CONCAVE FUNCTIONS, REARRANGEMENTS, AND BANACH LATTICES

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1. INTRODUCTION

A theory of the representation of concave functions on an abstract Boolean ring in terms of rearrangements of measures is given in Sections 2 and 3. These results are used to give a Banach lattice characterization (Section 4) of certain spaces of measurable functions introduced by G. G. Lorentz [8]. This application uses previous results of the authors [3], as well as the results of D. Maharam [9], L. Loomis [7], and H. Freudenthal [4].

2. INTEGRATION

Let B be a non-atomic σ-Boolean ring. A *positive function* x is a family $x(\alpha)$ of elements of B, defined for all $\alpha \geq 0$, for which

$$x(\alpha) \supset x(\beta) \text{ for } \beta \geq \alpha, \text{ and} \tag{1}$$

$$\bigcup_{\beta>\alpha} x(\beta) = x(\alpha). \tag{2}$$

(Unions and intersections of nested families in B indexed by reals exist, since cofinal sequences exist.)

A (real) *function* y is the formal difference $y = y_+ - y_-$ of two positive functions which are disjoint. We reserve the phrase *function on* B for the more usual meaning of a function with domain B. (We assume that all the functions on B in this paper take on more than three values.) A function Φ on B is *positive* if $\Phi(e) \geq 0$ and $\Phi(0) = 0$ (we allow the possibility that $\Phi(e) = +\infty$); Φ is *strictly positive* if in addition $\Phi(e) = 0 \Rightarrow e = 0$; Φ is a *measure* if it is positive and countably additive.

Let μ be a given strictly positive measure on B which is locally finite ($\mu(e) = +\infty$ implies that there exists an f such that $0 \neq f \subset e$ and $\mu(f) < +\infty$.) It follows that μ is full-valued [5]. An *admissible measure* is a measure ϕ defined on B', the σ-finite elements of B, which is finite when μ is finite.

If x is a positive function, it determines a measure ϕ_x on B by

$$\phi_x(e) = \int_0^{+\infty} \mu(x(\alpha) \cap e)\, d\alpha . \tag{3}$$

THEOREM 1 (Radon-Nikodym). *If ϕ is an admissible measure, then there exists a unique positive function x_ϕ such that* (3) *holds for all* e *in* B' (*that is,* $\phi = \phi_{x_\phi}$) [2].

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LEMMA 1. *If* x *is a positive function, then* ϕ_x *is admissible if and only if there exists an* $\alpha_0 < +\infty$ *for which* $\int_{\alpha_0}^{+\infty} \mu(x(\alpha))\, d\alpha < +\infty$.

Proof. By (3), $\phi_x(e) \leq \alpha_0 \mu(e) + \int_{\alpha_0}^{+\infty} \mu(x(\alpha))\, d\alpha$, and therefore $\phi_x(e)$ is finite on elements of finite μ-measure if the integral is finite.

Conversely, suppose that ϕ_x is admissible. We show first that there exists an $\alpha_0 < +\infty$ for which $\mu(x(\alpha_0)) < +\infty$. If not, then $\mu(x(4^n)) = +\infty$ for all n, and there exists an $e_n \subset x(4^n)$ for which $\mu(e_n) = 1/2^n$, since μ is full-valued. Putting $e = \bigcup_n e_n$, we have $\mu(e) \leq 1$. But $\phi_x(e) \geq \alpha\mu(x(\alpha) \cap e)$ for all α, by (1) and (3). Thus $\phi_x(e) \geq 4^n \mu(x(4^n) \cap e) \geq 4^n \mu(e_n) = 2$ for all n. Thus $\phi_x(e) = +\infty$ and ϕ_x is not admissible. Thus we have an α_0 for which $\mu(x(\alpha_0)) < +\infty$. Then the fact that ϕ_x is admissible implies that

$$+\infty > \phi_x(x(\alpha_0)) = \int_0^{+\infty} \mu(x(\alpha) \cap x(\alpha_0))\, d\alpha = \alpha_0 \mu(x(\alpha_0)) + \int_{\alpha_0}^{+\infty} \mu(x(\alpha))\, d\alpha .$$

COROLLARY. *If* ϕ *is admissible, then* ϕ *is absolutely continuous.*

3. REARRANGEMENTS, CONCAVE FUNCTIONS

Two positive functions x and y are *covariant* if for every $\alpha, \beta \geq 0$ either $x(\alpha) \subset y(\beta)$ or $x(\alpha) \supset y(\beta)$. The two functions are *rearrangements of each other* (with respect to μ) if $\mu(x(\alpha)) = \mu(y(\alpha))$ for all $\alpha \geq 0$. Two admissible measures ϕ and ψ are rearrangements if x_ϕ and x_ψ are rearrangements.

LEMMA 2. *If* x *and* y *are positive functions, then a necessary and sufficient condition for the existence of a positive function* z *which is covariant with* x *and is a rearrangement of* y *is that*

$$\mu\Big(\bigcap_{\mu(x(\alpha))=+\infty} x(\alpha)\Big) \geq \mu\Big(\bigcup_{\mu(y(\beta))<+\infty} y(\beta)\Big). \tag{4}$$

(If $\mu(x(\alpha)) < +\infty$ for all α, $\bigcap_{\mu(x(\alpha))=+\infty} x(\alpha)$ is taken as $y(0) \cup x(0)$. With this definition, (4) is satisfied in this case, and as we show, the lemma holds.)

Proof. Let $F_x = \{\alpha \geq 0 \mid \mu(x(\alpha)) < +\infty\}$. Then (4) reads

$$\mu\Big(\bigcap_{\alpha \notin F_x} x(\alpha)\Big) \geq \mu\Big(\bigcup_{\beta \in F_y} y(\beta)\Big).$$

Suppose there exists a positive function z which is covariant with x and is a rearrangement of y. Then, for each pair $\alpha, \beta \geq 0$ for which $\alpha \notin F_x$ and $\beta \in F_y$, we have $\mu(y(\beta)) = \mu(z(\beta)) < \mu(x(\alpha))$, and therefore $z(\beta) \subset x(\alpha)$. Then

$$\Big(\bigcap_{\alpha \notin F_x} x(\alpha)\Big) \supset \Big(\bigcup_{\beta \in F_y} z(\beta)\Big)$$

and

$$\mu\Big(\bigcap_{\alpha \notin F_x} x(\alpha)\Big) \geq \mu\Big(\bigcup_{\beta \in F_y} z(\beta)\Big) = \sup_{\beta \in F_y} \mu(z(\beta)) = \sup_{\beta \in F_y} \mu(y(\beta)) = \mu\Big(\bigcup_{\beta \in F_y} y(\beta)\Big).$$

Conversely, there exists, by Zorn's lemma, a maximal nested family of elements $\{e_\nu\}$ such that

(a) for $\alpha \in F_x$, $x(\alpha)$ is an e_ν, (b) $\mu(e_\nu) < +\infty$, and (c) $e_\nu \subset \bigcap_{\alpha \notin F_x} x(\alpha)$.

A slightly stronger statement of the full-valuedness of μ (the proof is identical) asserts that for

$$0 \leq k < \mu\Big(\bigcap_{\alpha \notin F_x} x(\alpha)\Big)$$

there exists an e_ν such that $\mu(e_\nu) = k$. The required positive function z is then given by

$$z(\beta) = x(0) \cup y(0), \text{ for } \beta \text{ such that } \mu(y(\beta)) = +\infty;$$

$$z(\beta) = \bigcap_{\alpha \notin F_x} x(\alpha), \text{ for } \beta \text{ such that } \mu(y(\beta)) = \mu\Big(\bigcap_{\alpha \notin F_x} x(\alpha)\Big) < +\infty;$$

$$z(\beta) = e_\nu, \text{ where } \mu(e_\nu) = \mu(y(b)) < \mu\Big(\bigcap_{\alpha \notin F_x} x(\alpha)\Big).$$

These categories are distinct, and by (4) they exhaust the set of $\beta \geq 0$.

If ϕ is a measure on B, we define $\phi^* = \phi_\mu{}^*$ by

(5) $$\phi^*(e) = \sup_{\substack{f \in B' \\ \mu(f) = \mu(e)}} \phi(f).$$

LEMMA 3. *If ϕ is an admissible measure, then*

(6) $$\phi^*(e) = \int_0^{+\infty} \min\{\mu(x_\phi(\alpha)), \mu(e)\}\, d\alpha, \quad \textit{and}$$

(7) *for* $e \in B'$, $\phi^*(e) = \sup \psi(e)$, *where the supremum is taken over all measures* ψ *which are rearrangements of* ϕ.

Proof. Since $\mu(x_\phi(\alpha) \cap f) \leq \min\{\mu(x_\phi(\alpha)), \mu(f)\}$, (3) and (5) give immediately that

$$\phi^*(e) \leq \int_0^{+\infty} \min\{\mu(x_\phi(\alpha)), \mu(e)\}\, d\alpha.$$

If there were an $f \in B'$ comparable to all the $x_\phi(\alpha)$ such that $\mu(f) = \mu(e)$, then $\phi(f)$ would be given by (6) (since then $\mu(x_\phi(\alpha) \cap f) = \min\{\mu(x_\phi(\alpha)), \mu(f)\}$) and the reverse inequality would follow. That is, we require a

$$z(\alpha) = z_f(\alpha) = \begin{cases} 0 & (1 \leq \alpha), \\ f & (0 \leq \alpha < 1), \end{cases}$$

covariant with $x = x_\phi(\alpha)$, and a rearrangement of $y = y_e(\alpha)$. The existence of such a z requires that x and y satisfy (4). Now (4) is certainly satisfied if $\mu(x_\phi(0)) < +\infty$ or if $\mu(e) = +\infty$. (In the latter case, a slight modification of the construction in Lemma 2 ensures that $f \in B'$.) If $\mu(x_\phi(0)) = +\infty$ and $\mu(e) < +\infty$, the required z_f need not exist. However, in this case, let

$$\bar{\alpha} = \sup\{\alpha \geq 0 \mid \mu(x_\phi(\alpha)) = +\infty\}.$$

By Lemma 1, $\bar{\alpha} < +\infty$. Define x_n by

$$x_n(\alpha) = \begin{cases} x_\phi(\alpha) & \text{for } 0 \leq \alpha < \bar{\alpha} - \frac{\bar{\alpha}}{2n}, \\ x_\phi\left(\bar{\alpha} - \frac{\bar{\alpha}}{2n}\right) & \text{for } \bar{\alpha} - \frac{\bar{\alpha}}{2n} \leq \alpha < \bar{\alpha} + \frac{\bar{\alpha}}{2n}, \\ x_\phi(\alpha) & \text{for } \bar{\alpha} + \frac{\bar{\alpha}}{2n} \leq \alpha. \end{cases}$$

Then

$$\mu\Big(\bigcap_{\alpha \notin F_{x_n}} x_n(\alpha)\Big) = \mu\left(x_\phi\left(\bar{\alpha} - \frac{\bar{\alpha}}{2n}\right)\right) = +\infty;$$

(4) holds, and by Lemma 2 there exists a z_n covariant with x_n and a rearrangement of y_e. That is, there exists an $f_n \in B'$ which is comparable with the $x_n(\alpha)$ and such that $\mu(f_n) = \mu(e)$. Then

$$\phi^*(e) \geq \int_0^{+\infty} \mu(x_\phi(\alpha) \cap f_n)\, d\alpha$$

$$\geq \int_0^{+\infty} \mu(x_n(\alpha) \cap f_n)\, d\alpha - \int_{\bar{\alpha}-\frac{\bar{\alpha}}{2n}}^{\bar{\alpha}+\frac{\bar{\alpha}}{2n}} \mu(x_n(\alpha) \cap f_n)\, d\alpha$$

$$= \int_0^{+\infty} \min\{\mu(x_n(\alpha)), \mu(f_n)\}\, d\alpha - \frac{\bar{\alpha}}{n}\mu(f_n)$$

$$\geq \int_0^{+\infty} \min\{\mu(x_\phi(\alpha)), \mu(e)\}\, d\alpha - \frac{\bar{\alpha}}{n}\mu(e).$$

Since this is true for all n, (6) follows.

Now if ψ is a rearrangement of the admissible measure ϕ, (6) implies that $\phi^*(e) = \psi^*(e) \geq \psi(e)$, and we have one half of (7). Moreover, for $x = x_e$, for each

$e \in B'$, and for $y = x_\phi$, (4) is satisfied, and there exists a z which is covariant with x and is a rearrangement of $y = x_\phi$. Then ϕ_z is a rearrangement of ϕ and

$$\phi_z(e) = \int_0^{+\infty} \mu(z(\alpha) \cap e)\, d\alpha = \int_0^{+\infty} \min\{\mu(z(\alpha)),\ \mu(e)\}\, d\alpha = \phi^*(e),$$

and (7) is proved.

THEOREM 2. *Two admissible measures* ϕ *and* ψ *are rearrangements if and only if* $\phi^* = \psi^*$.

Proof. If $\phi^* = \psi^*$, then for all $u \in [0, \sup_{e \in B} \mu(e)]$ we have

$$\int_0^{+\infty} \min\{\mu(x_\phi(\alpha)),\ u\}\, d\alpha = \int_0^{+\infty} \min\{\mu(x_\psi(\alpha)),\ u\}\, d\alpha .$$

Integrating by parts, we get $\int_0^u \alpha_\phi(t)\, dt = \int_0^u \alpha_\psi(t)\, dt$ for all u, where α_ϕ and α_ψ are the inverse functions of $\mu(x_\phi(\alpha))$ and $\mu(x_\psi(\alpha))$ (defined except at a countable set of values of t). Then $\alpha_\phi(t) = \alpha_\psi(t)$ almost everywhere, and since the functions $\mu(x(\alpha))$ are continuous on the right, we have $\mu(x_\phi(\alpha)) = \mu(x_\psi(\alpha))$ for all α. The converse follows immediately from (7).

A characterization of those functions Φ on B which can be represented as a ϕ^* is obtained by exhibiting a certain Φ-dominated measure on B. The existence of such measures is discussed in the following theorem. Here B need not be a σ-ring, nor non-atomic, and the measures need only be finitely additive.

THEOREM 3. (α) *If* Φ *is a finite-valued positive function on* B *and* $\bar{\phi}$ *is a measure on a subring* $\bar{B}$ *of* B *for which* $\bar{\phi}(e) \leq \Phi(e)$ *for* e *in* $\bar{B}$, *then if* (i) Φ *is increasing* ($e_1 \subset e_2$ *implies* $\Phi(e_1) \leq \Phi(e_2)$), *and*

(8) Φ *is multiply subadditive* (*if* e_k ($k = 1, \cdots, n$) *cover* e p *times, then* $p\Phi(e) \leq \sum_{k=1}^n \Phi(e_k)$),

there exists an extension ϕ *of* $\bar{\phi}$ *to a measure on* B *such that* $\phi(e) \leq \Phi(e)$ *for all* e *in* B.

(β) *If* Φ *is a finite-valued positive function on* B *which is* (i) *increasing and* (ii) *concave* ($\Phi(e_1 \cup e_2) + \Phi(e_1 \cap e_2) \leq \Phi(e_1) + \Phi(e_2)$), *and* $\mathcal{F}$ *is a nested family in* B, *then there exists a measure* ϕ *on* $\bar{B}$ *such that* $\phi(e) = \Phi(e)$ *for all* e *in* $\mathcal{F}$ *and* $\phi(e) \leq \Phi(e)$ *for* e *in* B.

Proof. Let $S(B)$ be the set of formal sums $\Sigma_{k=1}^n a_k e_k$ (e_k in B, a_k real), with the natural relations and vector lattice operations. Then $S(B)$ is a vector lattice and B is imbedded in $S(B)$. In [3], the authors obtained the following results. If Φ is a positive function on B, then

$$P_1(x) = \inf \sum_{k=1}^n a_k \Phi(e_k)$$

for all a_k and e_k such that $|x| = \Sigma_{k=1}^n a_k e_k$ ($a_k \geq 0$) is a lattice semi-norm ($|x| \leq |y|$ implies that $P_1(x) \leq P_1(y)$) on $S(B)$ if and only if Φ is increasing and multiply subadditive. Moreover, P_1 is additive on positive covariant elements of

S(B) if and only if Φ is increasing and concave. (Thus (i), together with (ii), implies (8)).

We use the following easily verified version of the Hahn-Banach theorem. If V is a vector lattice with lattice semi-norm P; $\overline{V}$ a subvector lattice; $\overline{p}$ a positive linear functional on $\overline{V}$ such that $|\overline{p}(\overline{v})| \leq P(\overline{v})$ for $\overline{v}$ in $\overline{V}$; then there exists an extension p of $\overline{p}$ to a positive linear functional on V satisfying $|p(v)| \leq P(v)$ for v in V.

To prove (α), we take $V = S(B)$, $P = P_1$, $\overline{V} = S(\overline{B})$, and we let $\overline{p}$ be the linear extension of $\overline{\phi}$ to $S(\overline{B})$. For $\overline{v}$ in $\overline{V}$ and $\overline{v} \geq 0$, there exist $a_k \geq 0$ and e_k such that

$$P_1(\overline{v}) + \varepsilon \geq \sum a_k \Phi(e_k) \geq \sum a_k \overline{\phi}(e_k) = \overline{p}(\overline{v})$$

and $\overline{p}(\overline{v}) \leq P_1(\overline{v})$. Clearly $\overline{V}$ is a subvector lattice, $\overline{p}$ is positive, and therefore

$$|\overline{p}(\overline{v})| = |\overline{p}(\overline{v}_+) - \overline{p}(\overline{v}_-)| \leq |\overline{p}(\overline{v}_+ + \overline{v}_-)| = \overline{p}(|\overline{v}|) \leq P_1(|\overline{v}|) = P_1(\overline{v}).$$

The required ϕ is the extension p restricted to B.

To prove (β), we take $V = S(B)$, $P = P_1$, $\overline{V} = C - C$, where C is the cone generated by $\mathscr{F}$ in $S(\underline{B})$, and $\overline{p}(c_1 - c_2) = P_1(c_1) - P_1(c_2)$. Then $\overline{p}$ is well-defined, linear, and positive on $\overline{V}$, since P_1 is additive on C. Moreover,

$$|\overline{p}(c_1 - c_2)| = |P(c_1) - P(c_2)| \leq P(c_1 - c_2).$$

Once again, the required ϕ is the extension p restricted to B.

THEOREM 4. *If* B *is a* σ-*Boolean ring and* μ *is a strictly positive measure on* B, *then a strictly positive function* Φ *on* B *satisfies the condition* $\Phi = \phi^*$ *for some admissible measure* ϕ *if and only if*

(i) Φ *is increasing,*

(ii) Φ *is concave,*

(iii) $\Phi(e)$ *depends only on* $\mu(e)$ *and is finite where* μ *is finite,*

(iv) *if* e_n *is nested and* $\mu(e_n) \to 0$ *then* $\Phi(e_n) \to 0$, *and*

(v) *if* $\mu(e) = +\infty$, *then* $\Phi(e) = \sup \Phi(e')$, $e' \subset e$, *and* $\mu(e') < +\infty$.

Proof. The necessity of the conditions follows easily from (6), Lemma 1, and the full-valuedness of μ. For example, (ii) follows from (6) when we note that for reals $0 \leq h \leq a \leq b$, $\min(u, a - h) + \min(u, b + h) \leq \min(u, a) + \min(u, b)$. While (6) implies continuity of ϕ^* at all e, we need hypothesize it only at 0 and $+\infty$; continuity elsewhere is then implied by (i) to (iii).

Conversely, suppose that Φ satisfies (i) to (v). Let $\overline{B}$ be the Boolean ring of μ-finite elements of B, and $\mathscr{F}$ a maximal nested family in $\overline{B}$. By (i) to (iii), Theorem 3 applies and there exists a finitely additive measure $\overline{\phi}$ on $\overline{B}$ such that $\overline{\phi} = \Phi$ on $\mathscr{F}$ and $\overline{\phi} \leq \Phi$ on $\overline{B}$. Then, by (iii), $\overline{\phi}$ is finite on $\overline{B}$, and by (iv), $\overline{\phi}$ is countably additive on $\overline{B}$. Thus $\overline{\phi}$ may be extended to a measure ϕ on B', and ϕ is an admissible measure. Moreover, $\Phi(e) = \Phi(e')$ if $\mu(e') = \mu(e)$, $e' \in B'$ by (iii), and therefore $\phi \leq \Phi$ implies that $\phi^* \leq \Phi$. For $\mu(e) < +\infty$, there exists an f in $\mathscr{F}$ such that $\mu(e) = \mu(f)$ (μ is full-valued and $\mathscr{F}$ is maximal) and $\phi^*(e) = \phi^*(f) \geq \phi(f) = \Phi(f) = \Phi(e)$. This reverse inequality then follows from (v) for $\mu(e) = +\infty$.

In the problem of representation of Banach lattices, we are given the σ-Boolean ring B and a function Φ defined on B, but not the measure μ. We are interested in whether there exists a strictly positive full-valued measure μ on B which allows the representation $\Phi = \phi^*$ for some μ-admissible measure ϕ. Clearly, by Theorem 4, we need consider this question only for those Φ which are increasing and concave. We have the following theorem.

THEOREM 5. *If* Φ *is a strictly positive, increasing, concave function on a non-atomic* σ*-Boolean ring* B, *then a strictly positive full-valued measure* μ *on* B *such that* (iii) *holds, exists if and only if*

(vi) *If* $e_1 \cap e_2 = 0$, *then* $\Phi(e_1 \cup e_2)$ *depends only on* $\Phi(e_1)$ *and* $\Phi(e_2)$;

(vii) *if* $e_n \downarrow 0$ *and* $\Phi(e_1) < \sup \Phi(e) = M$, *then* $\Phi(e_n) \to \Phi(0^+) = \inf_{e \neq 0} \Phi(e)$;

(viii) *if* $e \neq 0$, $\Phi(e) > \Phi(0^+)$; *if* $\Phi(e) = M$ *and* $\Phi(f) < M$, *then there exists an* $e' \subset e$ *such that* $\Phi(f) \leq \Phi(e') < M$.

Proof. If (iii) holds, then $\Phi(e) = F(\mu(e))$ where F is defined on some closed interval $[0, \beta]$ $(0 < \beta \leq +\infty)$. F is real on the real part of its domain and is a positive, increasing, concave function. If $z < +\infty$, and $0 \leq x < z \leq \beta$, then

$$F(z) \geq F\left(\frac{x+z}{2}\right) \geq \frac{F(x) + F(z)}{2},$$

and $F(x) = F(z)$ if $F(x) = F\left(\frac{x+z}{2}\right)$. Therefore if $x \in F^{-1}(r)$,

$$F^{-1}(r) = \{x\} \qquad \text{or} \qquad F^{-1}(r) \supset [x, +\infty) \cap [0, \beta].$$

(vi) now follows immediately. The first part of (viii) also follows, since Φ is assumed to take on more than three values. (vii) is a restatement of the theorem that $\mu(e_1) < +\infty$ and $e_n \downarrow 0$ imply $\mu(e_n) \to 0$. The last part of (viii) is a consequence of the assumption that μ is full-valued.

Suppose that Φ satisfies (vi), (vii), and (viii). Since B is non-atomic, for any $0 \neq e$ there exists a sequence $e_n \subset e$ such that $e_n \downarrow 0$.

(a) If $e_0 \neq 0$, $e_n \uparrow e_0$ $(e_n \downarrow e_0)$ and $\Phi(e_n) < M$, then $\Phi(e_n) \to \Phi(e_0)$. For there exists $f_n \downarrow 0$ such that $f_n \subset e_1$ $(f_n \subset e_0)$. Since Φ is increasing and concave, $\Phi(e_0) \geq \Phi(e_n)$ and

$$\Phi(e_n) + \Phi((e_0 - e_n) \cup f_n) \geq \Phi(e_0) + \Phi(f_n)$$

$(\Phi(e_n) \geq \Phi(e_0))$ and $\Phi(e_0) + \Phi((e_n - e_0) \cup f_n) \geq \Phi(e_n) + \Phi(f_n))$. The conclusion follows from (vii), since $(e_0 - e_n) \cup f_n \downarrow 0$ $((e_n - e_0) \cup f_n \downarrow 0)$.

(b) If $e \neq 0$, $\Phi(e) < M$, and $\Phi(0^+) < \delta \leq \Phi(e)$, there exists $f \subset e$ such that $\Phi(f) = \delta$. Let $G = \{g \subset e \mid \Phi(g) \leq \delta\}$. G is partially ordered by inclusion, and by Zorn's lemma there exists a maximal chain $\{g_\alpha\}$ in G. There exists an increasing sequence g_n which is cofinal with the chain $\{g_\alpha\}$, for otherwise there would exist an uncountable number of non-zero disjoint subelements of e and this contradicts (vii) and the first part of (viii). Putting $f = \bigcup_1^\infty g_n = \bigcup_\alpha g_\alpha$, we have $\Phi(f) \leq \delta$ by (a). If $\Phi(f) < \delta$, then $f \neq e$ and there exist $f_n \subset e$ such that $f_n \downarrow f$, and by (a), once again, the maximality of $\{g_\alpha\}$ would be contradicted.

(c) If $e \subset f$ and $\Phi(e) = \Phi(f) < \Phi(g) < M$, then $e = f$. Suppose first that $f \subset g$. Then if $f - e \neq 0$, there exists a finite sequence $f = f_0 \subset f_1 \subset \cdots \subset f_n = g$ such that $\Phi(f_{j+1} - f_j) \leq \Phi(f - e)$, by (b) and (vii). By (b) there exists $f' \subset f - e$ such that

$\Phi(f_1 - f) = \Phi(f')$, and $\Phi(f) \geq \Phi(e \cup f') = \Phi(f \cup (f_1 - f)) = \Phi(f_1)$. Thus $\Phi(f) = \Phi(f_1)$, and continuing, we see that $\Phi(f) = \Phi(g)$. But $\Phi(f) < \Phi(g)$ and therefore $e = f$. The case $f \not\subset g$ may be reduced to this one. First $\Phi(g - f \cap g) > \Phi(f - f \cap g)$; for otherwise $\Phi(g) \leq \Phi(f)$ by (b) and (vi). Then there exists $g' \subset g - f \cap g$ such that $\Phi(g') = \Phi(f - f \cap g)$ and $g'' = f \cup (g - g') \supset f$ and

$$\Phi(g'') = \Phi((f - f \cap g) \cup (f \cap g) \cup (g - g' - f \cap g)) = \Phi(g' \cup (f \cap g) \cup (g - g' - f \cap g)) = \Phi(g).$$

If we define $e \sim f$ when $\Phi(e) = \Phi(f)$, the theorem follows from the theorem of Maharam [9, Section 19]; for the existence of the unit is not essential and the countable chain condition is needed only for subelements of e where $\Phi(e) < M$. This last is implied by (vii) and (viii). The measure constructed in [9] is obtained by the techniques developed in [10]. We give a simpler construction for our case, using the special properties of Φ. Choose $0 \neq e \subset f$ such that $\Phi(e) < \Phi(f) < M$. Then for $\Phi(0^+) < \delta \leq \Phi(f - e)$, define $\psi(\delta) = \Phi(e \cup e') - \Phi(e)$, where $e' \subset f - e$ and $\Phi(e') = \delta$. If $G = (e_1, \cdots, e_n)$, where the e_i are disjoint and $\Phi(e_i) \leq \Phi(f - e)$, let $\psi(G) = \Sigma\, \psi(\Phi(e_i))$. Then $\mu(e)$ is defined to be sup $\psi(G)$ over all G such that $e_i \in G$ implies $e_i \subset e$. The proof that μ is indeed the desired measure follows reasonably easily and is not given.

By modifying (vii) and (viii) somewhat and combining Theorems 4 and 5, we get

THEOREM 6. *If Φ is a strictly positive function on a non-atomic σ-Boolean ring* B, *then a strictly positive full-valued measure μ on* B *and a μ-admissible measure ϕ such that $\Phi = \phi^*$ exist if and only if*

(i) *Φ is increasing,*

(ii) *Φ is concave,*

(vi) *if $e_1 \cap e_2 = 0$, then $\Phi(e_1 \cup e_2)$ depends only on $\Phi(e_1)$ and $\Phi(e_2)$,*

(ix) *if $e_n \downarrow 0$ and $\Phi(e_1) < M$, then $\Phi(e_n) \to 0$, and*

(x) *if $\Phi(e) = M$, then $\Phi(e) = \sup \Phi(e')$, for $e' \subset e$ and $\Phi(e') < M$.*

Proof. If the required μ and ϕ exist, then (i) and (ii) follow from Theorem 4, and (vi) from Theorem 5. Since $\Phi(e_1) < M$ implies that $\mu(e_1) < +\infty$, the countable additivity of μ and (iv) imply (ix). Condition (x) follows from (6), Lemma 1, and the full-valuedness of μ.

Conversely, (x) implies the second part of (viii). Moreover, since Φ is strictly positive, there exists an e in B such that $\Phi(e) > 0$. Thus $M > 0$, and by (x), there exists an e in B such that $0 < \Phi(e) < M$. Then there exists a sequence satisfying the hypothesis and therefore the conclusion of (ix), so that $\Phi(0^+) = 0$ and (ix) implies (vii). Moreover, the first part of (viii) follows since Φ is strictly positive. By Theorem 4 we have the existence of the required μ satisfying (iii). Then (ix) implies (iv), (x) and (iii) imply (v), and by Theorem 4 there exists a μ-admissible ϕ for which $\Phi = \phi^*$.

4. SPECIAL SPACES Λ

The space Y of all real functions $y = y_+ - y_-$ (y_+ and y_- positive and disjoint), forms in a natural way a relatively σ-complete vector lattice in which the σ-Boolean ring B is embedded [1].

If Φ is a function on B to $[0, +\infty]$, we define

$$\|y\| = \int_0^{+\infty} \Phi(y_+(\alpha) \cup y_-(\alpha))\, d\alpha . \tag{9}$$

The set $X = \{ y \mid \|y\| < +\infty \}$ forms a subvector lattice in which $\|y\|$ is a lattice norm on X if and only if Φ is strictly positive, increasing, and concave. X is a Banach lattice (complete) if and only if Φ satisfies (ix) as well. The space X together with the above norm is then called a Λ_Φ-space. These results together with a Banach lattice characterization of Λ_Φ-spaces are discussed in [3].

If μ is a strictly positive, locally finite measure on B and ϕ is a μ-admissible measure, we may define

$$\|y\| = \sup \int_0^{+\infty} \psi(y_+(\alpha) \cup y_-(\alpha))\, d\alpha , \tag{10}$$

where the supremum is taken over all measures ψ which are rearrangements of ϕ. We denote the set of elements for which (10) is finite by $\Lambda_{\phi,\mu}$. By Lemma 2, (with a slight modification of y, if necessary as in the proof of Lemma 3), we have

$$\begin{aligned} \sup \int_0^{+\infty} \psi(y_+(\alpha) \cup y_-(\alpha))\, d\alpha &= \int_0^{+\infty} \sup \psi(y_+(\alpha) \cup y_-(\alpha))\, d\alpha \\ &= \int_0^{+\infty} \phi^*(y_+(\alpha) \cup y_-(\alpha))\, d\alpha . \end{aligned} \tag{11}$$

Thus Theorem 6 implies

THEOREM 7. *Every* $\Lambda_{\phi,\mu}$ *space is a* Λ_Φ*-space. A* Λ_Φ*-space is a* $\Lambda_{\phi,\mu}$*-space for some* μ *and* ϕ *if and only if* Φ *in addition satisfies* (vi) *and* (x).

In [8] Lorentz considered the space $\Lambda_{g,\mu}$ of real-valued measurable functions on a measure space (Z, S, μ) for which

$$\|h\| = \sup_{g'} \int g' |h|\, d\mu < +\infty, \tag{12}$$

where g is a locally integrable function on Z and g' is any rearrangement of g [6]. (Functions which differ on sets of measure zero are identified.)

THEOREM 8. *Every* $\Lambda_{g,\mu}$*-space is a* $\Lambda_{\phi,\mu}$*-space, and conversely.*

Proof. To go from (Z, S, μ) to B, μ, we take B to be S, modulo sets of measure zero. Conversely, any σ-Boolean ring is equivalent to a σ-field S of subsets of a set Z modulo a σ-ideal [7]. The point-function h corresponds to the function $y_+(\alpha) = \{z \mid h(z) > \alpha\}$, $y_-(\alpha) = \{z \mid h(z) < -\alpha\}$. The abstract operations are defined just to correspond with this representation of point functions. The correspondence between ϕ and g is that of (3). Then $\phi_g(s) = \int_s g\, d\mu$, and it follows that the norms given by (10) and (12) are identical.

The final theorem relates these results to an abstract relatively σ-complete Banach lattice X with a unit (that is, an element $1 > 0$ such that $1 \cap y = 0$ implies $y = 0$). Then $B(1) = \{e \text{ in } X \mid 0 \leq e \leq 1 \text{ and } e \cap 1 - e = 0\}$ is a σ-Boolean algebra, and the mapping

$$y \to y_+ - y_-, \text{ and for } x \geq 0, \; x \to x(\alpha) = \bigcup_n \{n(x - \alpha 1)_+ \cap 1\}$$

is a representation X(1) of X as a Banach lattice of functions [4]. Combining these results with those of [3] and Theorems 7 and 8, we have

THEOREM 9. *If* X *is a relatively* σ*-complete Banach lattice, then* X(1) *is a* Λ_Φ*-space if and only if the norm is additive on positive covariant elements, or, equivalently, if and only if the norm is concave on* B(1) *and is the maximal extension to a norm on the finite step functions. If* B(1) *is non-atomic, then* X(1) *is a* $\Lambda_{g,\mu}$*-space if and only if it is a* Λ_Φ *space and the norm satisfies* (vi) *and* (x) *on* B(1).

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