

A NOTE ON CERTAIN CONNECTED METRIC DIVISION RINGS

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1. INTRODUCTION

A. Ostrowski has shown in [5] that if K is a connected metric field, with norm N such that $N(xy) = N(x)N(y)$ for all x and y , then K is a subfield of the complex field. In [2], the same conclusion was obtained with the assumption that $N(xy) = N(x)N(y)$ for all x and y weakened to $N(x^2) = N(x)^2$ for all x . Indeed, it was shown that if K is a connected metric division ring in which $N(x^2) = N(x)^2$ for all x , and $N(\dots xy \dots) = N(\dots yx \dots)$ for all x and y , then K is a division subring of the division ring of all real quaternions.

In this note some further embeddings of connected metric division rings in the quaternions are obtained, but with the special assumptions concerning the behavior of the norm now confined to a sufficiently large portion of the division ring. For instance, Theorem 3 indicates that a connected metric division ring K is a division subring of the quaternions if $N(x^2) = N(x)^2$ throughout some neighborhood of zero and $N(\dots xy \dots) = N(\dots yx \dots)$ for every x in some neighborhood of zero. Similarly, Theorem 4 shows that a connected metric division ring K may be embedded in the quaternions if it contains a set B which fails to be nowhere dense, such that $N(x_1 \dots x_r) = N(x_1) \dots N(x_r)$ whenever x_1, \dots, x_r are in B , and $N(\dots xy \dots) = N(\dots yx \dots)$ whenever x is in B .

2. THE SETS $\mathcal{L}(N)$ AND $\mathcal{G}(N)$

The notation and terminology of [2] are assumed known. A pseudonorm N on a ring R with unit e is said to be *unitary* if $N(e) = 1$. We shall now introduce some sets, similar to sets already considered in [1], which measure the extent to which a given pseudonorm resembles a pseudo absolute value.

Definition. If N is a pseudonorm for a ring R , we let $\mathcal{L}(N)$ denote the set of all c in R such that $N(cx) = N(c)N(x)$ for all x in R .

LEMMA 1. *Let N be a pseudonorm for a ring R . Then (i) $I(N) \subset \mathcal{L}(N)$; (ii) if $c, d \in \mathcal{L}(N)$, then $cd \in \mathcal{L}(N)$; (iii) if $c, cd \in \mathcal{L}(N)$ with $N(c) \neq 0$, then $d \in \mathcal{L}(N)$.*

LEMMA 2. *Let R be a ring with unit element e , and let N be a pseudonorm for R . Then (i) $\mathcal{L}(N) \neq I(N)$ only if N is unitary; (ii) $e \in \mathcal{L}(N)$ if and only if N is either unitary or is the zero pseudonorm.*

LEMMA 3. *Let R be a topological ring, and let N be a continuous pseudonorm for R . Then $\mathcal{L}(N)$ is a closed set in R .*

Proofs of these lemmas are left to the reader.

It is easily verified that N is a pseudo absolute value if and only if $\mathcal{L}(N)$ is the entire ring. Thus, the extent to which N resembles a pseudo absolute value is indicated by the size of $\mathcal{L}(N)$. As in [1], we find it useful to consider the *regular* elements of $\mathcal{L}(N)$ when N is a unitary pseudonorm. (See [1] for the basic terminology and notation relative to regular elements and to the group G of all regular elements.)

Definition. If N is a unitary pseudonorm for a ring R , we let $\mathcal{G}(N)$ denote the set of all regular elements c in R such that $N(c)N(c^{-1}) = 1$.

LEMMA 4. *Let R be a ring with unit e , and let N be a unitary pseudonorm for R . Then (i) $\mathcal{G}(N) = \mathcal{L}(N) \cap G$; (ii) $\mathcal{G}(N)$ is a subgroup of G .*

The proof is left to the reader.

LEMMA 5. *Let R be a connected topological ring with unit e and with continuous inversion. If N is a continuous unitary pseudonorm for R such that $\mathcal{L}(N)$ fails to be nowhere dense in R , then $\mathcal{G}(N)$ contains every component of G which it meets.*

Proof. $\mathcal{L}(N)$ is closed, by Lemma 3, so that $\mathcal{L}(N)$ contains a nonempty open set A , since $\mathcal{L}(N)$ fails to be nowhere dense. If the ideal $I(N)$ contained A , then Proposition 4 of [4; Chap. III, Section 2, No. 1] would imply that $I(N)$ is an open subgroup of the additive group of R and hence is also closed. Since R is connected, the nonempty open and closed set $I(N)$ would coincide with R , and N could not be unitary. Thus, $I(N)$ does not contain A , so that there exists an element c in A with $N(c) \neq 0$.

If B is the preimage of A relative to the continuous mapping $x \rightarrow cx$ of R into itself, then B is open and obviously contains e . Also, if x is in B , then cx is in A and therefore in $\mathcal{L}(N)$, so that Lemma 1 (iii) shows that x is in $\mathcal{L}(N)$. That is, $B \subset \mathcal{L}(N)$, so that $B \cap G \subset \mathcal{L}(N) \cap G = \mathcal{G}(N)$. Since $B \cap G$ is open in G and contains e , this element is an interior point of $\mathcal{G}(N)$ in the topological group G , whence $\mathcal{G}(N)$ is open in G by the previously cited result from [4], and $\mathcal{G}(N)$ is therefore open and closed in G . It follows that $\mathcal{G}(N)$ contains every connected component of G which it meets.

THEOREM 1. *Let K be a connected topological division ring with continuous inversion, and let N be a continuous unitary pseudonorm for K such that $\mathcal{L}(N)$ fails to be nowhere dense in K . Then K is algebraically isomorphic to a division subring of the division ring \mathfrak{D} of all real quaternions.*

Proof. By the preceding lemma, $\mathcal{G}(N)$ contains every component of G which it meets. If G is connected, then $\mathcal{G}(N) \supset G$, so that $\mathcal{G}(N) = G$. On the other hand, if G is not connected, then the additive group of K is a connected topological group in which the complement of 0 is not connected. It follows as in [4; Chap. V, Section 3, Exercise 4] that G , the complement of 0 , falls into two connected sets, G_1 and G_2 , such that the negative of any element of G_1 is in G_2 . Thus, e is in one of these sets and $-e$ is in the other; since $\mathcal{G}(N)$ contains both e and $-e$, $\mathcal{G}(N)$ meets both components of G , so that again $\mathcal{G}(N) \supset G$ and therefore $\mathcal{G}(N) = G$. In both cases, $G = \mathcal{G}(N) \subset \mathcal{L}(N)$, while 0 is always in $\mathcal{L}(N)$, so that $K = G \cup 0 \subset \mathcal{L}(N)$. This shows that N is a pseudo absolute value. But N is nonzero, so that the ideal $I(N)$ is not all of the division ring K and is consequently the zero ideal, whence N is an absolute value. It is easily seen that every set open in the N -topology is open, so that the N -topology of R is *less fine* than the given topology. Then K is connected in the N -topology, since K is connected. Since N is an absolute value, the result follows from Corollary 2 of Theorem 9 in [2], which was cited at the end of the first paragraph of this note.

3. CONNECTED METRIC DIVISION RINGS

In this section we give a few embedding theorems for some connected metric division rings. The terminology agrees with that in [3].

THEOREM 2. *Let K be a connected metric division ring, with norm N , such that K contains an N -stable, N -power multiplicative semigroup A which fails to be nowhere dense. Then K is algebraically isomorphic to a division subring of \mathfrak{D} .*

Proof. By the corollary of Theorem 3 in [3], if we choose a nonzero c in A , there exists a pseudonorm N' subordinate to N , with $N'(c) = N(c)$, such that N' is homogeneous on A . That is, $\mathcal{L}(N') \supset A$, so that $\mathcal{L}(N')$ fails to be nowhere dense in K . Also, $N'(c) = N'(ce) = N'(c)N'(e)$ since c is in $\mathcal{L}(N')$, and $N'(c) = N(c) \neq 0$ since N is a norm, so that $N'(e) = 1$. Thus, N' is unitary, while Lemma 5 of [2] indicates that N' is continuous since it is a subordinate pseudonorm of K . The theorem follows if we apply the preceding theorem to N' ; inversion in K , indeed in any metric ring with unit, is easily proved continuous, so that all hypotheses of Theorem 1 are satisfied.

THEOREM 3. *Let K be a connected metric division ring, with norm N , such that there exists a neighborhood U of 0 which is N -stable, and a neighborhood V of 0 for which $N(x^2) = N(x)^2$ for all x in V . Then K is algebraically isomorphic to a division subring of \mathfrak{D} .*

Proof. Let ε be a positive number less than 1 so small that $A = \{x \mid N(x) < \varepsilon\}$ is contained in U and in V . Then A is an N -stable semigroup, and $N(x^2) = N(x)^2$ for all x in A , so that A is N -power multiplicative by Theorem 1 of [3]. But A is a nonempty open set and therefore fails to be nowhere dense; the preceding theorem is then applied to A .

COROLLARY 1. *Let K be a connected metric division ring, with norm N , such that there exists a neighborhood U of 0 which is N -stable, and a neighborhood V of 0 for which $N(xy) = N(x)N(y)$ for all x and y in V . Then K is algebraically isomorphic to a division subring of \mathfrak{D} .*

COROLLARY 2. *Let K be a connected metric division ring, with norm N , such that there exists a neighborhood U of 0 which is N -stable, and a neighborhood V of 0 for which $N(x)N(x^{-1}) = 1$ whenever x is a nonzero element of V . Then K is algebraically isomorphic to a division subring of \mathfrak{D} .*

Proof. The condition $N(x)N(x^{-1}) = 1$ implies that

$$N(x)N(y) = N(x)N(x^{-1}xy) \leq N(x)N(x^{-1})N(xy) = N(xy),$$

so that $N(xy) = N(x)N(y)$ for all y whenever x is a nonzero element of V . Since $N(xy) = N(x)N(y)$ for all y if $x = 0$, we have $N(xy) = N(x)N(y)$ for all y whenever x is in V , and the first corollary may then be applied.

In order to avoid the assumption that A is a semigroup in Theorem 2, we introduce a stronger multiplicative property for the norm to have on A instead of merely being multiplicative.

Definition. If N is a pseudonorm for a ring R , and A is a set in R such that $N(x_1 \cdots x_r) = N(x_1) \cdots N(x_r)$ whenever x_1, \dots, x_r are elements of A , then A is said to be *strongly N -multiplicative*.

It is easily seen that a set A is strongly N -multiplicative if and only if the semigroup generated by A is N -multiplicative.

THEOREM 4. *Let K be a connected metric division ring, with norm N , which contains an N -stable, strongly N -multiplicative set B such that B fails to be nowhere dense. Then K is algebraically isomorphic to a division subring of \mathfrak{D} .*

Proof. If A is the semigroup generated by B , then A is N -multiplicative and therefore N -power multiplicative. Since $A \supset B$, A also fails to be nowhere dense. It is easily verified that the set of elements at which a pseudonorm is stable constitutes a semigroup, so that N is stable on A since it is stable on B . Theorem 2 may then be applied to the semigroup A to obtain the desired result.

Note. In all of the theorems and corollaries, the assumption that K is a field would render unnecessary any assumptions about stability of the norm, and in each case the conclusion would then be that K is algebraically isomorphic to a subfield of the field of all complex numbers.

We note that in Theorem 3 it would suffice to assume for each x in V that $N(x^{r(x)}) = N(x)^{r(x)}$ for some integer $r(x)$ greater than 1, since Lemma 3 of [3] would then imply that $N(x^2) = N(x)^2$ for each x in V .

REFERENCES

1. S. Aurora, *Multiplicative norms for metric rings*, Pacific J. Math. 7 (1957), 1279-1304.
2. ———, *On power multiplicative norms*, Amer. J. Math. 80 (1958), 879-894.
3. ———, *The embedding of certain metric fields*, Michigan Math. J. 7 (1960)
4. N. Bourbaki, *Éléments de mathématique, Livre III. Topologie générale*, Actualités Sci. Ind. nos. 916 (1942), 1045 (1948), 1084 (1949); Hermann et Cie, Paris.
5. A. Ostrowski, *Über einige Lösungen der Funktionalgleichung $\phi(x) \cdot \phi(y) = \phi(xy)$* , Acta Math. 41 (1918), 271-284.

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