfrak{P})$ at a point x if there exists a neighborhood U of x satisfying the following conditions:

(1) U^- is compact and $H_n(X, X - U) \approx \mathfrak{P}$.

(2) Whenever $y \in U$ and V is a neighborhood of y, there exists a neighborhood W of y, contained in $U \cap V$, such that the homomorphism of $H_k(X, X - U)$ into H (X, X - W) induced by the inclusion map is an isomorphism onto for k = n and is trivial for $k \neq n$.

A locally compact Hausdorff space is said to have the *property* $P^n(\mathfrak{P})$ if it has the property $P^n(\mathfrak{P})$ at each of its points. By a *homology* n-*manifold* we mean a connected locally compact Hausdorff space of finite homology dimension having the property $P^n(\mathfrak{P})$.

In [3] homology manifolds are defined by using dimension instead of homology dimension. However all results proved in [3] hold for homology manifolds in the present sense. Hence we have

(7.1) Let M, N be closed subsets of a homology n-manifold with $M \supset N$. Then $H_n(M, N) \neq 0$ if and only if M - N contains a non-null open set.

(7.2) Let X be a homology n-manifold, and let A be a closed subset of X. A is of homology dimension n if and only if the interior of A is not null. A is of homology dimension n - 1 if and only if A is nowhere dense and there exists a neighborhood U of a point x of A such that, whenever V is a neighborhood of x contained in U, V - A is not connected.

(7.3) If X is a homology n-manifold, $H_n(X)$ is isomorphic to $\mathfrak{P}$ or to $\mathfrak{C}_2$.

A homology n-manifold X is called *orientable* if $H_n(X)$ is isomorphic to $\mathfrak{P}$. A homeomorphism of a homology n-manifold X onto itself is said to *preserve the orientation* of X if it induces the identity homomorphism of $H_n(X)$.

(7.4) Let T be a periodic transformation on a homology n-manifold X, and let F be the fixed point set of T. Then F is of homology dimension $\leq n - 1$. If X is orientable and T preserves the orientation of X, then F is of homology dimension $\leq n - 2$.

Let X be a homology n-manifold, and let G be a p-adic group acting *effectively* on X. Let G_i be the open subgroup of G with G/G_i of order $[i]$, let F_i be the stationary point set of G_i, and let Q_i be the interior of F_i. Notice that every F_i is a proper closed subset of X.

(7.5) LEMMA. $Q_i - Q_{i-1}$ *is open.*

Proof. If our assertion is false, then there exists a point $y \in Q_i \cap (\bar{Q}_{i-1} - Q_{i-1})$. Let Y be the component of Q_i containing y, and let T be an element of $G_{i-1} - G_i$. Then Y is a homology n-manifold, and T is a periodic transformation on Y leaving every point of $Y \cap Q_{i-1}$ fixed. By (7.4), T must be the identity transformation on Y, so that $Y \subset Q_{i-1}$, contrary to our assumption that $y \in \bar{Q}_{i-1} - Q_{i-1}$.

Now suppose that X is orientable and that every element of G preserves the orientation of X. Let M be a G-invariant closed subset of X, let $E_i = F_i \cap M$, and let U_i be the interior of E_i. It follows from (7.5) that $V_i = U_i - U_{i-1}$ is open.

(7.6) LEMMA. $V_{i+1} \cap F_i$ *is of homology dimension* $\leq n - 2$, *and for every component* C *of* V_{i+1}, $\bar{C} - C$ *is not contained in* F_i.

Proof. The first part is a direct consequence of (7.4).

Suppose that C is a component of V_{i+1} with $\bar{C} - C$ contained in F_i. Since $C \cap F_i$ is of homology dimension $\leq n - 2$, it follows from (7.3) that $G_i(C - F_i)/G_i$ is connected. Let I be the natural image of $H_n(X)$ in $H_n(G_i\bar{C}, G_i(\bar{C} - C))$. Since X is orientable and every element of G_i preserves the orientation of X, we infer that I is isomorphic to $\mathfrak{P}$ and that the homomorphism induced by the projection of $G_i\bar{C}$ onto $G_i\bar{C}/G_i$ maps I onto $H_n(G_i\bar{C}/G_i, G_i(\bar{C} - C)/G_i)$, with its kernel intersecting I at a cyclic group of order p. By assumption, $\bar{C} - C$ is contained in F_i, so that the projection of $G_i(\bar{C} - C)$ onto $G_i(\bar{C} - C)/G_i$ is a homeomorphism onto. It follows that the boundary homomorphism $H_n(G_i\bar{C}, G_i(\bar{C} - C)) \to H_{n-1}(G_i(\bar{C} - C))$ is not one-one. Hence $H_n(G_i\bar{C}) \neq 0$, contrary to (7.1).

(7.7) LEMMA. $I_n(M/G_i) = 0$ *for all integer* $i \geq 0$.

Proof. From definition we can easily see that the inclusion map of E_{i+1} into M induces an isomorphism of $I_n(E_{i+1}/G_i)$ onto $I_n(M/G_i)$. Hence we have only to show that $I_n(E_{i+1}/G) = 0$.

By (3.6), E_{i+1}/G_i is of homology dimension $\leq n$. It follows from (4.18) that $H_{n+1}(E_{i+1}/G_i) = 0$ so that, by (4.16), $\theta_{i*}: I_n(E_{i+1}/G_i) \to H_n(E_{i+1}/G)$ is an isomorphism into. Hence it is sufficient to show that $H_n(E_{i+1}/G) = 0$.

Suppose that $H_n(E_{i+1}/G) \neq 0$. Then for some integer j $(0 \leq j \leq i)$,

$$H_n(E_{j+1}/G, E_j/G) \neq 0.$$

Since V_{j+1} is the interior of $E_{j+1} - E_j$, it follows that

$$H_n(\overline{V}_{j+1}/G, (\overline{V}_{j+1} \cap F_j)/G \neq 0.$$

Since G/G_{j+1} acts freely on $V_{j+1} - F_j$, it follows that $H_n(\overline{V}_{j+1}, \overline{V}_{j+1} \cap F_j) \neq 0$. Hence, by (7.1), there exists a component C of V_{j+1} such that $\overline{C} - C$ is contained in F_j, contrary to (7.6).

Because of (7.7), we can strengthen (5.3), (5.4) and (6.4). In fact, we have

(7.8) LEMMA. *Whenever* $k \geq n$, $I_k(M/G_i) = 0$ *for all* i. *Hence*

$$\iota_*: \overline{H}_k^\sigma(M) \to H_k^\sigma(M) \quad \textit{and} \quad \kappa_{i*}: H_{k+1}(M/G) \to H_{k+1}(M/G_i) \quad (i = 0, 1, \cdots)$$

are isomorphisms onto when $k \geq n$, *and isomorphisms into when* $k = n - 1$.

(7.9) LEMMA. *There exists an exact sequence*

$$H_n(M/G) \xleftarrow{\pi_*} H_n(M) \xleftarrow{\omega_* \iota_*} H_n(M) \longleftarrow H_{n+1}(M/G) \longleftarrow \cdots.$$

(7.10) LEMMA. (i) $H_{n+3}(M/G) = 0$.

(ii) *If the projection of* X *into* X/G *induces an isomorphism of* $H_n(X)$ *into* $H_n(X/G)$, *then* $H_{n+2}(X/G)$ *contains a subgroup isomorphic to* G.

(7.11) THEOREM. *If* G *is a* p-*adic group acting effectively on a homology* n-*manifold* X, *then the orbit space* X/G *is of homology dimension* n + 2.

Proof. If F_i is null for all $i \geq 0$, we let y be any point of X. Then G acts freely on G(y). If there exists an $F_j \neq \emptyset$, we take a point $x \in F_j - Q_j$. It follows from (7.4) that no F_i contains x as an interior point. Hence there is a point y not contained in any F_i. Again, G acts freely on G(y).

As in the proof of (6.6), there exists an orientable connected neighborhood X' of y which is invariant under an open subgroup G' of G. It is clear that X' is an orientable homology n-manifold and that G' acts effectively on X'. If we take G' so small that every element of G' preserves the orientation of X' and that the projection of X' onto X'/G' induces an isomorphism of $H_n(X')$ into $H_n(X'/G')$, it follows from (7.10), (ii) that X'/G' is of homology dimension $\geq n + 2$. Hence, by results of Section 3, X/G is of homology dimension $\geq n + 2$. Using (7.10), (i), we can easily see that X/G is of homology dimension $\leq n + 2$. Hence our theorem is proved.

(7.11) COROLLARY. *If* G *is a* p-*adic solenoid group acting effectively on a homology* n-*manifold* X, *then the orbit space* X/G *is of homology dimension* n + 1.

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