ON SEQUENCES OF SUBORDINATE FUNCTIONS

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Let f(z) and g(z) be two functions regular in the disk $|z| < 1$. If there exists a function $\phi(z)$ that is regular in $|z| < 1$ and satisfies $|\phi(z)| < 1$ and $\phi(0) = 0$, such that

$$g(z) = f(\phi(z))$$

in $|z| < 1$, then g(z) is called *subordinate* to f(z) (see for instance [2, p. 163]). The condition implies that $g(0) = f(0)$ and $|g'(0)| \leq |f'(0)|$. The relation of subordination is transitive. We shall prove the following theorems:

THEOREM 1. *Let the functions* $f_n(z)$ *be regular in* $|z| < 1$, *let* $\alpha_n = f_n'(0)$ *be positive, and let* $f_n(z)$ *be subordinate to* $f_{n+1}(z)$. *Then the condition*

$$\alpha = \lim_{n \to \infty} \alpha_n < \infty$$

is necessary and sufficient in order that $\{f_n(z)\}$ *converges uniformly in* $|z| \leq r$ *for every* $r < 1$.

THEOREM 2. *Let the functions* $f_n(z)$ *be regular in* $|z| < 1$, *let* $\alpha_n = f_n'(0)$ *be positive, and let* $f_{n+1}(z)$ *be subordinate to* $f_n(z)$. *Then the sequence* $\{f_n(z)\}$ *converges uniformly in* $|z| \leq r$ *for every* $r < 1$. *The limit function is constant if and only if*

$$\alpha = \lim_{n \to \infty} \alpha_n = 0.$$

Remarks. 1. Note the difference in the assumptions: In Theorem 1 we assume that $f_n(z)$ is subordinate to $f_{n+1}(z)$, whereas in Theorem 2 we assume the reverse relationship.

2. In Theorem 1 we have $\alpha_{n+1} \geq \alpha_n$. Therefore either the limit α exists or $\alpha_n \to \infty$. In Theorem 2 we have $\alpha_{n+1} \leq \alpha_n$. Hence the limit α always exists and is nonnegative.

3. In Theorem 2, it is of course essential to assume that $f_n'(0)$ is real and nonnegative, as the example $f_n(z) = (-1)^n z$ shows.

4. Theorem 1 implies that if its hypothesis is satisfied and if $|f_n(z)| \leq K$ in some neighborhood of $z = 0$, then $|f_n(z)| \leq M(r)$ in $|z| \leq r$ for every $r < 1$. The functions $f_n(z) = e^{nz}$ give an example for Theorem 1 with $\alpha_n = n \to \infty$.

We need two lemmas. We denote by A the class of all functions $\phi(z)$ that are regular in $|z| < 1$ and satisfy the conditions $|\phi(z)| < 1$ and $\phi(0) = 0$.

LEMMA 1. *Every function* $\zeta = \phi(z)$ *of class* A *with* $\phi'(0) \geq \sigma > 0$ *maps the disk*

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$$|z| < \rho = \frac{\sigma}{1 + \sqrt{1 - \sigma^2}}$$

one-to-one onto a region that contains the disk $|\zeta| < \rho^2$.

Proof. For the case where $\sigma = 1$, the lemma is trivial. If $0 < \sigma < 1$, then by [3, p. 167],

$$|\phi(\rho e^{i\theta})| \geq \rho \frac{\phi'(0) - \rho}{1 - \phi'(0)\rho} \geq \rho \frac{\sigma - \rho}{1 - \sigma\rho} = \rho^2.$$

Since $\phi(z)$ is univalent in $|z| < \rho$ [3, p. 171], we get the desired result.

LEMMA 2. *To every* $\delta > 0$ *and* $r < 1$, *there corresponds an* $\eta > 0$ *such that, for every function* $\phi(z)$ *of class* A *with* $\phi'(0) \geq 1 - \eta$, *we have*

$$|\phi(z) - z| \leq \delta$$

in $|z| \leq r$.

Proof (see [1, p. 20]). Let

$$\phi(z) = \sum_{n=1}^{\infty} b_n z^n.$$

Since $|\phi(z)| < 1$ in the unit disk, $\sum_{n=1}^{\infty} |b_n|^2 \leq 1$. For $b_1 = \phi'(0) \geq 1 - \eta$, we have

$$\sum_{n=2}^{\infty} |b_n|^2 \leq 1 - |b_1|^2 \leq 1 - (1 - \eta)^2 \leq 2\eta.$$

From Schwarz's inequality we obtain

$$\begin{aligned} |\phi(z) - z|^2 &= \left|(b_1 - 1)z + \sum_{n=2}^{\infty} b_n z^n\right|^2 \\ &\leq \left((1 - b_1)^2 + \sum_{n=2}^{\infty} |b_n|^2\right) \sum_{n=1}^{\infty} r^{2n} \\ &\leq (\eta^2 + 2\eta)\frac{r^2}{1 - r^2}. \end{aligned}$$

and we can make this expression less than or equal to δ^2 by choosing $\eta > 0$ small enough.

Proof of Theorem 1. (a) If $\{f_n(z)\}$ converges uniformly in $|z| \leq 1/2$, we see immediately that $\{\alpha_n\}$ is bounded. Since the sequence $\{\alpha_n\}$ is increasing, it converges to a finite limit.

Now we assume that $\alpha_n \to \alpha$ with $0 < \alpha < \infty$ (if $\alpha = 0$, we have $\alpha_n \equiv 0$, contrary to the hypothesis of the theorem). Since $f_m(z)$ is subordinate to $f_n(z)$ for $n \geq m$, there exist functions $\phi_{mn}(z)$ of class A such that

$$f_m(z) = f_n(\phi_{mn}(z)) \qquad (n \geq m). \tag{1}$$

If r is given $(0 < r < 1)$, let σ be obtained from the equation

$$\sqrt{r} = \frac{\sigma}{1 + \sqrt{1 - \sigma^2}}. \tag{2}$$

Since $\alpha_n \to \alpha$ and $0 < \alpha < \infty$, we can find an integer m such that

$$\alpha_m / \alpha_n \geq \sigma$$

for $n \geq m$. From equation (1) we obtain

$$\alpha_m = f_m'(0) = f_n'(0)\,\phi_{mn}'(0) = \alpha_n\, \phi_{mn}'(0),$$

and therefore

$$\phi_{mn}'(0) = \alpha_m/\alpha_n \geq \sigma.$$

Next, applying Lemma 1 and equation (2), we see that the function $\zeta = \phi_{mn}(z)$ maps $|z| < \sqrt{r}$ onto a region that contains the disk $|\zeta| < r$. Let $z = \psi_{mn}(\zeta)$ be the inverse function to $\zeta = \phi_{mn}(z)$. Then $\psi_{mn}(\zeta)$ is regular in $|\zeta| < r$ and satisfies the inequality $|\psi_{mn}(\zeta)| \leq \sqrt{r}$. Therefore we get from equation (1) that for $n \geq m$

$$f_n(\zeta) = f_m(\psi_{mn}(\zeta))$$

in $|\zeta| < r$. This implies that

$$\max_{|\zeta| \leq r} |f_n(\zeta)| \leq \max_{|z| \leq \sqrt{r}} |f_m(z)|$$

for all $n \geq m = m(r)$. Hence there exists an $M(r)$ such that, for all n,

$$|f_n(\zeta)| \leq M(r) \tag{3}$$

in $|z| \leq r$, and the sequence $\{f_n(z)\}$ is normal in $|z| < 1$.

(b) Therefore there exists a subsequence $\{f_{n_\nu}(z)\}$ that converges uniformly in $|z| \leq R$ for every $R < 1$. Let $f(z)$ be the limit function. Let $\varepsilon > 0$ and $r < 1$ be given. We choose a ν_0 such that

$$|f_{n_\nu}(z) - f(z)| < \varepsilon/2 \tag{4}$$

for $\nu \geq \nu_0$ and $|z| \leq r$. Because of inequality (3), the sequence $\{f_n'(z)\}$ is collectively bounded, also, and we can find a positive number δ such that

$$|f_n(z') - f_n(z'')| < \varepsilon/2 \tag{5}$$

for $|z' - z''| \leq \delta$, $|z'| \leq r + \delta$, $|z''| \leq r + \delta$, and all n.

Applying Lemma 2 to the function $\phi_{kn}(z)$ with $k \leq n$, we see that

$$(6) \qquad |\phi_{kn}(z) - z| \leq \delta$$

in $|z| \leq r$ if

$$\phi'_{kn}(0) = \alpha_k/\alpha_n \geq 1 - \eta,$$

for a certain $\eta > 0$. This inequality is valid if $n \geq k \geq N$, for a suitably chosen integer N.

Now we can complete the proof. If $|z| \leq r$ and $k \geq N$, we choose some $n_\nu \geq k$ with $\nu \geq \nu_0$. Then (6) is true with $n = n_\nu$. Because

$$f_k(z) = f_{n_\nu}(\phi_{kn_\nu}(z)),$$

we thus deduce from (5) that

$$|f_k(z) - f_{n_\nu}(z)| < \varepsilon/2.$$

Combining this inequality with (4), we finally get

$$|f(z) - f_k(z)| < \varepsilon$$

for $|z| \leq r$, $k \geq N$.

Proof of Theorem 2. (a) Since $f_{n+1}(z)$ is subordinate to $f_n(z)$ and hence to $f_1(z)$, we have $f_n(0) = f_1(0)$ for every n. Therefore we can assume that $f_n(0) = 0$ for all n. The maximum

$$M_n(r) = \max_{|z| \leq r} |f_n(z)|$$

satisfies [2, p. 164]

$$(7) \qquad M_{n+1}(r) \leq M_n(r).$$

Therefore $M_n(r) \leq M_1(r)$, and the sequence $\{f_n(z)\}$ is normal. For $\alpha_n \to \alpha > 0$, the proof of the first part of Theorem 2 is similar to part (b) of the proof of Theorem 1. The limit function f(z) is not constant, because

$$f'(0) = \lim_{n\to\infty} f'_n(0) = \lim_{n\to\infty} \alpha_n = \alpha \neq 0.$$

(b) Let $\alpha_n \to 0$. Then there exists an increasing sequence $\{n_\nu\}$ such that

$$q_\nu = \alpha_{n_{\nu+1}}/\alpha_{n_\nu} \to 0.$$

Since $f_{n_{\nu+1}}(z)$ is subordinate to $f_{n_\nu}(z)$, we have

$$(8) \qquad f_{n_{\nu+1}}(z) = f_{n_\nu}(\chi_\nu(z)),$$

with functions $\chi_\nu(z)$ of class A. We have

$$\alpha_{n_{\nu+1}} = f'_{n_{\nu+1}}(0) = f'_{n_\nu}(0)\chi'_\nu(0) = \alpha_{n_\nu}\chi'_\nu(0)$$

and

$$\chi'_\nu(0) = q_\nu \to 0\,.$$

If $r < 1$ is given, we choose a μ such that

$$q_\nu \leq \sqrt{r} - r$$

for $\nu \geq \mu$. Then we have, for $|z| \leq r$ [3, p. 167],

$$|\chi_\nu(z)| \leq r\,\frac{q_\nu + r}{1 + q_\nu r} \leq r(\sqrt{r} - r + r) = r^{3/2}\,.$$

From equation (8) we deduce that

$$\max_{|z| \leq r} |f_{n_{\nu+1}}(z)| \leq \max_{|\zeta| \leq r^{3/2}} |f_{n_\nu}(\zeta)|\,.$$

Therefore we have, for $\nu \geq \mu$,

$$M_{n_\nu}(r) \leq M_{n_\mu}\left(r^{(3/2)^{\nu-\mu}}\right)$$

and

$$\limsup_{\nu \to \infty} M_{n_\nu}(r) \leq M_{n_\mu}(0) = |f_{n_\mu}(0)| = 0\,.$$

Using inequality (7), we see that $M_n(r) \to 0$ for every $r < 1$, so that $f_n(z) \to 0$ uniformly in each circle $|z| \leq r < 1$.

REFERENCES

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