THE STRUCTURE OF A LATTICE-ORDERED GROUP WITH A FINITE NUMBER OF DISJOINT ELEMENTS

Paul Conrad

1. DEFINITIONS AND STATEMENT OF THE MAIN THEOREM

Throughout this paper, the terminology and results of Chapter XIV of Birkhoff's book [1] will be used. $L = L(+, \cap, \cup, <)$ will always denote an l-group. An element a in L will be called *positive* if $a \geq 0$, and *strictly positive* if $a > 0$.

Definition 1. The elements $a_1, \cdots, a_n$ of L are *disjoint* if they are strictly positive and $a_i \cap a_j = 0$ for all $i \neq j$. Clearly, L is a linearly ordered group (notation: o-group) if and only if it does not contain a pair of disjoint elements.

Definition 2. L is a *cardinal sum* of l-ideals $A_1, \cdots, A_n$ (notation: $L = A_1 \boxplus \cdots \boxplus A_n$) if L is the direct sum of the A_i (notation: $L = A_1 \oplus \cdots \oplus A_n$) and if for $a_i \in A_i$, $a_1 + \cdots + a_n \geq 0$ if and only if $a_i \geq 0$ for $i = 1, \cdots, n$.

Definition 3. L is a *lexico-extension* of an l-group S (notation: $L = \langle S \rangle$) if S is an l-ideal of L, L/S is an o-group, and every positive element in L but not in S exceeds every element in S. In particular, $L = \langle L \rangle$.

Let S be an l-ideal of L. It is easy to verify that $L = \langle S \rangle$ if and only if each nonzero coset of L/S consists entirely of positive elements or of negative elements. If $S = 0$, then $L = \langle S \rangle$ if and only if L is an o-group.

LEMMA 1.1. *If* $S \neq 0$, *then* $L = \langle S \rangle$ *if and only if each* a $(0 < a \in L \setminus S)$ *exceeds every element in* S.

Proof. Suppose that each a $(0 < a \in L \setminus S)$ exceeds every element in S, and assume (by way of contradiction) that L/S is not an o-group. Then there exist strictly positive elements X and Y in L/S for which $X \cap Y = S$. Let $X = x + S$ and $Y = y + S$, where $0 < x \in L \setminus S$ and $0 < y \in L \setminus S$; then $S = X \cap Y = x \cap y + S$. Thus $x \cap y \in S$, and since $S \neq 0$ there exists an element z in S such that $z > x \cap y$. But x and y exceed z and therefore $x \cap y \geq z$, a contradiction.

Let $A_1, A_2, \cdots, A_n$ be o-groups. Then, by a finite alternating sequence of cardinal summations and lexico-extensions, we can construct l-groups from the A_i, in which each A_i is used exactly once to make a cardinal extension, and in which the lexico-extensions are arbitrary. We shall call such groups *lexico-sums of the* A_i. For example, if $n = 3$, then there are two ways of constructing lexico-sums of A_1, A_2, and A_3 in this order, namely $\langle A_1 \boxplus \langle A_2 \boxplus A_3 \rangle\rangle$ and $\langle\langle A_1 \boxplus A_2 \rangle \boxplus A_3 \rangle$. Let S be a lexico-sum of $A_1, \cdots, A_n$, and pick one strictly positive element a_i in each A_i. Then for $n \geq 2$ the following two propositions are easily verified.

I. $a_1, \cdots, a_n$ are disjoint, and S does not contain $n + 1$ disjoint elements.

II. $S = \langle A \boxplus B \rangle$, where A is a lexico-sum of $A_1, \cdots, A_s$, and where B is a lexico-sum of $A_{s+1}, \cdots, A_n$ for a suitable ordering of the subscripts.

The following is the main theorem proved in this paper.

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THEOREM 1. *Let* L *be an* l-*group which contains* n *disjoint elements* $a_1, \cdots a_n$ *but does not contain* n + 1 *such elements. Let* A_i *be the subgroup of* L *generated by* $\{ x \in L: x \cap a_j = 0$ *for all* $j \neq i\}$. *Then the* A_i *are* o-*groups, and* L *is a lexico-sum of the* A_i.

COROLLARY I. *An* l-*group* L *is a lexico-sum of* n *ordered subgroups if and only if* L *contains* n *disjoint elements but does not contain* n + 1 *such elements.*

The corollary is an immediate consequence of the theorem and Proposition I. In [2], this theorem is proved for the case where n = 2; if n = 1, then L itself is an o-group and the theorem is trivially true.

2. PRELIMINARY LEMMAS

If S is a subset of a group G, then [S] will always denote the subgroup of G that is generated by S.

LEMMA 2.1. *Let* A *be a subsemigroup of a group* G. *Then the following are equivalent.*

a) $[A] = \{x - y: x, y \in A\}$.

b) If $a, b \in A$, then there exist x, y in A such that $a + x = b + y$. That is, every pair of elements in A has a common right multiple.

This is a rather trivial corollary of a result of Ore [5], but since it is used in some of the key steps in the proof of our theorem, we shall give a proof. Let $B = \{x - y: x, y \in A\}$, and first suppose that $[A] = B$. If $a, b \in A$, then $-a + b \in [A]$, and hence $-a + b = x - y$ for some x, y in A. Thus $a + x = b + y$.

Conversely, suppose that (b) is satisfied. If $a \in A$, then $a = 2a - a \in B$. Thus $[A] \supseteq B \supseteq A$, and it suffices to show that B is a group. Clearly, $0 \in B$ and B is closed with respect to inverses. Consider $a - b$ and $c - d$ in B. Pick $x, y \in A$ such that $b + x = c + y$. Then $x - y = -b + c$, and

$$a - b + c - d = a + x - y - d = a + x - (d + y) \in B.$$

Thus B is a group.

A subset S of L is *convex* if

(i) $a < x < b$ and $a, b \in S$ imply that $x \in S$, and

(ii) $a \cup 0 \in S$ for all $a \in S$.

Clearly the intersection of convex subsets is a convex subset. Also note that a set of positive elements is convex if and only if (i) is satisfied.

LEMMA 2.2. *Let* S *be a normal subgroup of* L. *Then the following are equivalent.*

(a) S *is an* l-*ideal.*

(b) S *is convex.*

Birkhoff proved that (a) implies (b) [1, p. 222], and the converse is an immediate consequence of his Theorem 2 on page 215.

LEMMA 2.3. *If* A *is a convex subsemigroup of positive elements of* L *that contains* 0, *then* $[A] = \{x - y: x, y \in A\}$, $[A]$ *is convex, and* A *is the semigroup of all positive elements of* $[A]$.

Proof. Consider $x, y \in A$. Because A is convex, $x \cap y \in A$. Also,

$$x = x \cap y + x' \geq x' \geq 0, \qquad y = x \cap y + y' \geq y' \geq 0$$

and $x' \cap y' = 0$. Thus $x', y' \in A$ and

$$x' + y' = x' \cup y' = y' \cup x' = y' + x'.$$

Furthermore,

$$x + y' = x \cap y + x' + y' = x \cap y + y' + x' = y + x'.$$

Thus by Lemma 2.1, $[A] = \{x - y: x, y \in A\}$. If $a_1 - a_2 < x < b_1 - b_2$, where $a_i, b_i \in A$, then $0 < a_1 < x + a_2 < b_1 - b_2 + a_2 < b_1 + a_2$. Thus $x + a_2 \in A$, and hence

$$x = x + a_2 - a_2 \in [A].$$

Next consider $a = a_1 - a_2$ in $[A]$, where $a_i \in A$. Since $a_1 + a_2 \geq a_1 - a_2$ and $a_1 + a_2 \geq 0$, $0 \leq a \cup 0 \leq a_1 + a_2 \in A$, and hence $a \cup 0 \in A$. Therefore $[A]$ is convex. In particular, if $0 \leq a \in [A]$, then $a \cup 0 = a \in A$.

Let $L_1, \cdots, L_n$ be subsemigroups of L, and let A be the subgroup of L that is generated by the L_i. Then $A = L_1 \oplus \cdots \oplus L_n$ will mean that $A = L_1 + \cdots + L_n$, that $L_i \cap (L_1 + \cdots + L_{i-1} + L_{i+1} + \cdots + L_n) = 0$ for all i, and that $a_i + a_j = a_j + a_i$ for all $a_i \in A_i$ and $a_j \in A_j$ provided that $i \neq j$.

THEOREM 2. *Let* $L_1, \cdots, L_n$ *be convex subsemigroups of positive elements of* L *such that* $L_i \cap L_j = 0$ *for all* $i \neq j$, *and let* A *be the subsemigroup of* L *that is generated by the* L_i.

a) $A = L_1 \oplus \cdots \oplus L_n$; *and if* $x = x_1 + \cdots + x_n$ *for* $x_i \in L_i$, *then* $x = x_1 \cup \cdots \cup x_n$, *and this representation is unique.*

b) $[A] = \{a - b: a, b \in A\}$ *and* $[L_i] = \{x - y: x, y \in L_i\}$ *for* $i = 1, \cdots, n$. $[A]$ *is convex, and* A *is the convex subsemigroup of all positive elements of* $[A]$.

c) $[A] = [L_1] \boxplus \cdots \boxplus [L_n]$.

Proof. If $x \in L_i$ and $y \in L_j$, where $i \neq j$, then $x \cap y \in L_i \cap L_j = 0$ because L_i and L_j are convex. Thus $x \cap y = 0$, and hence $x + y = x \cup y = y \cup x = y + x$. It follows that $A = L_1 + \cdots + L_n$. To complete the proof of (a), we use induction on n. Let

$$x \in L_i \cap (L_1 + \cdots + L_{i-1} + L_{i+1} + \cdots + L_n).$$

Then, by induction,

$$x = x_i = x_1 \cup \cdots \cup x_{i-1} \cup x_{i+1} \cup \cdots \cup x_n,$$

where the $x_j \in L_j$. But then

$$0 = x_i \cap x_j = (x_1 \cap x_j) \cup \ldots \cup (x_n \cap x_j) = x_j$$

for all $j \neq i$. Thus $x = 0$ and $A = L_1 \oplus \cdots \oplus L_n$. If $x = x_1 + \cdots + x_n$, where the $x_i \in L_i$, then by induction $x = (x_1 \cup \cdots \cup x_{n-1}) + x_n$. But

$$(x_1 \cup \cdots \cup x_{n-1}) \cap x_n = (x_1 \cap x_n) \cup \cdots \cup (x_{n-1} \cap x_n) = 0,$$

and therefore $x = x_1 \cup \cdots \cup x_n$. If $x = x_1 \cup \cdots \cup x_n = y_1 \cup \cdots \cup y_n$, where $y_i, x_i \in L_i$, then $y_i = x \cap y_i = x_i \cap y_i = y_i \cap x_i = x \cap x_i = x_i$. Therefore the representation is unique.

To prove (b), it suffices by Lemma 2.3 to show that A is convex. If

$$0 < x < b \in A,$$

then $0 < x < b = b_1 \cup \cdots \cup b_n$, where the $b_i \in L_i$. Thus

$$x = x \cap (b_1 \cup \cdots \cup b_n) = (x \cap b_1) \cup \cdots \cup (x \cap b_n) \in L_1 + \cdots + L_n.$$

Therefore A is convex.

Finally assume that (c) is true for all $m < n$, where $n \geq 2$, and let

$$B = L_2 \oplus \cdots \oplus L_n.$$

Then, by induction, $[B] = [L_2] \boxplus \cdots \boxplus [L_n]$, and B and $[B]$ are convex. By Lemma 2.3, $[L_1]$ is convex and hence $[L_1] \cap [B]$ is convex. Thus either $[L_1] \cap [B] = 0$ or $[L_1] \cap [B]$ contains a strictly positive element z. In the latter case

$$z = a_1 - a_2 = b_1 - b_2,$$

where $a_i \in L_1$ and $b_i \in B$. Thus $a_1 = z + a_2 \geq z > 0$ and $b_1 = z + b_2 \geq z > 0$. Since L_1 and B are convex, $z \in L_1 \cap B$, a contradiction. Therefore $[L_1] \cap [B] = 0$. Next consider $x \in [L_1]$ and $y \in [B]$. Here $x = x_1 - x_2$ and $y = y_1 - y_2$, where $x_i \in L_1$ and $y_i \in B$. Since the x_i and the y_i commute, $x + y = y + x$ and hence $[A] = [L_1] \oplus [B]$. If $0 \leq x + y$, then $0 \leq x + y \leq |x| + |y|$, and by [1, p. 245, Lemma 3], $x + y = u + v$, where $0 \leq u \leq |x|$ and $0 \leq v \leq |y|$. Since $[L_1]$ and $[B]$ are convex, $u \in [L_1]$ and $v \in [B]$. Therefore $x = u \geq 0$ and $y = v \geq 0$, and hence

$$[A] = [L_1] \boxplus [B] = [L_1] \boxplus \cdots \boxplus [L_n].$$

COROLLARY. *If* $A_1, \cdots, A_n$ *are convex subgroups of* L *and if the subgroup* G *of* L *that is generated by the* A_i *is* $\sum_{i=1}^{n} \oplus A_i$, *then* $G = \sum_{i=1}^{n} \boxplus A_i$.

Proof. Let $L_i = \{x \in A_i : x \geq 0\}$. Then the L_i are convex subsemigroups of positive elements, and $L_i \cap L_j = 0$ if $i \neq j$. Moreover, $[L_i] = A_i$. Thus, by (c), $G = \sum_{i=1}^{n} \boxplus A_i$.

Remark. Let A be the semigroup described in Theorem 2, and suppose that $[A]$ contains m disjoint elements, but does not contain $m + 1$ such elements. Then $m = m_1 + \cdots + m_n$, where L_i contains m_i disjoint elements but not $m_i + 1$ such elements. If $a_1, \cdots, a_n$ are disjoint elements of $[A]$, then m_i of the a_j belong to L_i for $i = 1, \cdots, n$.

3. PROOF OF THEOREM 1

We assume that $n \geq 2$. The proof consists of thirteen steps.

(1) $L_i = \{x \in L: x \cap a_j = 0$ *for all* $j \neq i\}$ *is a linearly ordered convex subsemigroup of* L. $[L_i] = \{x - y: x, y \in L_i\}$ *is a convex* o*-subgroup, and*

$$L_i = \{x \in [L_i]: x \geq 0\}.$$

For if $0 \leq x \leq a \in L_i$, then $0 \leq x \cap a_j \leq a \cap a_j = 0$ for all $j \neq i$. Thus $x \in L_i$, and hence L_i is convex. L_i is the intersection of the $n - 1$ semigroups

$$\{x \in L: x \cap a_j = 0\}$$

for all $j \neq i$ (see [1, p. 219] for a proof that these are semigroups). Thus L_i is a semigroup. Let x and y be nonzero elements in L_i. Then $x \cap y > 0$, for otherwise x, y, a_j for all $j \neq i$ are $n + 1$ disjoint elements. Therefore L_i contains no disjoint elements. Proposition (1) now follows from Lemma 2.3.

If $x \in L_i \cap L_j$, where $i \neq j$, then $x \cap a_i = 0$ for all i. Thus $x = 0$, for otherwise x and the a_i are $n + 1$ disjoint elements of L. Therefore $L_i \cap L_j = 0$ for all $i \neq j$, and by Theorem 2 we have:

(2) *The subsemigroup* A *of* L *that is generated by all the* L_i *is* $L_1 \oplus \cdots \oplus L_n$. $[A] = [L_1] \boxplus \cdots \boxplus [L_n] = \{a - b: a, b \in A\}$, $[A]$ *is convex, and* A *is the convex subsemigroup of all positive elements of* $[A]$. A similar statement holds for each partial sum of the L_i.

(3) *Pick* $0 < b_i \in L_i$ *for* $i = 1, \cdots, n$ *and define* $H_i = \{x \in L: x \cap b_j = 0$ *for all* $j \neq i\}$. *Then* $L_i = H_i$ *for* $i = 1, \cdots, n$. For if $x \in L_i$, then since $b_j \in L_j$, $x \cap b_j = 0$ for all $j \neq i$. Thus $x \in H_i$ and hence $L_i \subseteq H_i$. In particular, $0 < a_i \in H_i$. Thus, by reversing the argument, we have $H_i \subseteq L_i$.

Note that for any $x \in L$, $x \cap a_i = 0$ if and only if $x \cap b_i = 0$. This is a consequence of the fact that L_i is linearly ordered and convex.

(4) $B_i = \{x \in L: x \cap a_i = 0\}$ *is a convex subsemigroup of* L. *The subsemigroup* B *of* L *that is generated by* L_i *and* B_i *is* $L_i \oplus B_i$. $[B] = [L_i] \boxplus [B_i]$, $[B]$ *is convex, and* $B = \{x \in [B]: x \geq 0\}$. *For each* i, $[B_i]$ *contains the* $n - 1$ *disjoint elements* $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n$ *but does not contain* n *such elements. Each* B_i *is independent of the particular choice of the* a_j *in the* A_j. Clearly, B_i is convex, and if $x \in L_i \cap B_i$, then $x \cap a_j = 0$ for all j. Therefore $x = 0$ and $L_i \cap B_i = 0$. Thus all statements except the last follow from Theorem 2. The last statement follows from the remark after (3).

Let $N = 1, \cdots, n$, and let S be a nonvoid subset of N that contains s elements. Let $G_S = \{x \in L^+: x \cap a_i = 0$ for all $i \in N \setminus S\}$. It is easy to show that G_S is a convex subsemigroup of positive elements of L that contains $\Sigma_{i \in S} \oplus L_i$, and that G_S is independent of the particular choice of the a_i in the L_i. Moreover, G_S contains s disjoint elements but not $s + 1$ such elements.

(5) *If* $0 < a \in G_S \setminus \Sigma_{i \in S} \oplus L_i$, *then there exist* $i, j \in S$ *such that* $a > L_i \oplus L_j$. *In particular, if* $S = N$, *then* $G_S = L^+$, *and thus, if* $0 < a \in L \setminus A$, *then there exist* $i, j \in N$ *such that* $a > L_i \oplus L_j$. We use induction on s. If $s = 1$, then G_S is one of the L_i, and the statement is trivial. Suppose that $s > 1$ and that (5) is satisfied by all subsets T of N that contain $s - 1$ elements. Consider $0 < a \in G_S \setminus \Sigma_{i \in S} \oplus L_i$. For each $i \in S$,

$$a = a \cap a_i + a', \qquad a_i = a \cap a_i + a_i^!, \qquad a_i^! \in L_i, \qquad a' \cap a_i^! = 0.$$

Case I. There exists an $i \in S$ such that $a_i^! > 0$. Here, $a' \cap a_i = 0$, for otherwise $a_i^!$ and $a' \cap a_i$ are positive elements in the o-group $[L_i]$ and hence

$$0 < a_i^! \cap a' \cap a_i = 0 \cap a_i = 0.$$

Since $0 \leq a' \leq a$, $0 \leq a' \cap a_j \leq a \cap a_j = 0$ for all $j \in N \setminus S$. Thus $a' \in G_S$. Let $T = S \setminus \{i\}$. Then T contains $s - 1$ elements and $a' \in G_T$. If

$$a' \in \sum_{i \in T} \oplus L_i \subseteq \sum_{i \in S} \oplus L_i,$$

then $a = a \cap a_i + a' \in \Sigma_{i \in S} \oplus L_i$, a contradiction. Thus

$$0 < a' \in G_T \setminus \sum_{i \in T} \oplus L_i,$$

and hence by induction there exist $h, k \in T \subseteq S$ such that $a > L_h \oplus L_k$.

Case II. $a_i^! = 0$ for all $i \in S$. Here $a_i < a$ for all $i \in S$, and hence $a > \Sigma_{i \in S} + a_i$. Therefore either $a > \Sigma_{i \in S} \oplus L_i$, or we can replace the a_i by b_i $(0 < b_i \in L_i)$ for all $i \in S$, and one of the resulting $b_i^!$ is strictly positive. Thus we have Case I again.

(6) *If* $b_1, \cdots, b_n$ *are disjoint elements of* L, *then they belong to* A. *Moreover*, $b_i \in L_i^!$ *for some permutation* $i \to i'$ *of* N. For if $i \neq j$, $b_k \cap a_i \neq 0$ and $b_k \cap a_j \neq 0$, then

$$b_1, \cdots, b_{k-1}, b_k \cap a_i, b_k \cap a_j, b_{k+1}, \cdots, b_n$$

are disjoint. Therefore $b_k \cap a_i \neq 0$ for at most one i, and hence $b_k \in L_i$ for some i. But since the L_i are linearly ordered, no two of the b_k can belong to the same L_i.

(7) $[A]$ *is invariant with respect to* o-*automorphisms of* L. *In particular*, $[A]$ *is a normal subgroup of* L. Let α be an o-automorphism of L. Then, since $[A] = \{a - b\colon a, b \in A\}$, it suffices to show that $A\alpha \subseteq A$. Now

$$a_i\alpha \cap a_j\alpha = (a_i \cap a_j)\alpha = 0\alpha = 0,$$

provided that $i \neq j$. Thus the $a_i\alpha$ are disjoint, and hence by (6) $0 < a_i\alpha \in L_i^!$ where $i \to i'$ is a permutation of N. Thus by (3)

$$A\alpha = (L_1 \oplus \cdots \oplus L_n)\alpha \subseteq L_{1'} \oplus \cdots \oplus L_{n'} = A.$$

Let $G = L/[A]$, with the natural order.

(8) G *is an* l-*group with fewer than* n *disjoint elements*. For suppose (by way of contradiction) that $X_1, \cdots, X_n$ are disjoint elements in G. Then $X_i = x_i + [A]$ $(0 < x_i \in L \setminus [A]$, and $x_i \cap x_j \in [A]$ for all $i \neq j)$. By (5), for each i there exist $h, k \in N$ such that $x_i > L_h \oplus L_k$. It follows that there exist $i, j, k \in N$ such that $x_i > L_k$, $x_j > L_k$ and $i \neq j$. Thus $x_i \cap x_j > L_k$, and hence, since

$$[A] = [L_1] \boxplus \cdots \boxplus [L_n], \quad x_i \cap x_j \notin [A],$$

a contradiction.

Now we use induction on the number of disjoint elements. We assume that Theorem 1 is true for all l-groups with fewer than n disjoint elements. Thus G is a lexico-sum of fewer than n o-groups. Let π be the natural o-homomorphism of L onto G.

Case I. G is an o-group. Here the $B_i\pi$ are convex semigroups of positive elements that contain 0, and thus they form a chain. Without loss of generality, we may assume that $B_1\pi \subseteq B_2\pi \subseteq \cdots \subseteq B_n\pi$. Let B be the semigroup generated by L_n and B_n. Then, by (4), $[B] = [L_n] \boxplus [B_n]$, [B] is a convex subgroup of L, and $B = L_n \oplus B_n$ is the convex semigroup of positive elements of [B]. Since B_n contains $a_1, \cdots, a_{n-1}$ and does not contain n disjoint elements, $[B_n]$ is a lexico-sum of $[L_1], \cdots, [L_{n-1}]$ and thus [B] is a lexico-sum of all the $[L_i]$. We now must show that $L = <[B]>$.

(9) [B] *is invariant with respect to* o-*automorphisms of* L. For let α be an o-automorphism of L. Then the mapping β of $[A] + a \in G$ upon $[A] + a\alpha$ is an o-automorphism of G. By the proof of (7), α induces a permutation of the L_i. Thus α induces a permutation of the B_i, and $B_i\pi\beta = B_i\alpha\pi = B_j\pi$. Thus β induces a permutation of the $B_i\pi$. Since $B_n\pi \supseteq B_i\pi$ for all $i \in N$, $B_n\pi\beta = B_n\pi$. Finally

$$[B]/[A] = [B]\pi = ([B_n] + [L_n])\pi = [B_n]\pi = [B_n]\pi\beta = [B_n]\alpha\pi = ([B_n]\alpha + [A])\pi$$

$$= ([B_n]\alpha + [A]\alpha)\pi = ([B_n] + [A])\alpha\pi = [B]\alpha\pi = [B]\alpha/[A].$$

Therefore $[B] = [B]\alpha$.

To show that $L = <[B]>$, it suffices by Lemma 1.1 to prove:

(10) *If* $0 < a \in L \setminus [B]$, *then* a *exceeds every element in* [B]. $a = a \cap a_i + a'$, and $a_i = a \cap a_i + a_i^!$, where $a' \cap a_i^! = 0$. If $a_i^! \neq 0$, then $a' \in B_i$ because $0 < a_i^! \in L_i$ and $a' \cap a_i^! = 0$. Thus $a'\pi = [A] + a' \in B_i\pi \subseteq B_n\pi$, and thus $a' = b + c$, where $b \in B_n$ and $c \in [A]$. Thus $a = a \cap a_i + a' \in [B]$, a contradiction. Therefore $a_i = a \cap a_i$, and hence $a > a_i$ for all $i \in N$. Thus, by (3), $a > c_i$ for all $c_i \in L_i$ and all $i \in N$. It follows that $a > [A]$. Now consider $b \in [B]$. Since $a\pi > b\pi$, $a > d + b$ for some $d \in [A]$. Also, $0 < a - b - d \in L \setminus [B]$. Therefore $a - b - d > - d$, and hence $a > b$.

We have shown that [B] is a lexico-sum of the $[L_i]$ and that L is a lexico-extension of [B]. Therefore L is a lexico-sum of the $[L_i]$. This completes the proof of Case I.

Case II. G is not an o-group. Here, since G is a lexico-sum of fewer than n o-groups, $G = <U \boxplus V>$, where U and V are lexico-sums of n_u and n_v o-groups, respectively, and where $n_u + n_v < n$ and $n_u \neq 0 \neq n_v$. Suppose first that there exists an $i \in N$ such that $[B_i]\pi \supseteq U \boxplus V$. If C is any convex subgroup of G, then either $C \subseteq U \boxplus V$ or $C \supseteq U \boxplus V$. Also, the convex subgroups between G and $U \boxplus V$ form a chain, because $G/(U \boxplus V)$ is an o-group. Thus, without loss of generality, we may assume that $[B_n]\pi \supseteq [B_i]\pi$ for all $i \in N$. Now, if we repeat the argument in Case I, it follows that $[B] = [L_n] \boxplus [B_n]$ is a lexico-sum of the $[L_i]$ and that L is a lexico-extension of [B]. Therefore L is a lexico-sum of the $[L_i]$.

Suppose that $[B_i]\pi \subset U \boxplus V$ *for all* $i \in N$. Let $\mathfrak{U} = U\pi^{-1}$ and let $\mathfrak{V} = V\pi^{-1}$. By (5), if $0 < a \in \mathfrak{U} \setminus [A]$, then there exist $i, j \in N$ such that $a > L_i \oplus L_j$. Let

$$N_u = \{i \in N\text{: there exists an } a \in \mathfrak{U} \setminus [A] \text{ such that } a > L_i\},$$

$$N_v = \{i \in N\text{: there exists an } a \in \mathfrak{V} \setminus [A] \text{ such that } a > L_i\}.$$

$N_u \neq \square \neq N_v$, because $U \neq [A] \neq V$. In fact, N_u and N_v both contain at least two elements.

(11) $N_u \cap N_v = \square$, *or* $N_u \cap N_v$ *is void.* For suppose that $i \in N_u \cap N_v$; then there exist $a, b \in L$ such that $0 < a \in \mathfrak{U} \setminus [A]$, $0 < b \in \mathfrak{V} \setminus [A]$, $a > L_i$ and $b > L_i$. Therefore $a\pi \in U$, $b\pi \in V$, and thus $a \cap b + [A] = a\pi \cap b\pi = [A]$. But this is impossible, because $a \cap b > L_i$.

Thus without loss of generality we may assume that $N = 1, \cdots, n$, $N_u = 1, \cdots, s$ and $N_v = t + 1, \cdots, n$, where $s < t + 1$. Let H be the subsemigroup of L that is generated by G_{N_u}, G_{N_v} and $K = \Sigma_{i=s+1}^{t} \oplus L_i$.

(12) $[H] = [G_{N_u}] \boxplus [K] \boxplus [G_{N_v}]$, $[H]$ *is convex, and* H *is the semigroup of all positive elements of* $[H]$. Since the semigroups G_{N_v}, K and G_{N_u} are convex, it suffices by Theorem 2 to show that they are pairwise disjoint. If $x \in G_{N_u} \cap G_{N_v}$, then $x \cap a_i = 0$ for all i, and hence $x = 0$. If $x \in G_{N_u} \cap K$, then $x = x_{s+1} \cup \cdots \cup x_t$, where $x_i \in L_i$, and $x \cap a_i = 0$ for $i = s + 1, \cdots, t$. Thus $0 = x \cap a_i = x_i \cap a_i$ for $i = s + 1, \cdots, t$. Therefore, since the L_i are ordered, the $x_i = 0$ and hence $x = 0$. Thus $G_{N_u} \cap K = 0$, and by a similar argument, $G_{N_v} \cap K = 0$.

Now $[H]$ is convex and contains $a_1, \cdots, a_n$, and hence it contains exactly n disjoint elements. It follows that G_{N_u}, K, G_{N_v} contain s, t - s, n - t disjoint elements, respectively. Thus by induction $[G_{N_u}]$, $[K]$ and $[G_{N_v}]$ are lexico-sums of the $[L_i]$ that they contain. Therefore $[H]$ is a lexico-sum of the $[L_i]$ for all $i \in N$. To complete the proof of Theorem 1, we need only show that $L = \langle [H] \rangle$.

(13) $[H]\pi = U \boxplus V$ *and* $[H] = (U \boxplus V)\pi^{-1}$. To prove that $[H]\pi \subseteq U \boxplus V$, it suffices to show that if $h \in G_{N_v}$, then $h\pi \in U \boxplus V$. By the definition of $\mathfrak{U}$, there exist $u_1, \cdots, u_s \in \mathfrak{U}$ such that $u_i > L_i$. Thus $L_1 \oplus \cdots \oplus L_s < u_1 + \cdots + u_s = u$. Now suppose (by way of contradiction) that $h\pi \in G \setminus U \boxplus V$. Then $h\pi > U \boxplus V$, and so $h + [A] = h\pi > u\pi = u + [A]$. But this means that there exist $y_i \in [L_i]$ such that $h + y_1 + \cdots + y_n > u > L_1 \oplus \cdots \oplus L_s$. Thus

$$L_1 \oplus \cdots \oplus L_s < h + y_{s+1} + \cdots + y_n = k \in G_{N_v} \oplus K.$$

Therefore $a_1 < k$ and $a_1 \cap k = 0$, a contradiction.

To prove that $[H]\pi \supseteq U \boxplus V$, it suffices to show that $[H]\pi$ contains the positive part of V (for then $[H]\pi$ contains the difference group V, and by symmetry $[H]\pi$ contains U). Consider the positive element $a + [A]$ in V. $0 < a \in \mathfrak{V} \setminus [A]$. Thus $a \not> L_i$ for $i = 1, \cdots, t$ and therefore for each $i = 1, \cdots, t$ there exists a b_i $(0 < b_i \in L_i)$ such that $a \cap b_i < b_i$. By (3) we may assume that $a \cap a_i < a_i$ for $i = 1, \cdots, t$. Now

$$a = a \cap a_i + a', \qquad a_i = a \cap a_i + a_i', \qquad 0 < a_i' \in L_i, \qquad a' \cap a_i' = 0, \qquad a' \neq 0,$$

for otherwise $a \leq a_i$ and hence $a \in L_i$. In particular, for $i = 1$, $a \equiv a' \bmod [A]$, $a' \cap a_1 = 0$ and $0 < a' \leq a$. Let $b_1 = a'$, and repeat the above process on b_1 and 2. We get a b_2 such that $a \equiv b_2 \bmod [A]$, $b_2 \cap a_2 = 0$ and $0 < b_2 \leq b_1 \leq a$, and thus $b_2 \cap a_1 = 0$. Continuing in this way, we get a b_t such that $a \equiv b_t \bmod [A]$ and $b_t \cap a_i = 0$ for $i = 1, \cdots, t$. Thus $b_t \in G_{N_v}$ and $b_t\pi = a + [A]$. Therefore $G_{N_v}\pi$ contains the positive elements in V. Finally, consider $d + [A] \in U \boxplus V$. Since $[H]\pi \supseteq U \boxplus V$, there exists an $h \in [H]$ such that $d + [A] = d\pi = h\pi = h + [A]$. Thus

$d = h + a$, where $a \in [A] \subseteq [H]$, and hence $d \in [H]$. Therefore $[H] = (U \boxplus V)\pi^{-1}$. This completes the proof of (13).

$[H]$ is normal in L, because $[H] = (U \boxplus V)\pi^{-1}$ and $U \boxplus V$ is normal in G. $L/[H]$ is an o-group, because it is isomorphic to $G/(U \boxplus V)$, which is an o-group. Since $[B_i]\pi \subseteq U \boxplus V$ and $[H] = (U \boxplus V)\pi^{-1}$, $[B_i] \subseteq [H]$ for all $i \in N$. By repeating verbatim the argument in (10) with $[B]$ replaced by $[H]$, we can show that if $0 < a \in L \setminus [H]$, then a exceeds every element in $[H]$. Therefore L is a lexico-extension of $[H]$, and hence L is a lexico-sum of the $[L_i]$. This completes the proof of Theorem 1.

As in Theorem 1, assume that L contains n disjoint elements $a_1, \cdots, a_n$ but not $n + 1$ such elements.

COROLLARY II. *If* $L/[A]$ *is an* o-*group, then*

$$L = < \cdots <<<[L_1] \boxplus [L_2]> \boxplus [L_3]> [L_4]> \boxplus \cdots \boxplus [L_n]>.$$

Proof. By the proof for Case I, $L = <[B_n] \boxplus [L_n]>$, where the lexico-extension may be trivial. $[B_n]/\Sigma_{i=1}^{n-1} \boxplus [L_i]$ is an o-group; thus

$$[B_n] = <[\{x \in B_n : x \cap a_j = 0 \text{ for all } j \neq n-1\}] \boxplus [L_{n-1}]>.$$

The corollary now follows by finite induction.

4. EXAMPLES OF LEXICO-EXTENSIONS

In this section, let $N = 0, a, b, c, \cdots$ be an l-group, and let $\Delta = \theta, \alpha, \beta, \gamma, \cdots$ be an o-group. There exists at least one lexico-extension of N by Δ, namely the direct sum $\Delta \oplus N$, where we define (α, a) to be positive if $\alpha > \theta$ or $\alpha = \theta$ and $a > 0$. In general, by the extension theory of Schreier (see [3] or [4]), any lexico-extension G of N by Δ has the following representation. $G = \Delta \times N$; (α, a) is positive if $\alpha > \theta$ or $\alpha = \theta$ and $a > 0$;

$$(\alpha, a) + (\beta, b) = (\alpha + \beta, f(\alpha, \beta) + ar(\beta) + b),$$

where r is a mapping of Δ into the group A(N) of all order-preserving automorphisms of N, f is a mapping of $\Delta \times \Delta$ into N, and r and f satisfy

(1) $[ar(\alpha)]r(\beta) = -f(\alpha, \beta) + ar(\alpha + \beta) + f(\alpha, \beta)$ and $ar(\theta) = a$,

(2) $f(\alpha, \theta) = f(\theta, \beta) = 0$,

(3) $f(\alpha, \beta + \gamma) + f(\beta, \gamma) = f(\alpha + \beta, \gamma) + f(\alpha, \beta)r(\gamma)$

for all $a \in N$ and all $\alpha, \beta, \gamma \in \Delta$.

If G splits over N, then we can choose f so that $f(\alpha, \beta) \equiv 0$. Thus G is determined by Δ, N, and a homomorphism r of Δ into A(N). If N is in the center of G, then we can choose r so that $ar(\alpha) \equiv a$. Thus G is determined by Δ, N (where N is abelian), and a mapping f of $\Delta \times \Delta$ into N such that (2) and

(3') $f(\alpha, \beta + \gamma) + f(\beta, \gamma) = f(\alpha + \beta, \gamma) + f(\alpha, \beta)$

are satisfied. Note that here order does not rear its ugly head—every central extension of N by Δ is a lexico-extension under the above ordering of G.

Example I. Let Δ be the group of integers. Let $N = U \boxplus V \boxplus D$, where D is an l-group and $U = V = \Delta$. For $\alpha, \beta \in \Delta$ and $(u, v, d) \in N$ we define $f(\alpha, \beta) \equiv 0$,

$$(u, v, d)\, r(\alpha) = \begin{cases} (u, v, d) & \text{if } \alpha \text{ is even,} \\ (v, u, d) & \text{if } \alpha \text{ is odd.} \end{cases}$$

Then r is a homomorphism of Δ into A(N), and so $G = \Delta \times N$ is a splitting lexico-extension of N by Δ.

Example II. Let F be an l-group, and let Δ be an o-group. For each $\delta \in \Delta$, let $F_\delta = F$. Let $N = \Sigma_{\delta \in \Delta} \boxplus F$ (the small or the large direct sum). For $(\cdots, f_\delta, \cdots) \in N$ and $\alpha, \beta \in \Delta$, we define $f(\alpha, \beta) \equiv 0$ and $(\cdots, f_\delta, \cdots) r(\alpha) = (\cdots, f_{\delta+\alpha}, \cdots)$. That is, the element in the $(\delta + \alpha)$th component is replaced by the element in the α'th component. Then r is a homomorphism of Δ into A(N), and again $G = \Delta \times N$ is a splitting lexico-extension of N by Δ.

Example III. Let $N = R \boxplus D$, where D is an abelian l-group and R is the additive group of rational numbers. Let $\Delta = R_1 \oplus \cdots \oplus R_n$, ordered lexico-graphically, where $R_i = R$ for $i = 1, \cdots, n$. Let A be an n-by-n rational matrix. For $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ in Δ, define $f(\alpha, \beta) = \alpha A \beta^t$, where β^t is the transpose of β. Then f satisfies (3') and hence $G = \Delta \times N$ is a central lexico-extension of N by Δ. N is a direct summand of G if and only if there exists a mapping g of Δ into N such that $g(\theta) = 0$ and $f(\alpha, \beta) = - g(\alpha + \beta) + g(\alpha) + g(\beta)$ for all α and β in Δ. Note that $- g(\alpha + \beta) + g(\alpha) + g(\beta)$ is a symmetric function. Thus if A is not symmetric, then N is not a direct summand.

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Tulane University