THE EMBEDDING OF CERTAIN METRIC FIELDS

Silvio Aurora

1. INTRODUCTION

Mazur's Theorem, first stated in [5], can be formulated for the commutative case in the following way: *If* K *is a metric field which contains the real numbers and is such that the norm of* x *equals the ordinary absolute value of* x *whenever* x *is a real number, then* K *is the real field or the complex field.* In Section 4 of this paper, some results are obtained in which K is no longer assumed to contain the real field, and the condition on the norm is assumed only for a portion of the prime field of K; when K does not contain the real field, it might be one of the subfields of the complex field other than the reals or complexes; therefore our results show only that K is some subfield of the complex field.

For instance, Corollary 2 of Theorem 5 asserts that if K is a metric field such that the norm of $n \cdot e$ equals n, for every natural number n which is sufficiently large, then K is isomorphic to a subfield of the complex field. (Here e denotes the unit element of K, and $n \cdot e$ is the n-fold sum $e + \cdots + e$.) Similarly, Theorem 7 asserts that if K is a metric field of characteristic zero, such that the norm of x equals the ordinary absolute value of x whenever x is a positive rational number in K that is sufficiently small, then K is a subfield of the complex field.

In order to obtain these results, we develop in Section 3 some generalizations of Theorem 1 of [2]. These generalizations allow us to obtain, from a pseudonorm N which is sufficiently well-behaved on a semigroup A, a pseudonorm N' closely related to N, having the same desirable properties as N, and such that

$$N'(cx) = N'(c)\,N'(x)$$

for all c in A and for all x (in this paper, a *semigroup* is understood to be a nonempty set, contained in a ring, which is closed under the ring multiplication).

2. PSEUDONORMS AND SUBORDINATE PSEUDONORMS

The terminology and notation employed in [2] are assumed known. See also [1] and [4] for further remarks about norms and metric rings.

A pseudonorm N is said to be *stable* at an element c, and c is said to be N-*stable*, if $N(\cdots cx \cdots) = N(\cdots xc \cdots)$ for all x; if N is stable at every element of a set A, then A is said to be N-*stable*, and we say that N is *stable on* A. Pseudo absolute values are stable on the entire ring, and every pseudonorm is stable on the center.

If N is a pseudonorm of a ring R and c is an element of R such that $N(c^r) = N(c)^r$ for $r = 1, 2, \cdots$, then we say that N is *power multiplicative at* c and that c is N-*power multiplicative*. A pseudo absolute value is power multiplicative at all elements, for instance.

Under certain circumstances it is possible to replace a pseudonorm by a subordinate pseudonorm having special properties. First, we note that if N is a

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pseudonorm of a ring R and c is an element of R which is not N-null, then, for each x in R,

$$N(x) \geq N(xc)/N(c) \geq N(xc^2)/N(c)^2 \geq \cdots \geq 0,$$

so that $N_c(x) = \lim_{r \to \infty} N(xc^r)/N(c)^r = \inf_r N(xc^r)/N(c)^r$ exists for each x.

We indicate next the main results concerning the function N_c. Proofs are left to the reader.

LEMMA 1. *Let* N *be a pseudonorm of a ring* R, *and let* c *be an element of* R *which is* N-*stable and not* N-*null. Then* N_c *is a pseudonorm subordinate to* N, *and it is distinct from the zero pseudonorm if and only if* c *is* N-*power multiplicative.*

LEMMA 2. *Let* N *be a pseudonorm of a ring* R, *and let* c *be an element of* R *which is* N-*stable and* N-*power multiplicative, but not* N-*null. Then* N_c *is a non-zero pseudonorm subordinate to* N *such that*

(1) $N_c(c) = N(c)$,

(2) $N_c(cx) = N_c(c)\, N_c(x)$ *for all* x *in* R,

(3) *if* N *is stable at* d, *then* N_c *is stable at* d.

3. STRUCTURE THEOREMS FOR PSEUDONORMS

In this section we show that certain pseudonorms have subordinate pseudonorms with special multiplicative properties.

A pseudonorm N of a ring R is said to be *multiplicative (power multiplicative)* on a subset A if $N(xy) = N(x)\,N(y)$ for all x and y in A (if N is power multiplicative at x for each x in A); in this case A is said to be N-*multiplicative* (N-*power multiplicative*). A pseudonorm N is said to be *homogeneous* on a set A if $N(ax) = N(a)\,N(x)$ for all x whenever a is in A. The results below permit us to replace a pseudonorm which is multiplicative or power multiplicative on a semigroup by a subordinate pseudonorm which equals the original pseudonorm over a portion of the semigroup but is homogeneous on the semigroup. We first obtain some simple criteria for a pseudonorm to be power multiplicative on an element or a semigroup.

LEMMA 3. *Let* N *be a pseudonorm for a ring* R, *and let* x *be an element of* R. *If* $N(x^r) = N(x)^r$ *for some natural number* r, *then* $N(x^s) = N(x)^s$ *for every natural number* s *less than* r. *In particular,* N *is power multiplicative at an element* x *if and only if there exist infinitely many natural numbers* r *such that* $N(x^r) = N(x)^r$.

Proof. If $N(x^r) = N(x)^r$ and $s < r$, then

$$N(x)^r = N(x^r) = N(x^s \cdot x^{r-s}) \leq N(x^s)\, N(x)^{r-s},$$

whence $N(x)^s \leq N(x^s)$ and therefore $N(x^s) = N(x)^s$. The final statement in the lemma follows immediately.

THEOREM 1. *Let* N *be a pseudonorm for a ring* R, *and let* A *be a semigroup in* R. *Then* N *is power multiplicative on* A *if and only if* $N(x^2) = N(x)^2$ *for all* x *in* A.

Proof. Iterated application of the condition $N(x^2) = N(x)^2$ yields $N(x^s) = N(x)^s$ for x in A, whenever s is a power of 2. The final statement in Lemma 3 then shows that N is power multiplicative at x for each x in A, hence A is N-power multiplicative if $N(x^2) = N(x)^2$ for all x in A. The converse is obvious.

We note that if $N(x^{r(x)}) = N(x)^{r(x)}$ for some integer $r(x)$ greater than 1, then $N(x^2) = N(x)^2$. This leads to the following corollary.

COROLLARY. *Let* N *be a pseudonorm of a ring* R, *and let* A *be a semigroup in* R. *Then* N *is power multiplicative on* A *if and only if for each* x *in* A *there exists an integer* $r(x)$ *greater than* 1, *with* $N(x^{r(x)}) = N(x)^{r(x)}$.

If A is a nonempty set in a ring R and $\mathscr{N}$ is a nonempty set of pseudonorms of R such that each N in $\mathscr{N}$ is stable and power multiplicative on A, then $\mathscr{N}$ is said to be A-*hereditary* if, whenever N is in $\mathscr{N}$ and c is in A but not N-null, then N_c is in $\mathscr{N}$.

LEMMA 4. *Let* A *be a nonempty set in a ring* R, *and let* $\mathscr{N}$ *be an* A-*hereditary system of pseudonorms for* R. *If* $\mathscr{N}$ *has a minimal element* N, *then* N *is homogeneous on* A.

Proof. Suppose that N is a minimal element of $\mathscr{N}$. Let c be in A, and let x be any element of R. If $N(c) = 0$, then $N(cx) = N(c)\,N(x)$, since the right side and hence both sides of the equation are zero. On the other hand, if $N(c) \neq 0$, then the pseudonorm N_c exists and is such that $N_c(cx) = N_c(c)\,N_c(x)$. But N_c is subordinate to N and belongs to $\mathscr{N}$, since $\mathscr{N}$ is A-hereditary. Since N is a minimal element of $\mathscr{N}$, we have $N_c = N$. Thus, $N(cx) = N(c)\,N(x)$. This shows that N is homogeneous on A.

LEMMA 5. *Let* N *be a pseudonorm on a ring* R, *and let* A *be an* N-*stable*, N-*power multiplicative semigroup in* R. *Let* B *be an* N-*multiplicative semigroup contained in* A. *Then there exists a pseudonorm* N' *of* R *subordinate to* N, *and equal to* N *on* B, *such that* N' *is stable and power multiplicative on* A *and homogeneous on* B.

Proof. Let $\mathscr{N}$ be the set of all pseudonorms subordinate to N, equal to N on B, stable on A, and power multiplicative on A. Then $\mathscr{N}$ is not empty, since it contains N. It is readily proved that $\mathscr{N}$ is B-hereditary. Zorn's Lemma is used as in the proof of Lemma 4 of [2] to show that $\mathscr{N}$ has a minimal element N'. Then Lemma 4 shows that N' is homogeneous on B, and the result is proved.

THEOREM 2. *Let* N *be a pseudonorm of a ring* R, *and let* A *be an* N-*stable*, N-*multiplicative semigroup in* R. *Then there exists a pseudonorm* N' *of* R, *subordinate to* N, *and equal to* N *on* A, *such that* N' *is stable and homogeneous on* A.

Proof. Use Lemma 5 with A and B identical.

THEOREM 3. *Let* N *be a pseudonorm of a ring* R, *and let* A *be an* N-*stable*, N-*power multiplicative semigroup in* R. *Let* B *be an* N-*multiplicative semigroup contained in* A. *Then there exists a pseudonorm* N' *of* R, *subordinate to* N, *equal to* N *on* B, *and such that* N' *is stable and homogeneous on* A.

Proof. Let $\mathscr{N}$ be the set of all pseudonorms subordinate to N, equal to N on B, stable and power multiplicative on A, and homogeneous on B. Then $\mathscr{N}$ is not empty, by Lemma 5. It is easily verified that $\mathscr{N}$ is A-hereditary and contains a minimal element N'. Then N' is homogeneous on A, by Lemma 4. This completes the proof.

COROLLARY. *Let* N *be a pseudonorm of a ring* R, *and let* A *be an* N-*stable*, N-*power multiplicative semigroup in* R. *If* c *is an element of* A, *then there exists a pseudonorm* N', *subordinate to* N, *with* $N'(c) = N(c)$, *such that* N' *is stable and homogeneous on* A.

Proof. Use the theorem, with B taken as the semigroup generated by c.

This result, with $A = R$, quickly yields Theorem 1 of [2].

4. EMBEDDING THEOREMS FOR METRIC FIELDS

We can now show that metric fields can be embedded in the field $\mathfrak{C}$ of all complex numbers, provided that the norm acts as the ordinary absolute value over a sufficiently large portion of the prime field. First, Mazur's Theorem is recast in a slightly different form.

THEOREM 4. *Let* K *be a metric field which contains the field* P *of all rational numbers. If* $N(x) = |x|$ *for all* x *in* P, *then* K *is algebraically isomorphic to a subfield of* $\mathfrak{C}$.

Proof. K may easily be shown to be a commutative normed algebra over P in the sense of [3], hence, if R is the completion of K, then R is a complete normed algebra over the real field, according to [3; Chap. IX, Section 3, No. 7, p. 51]. It is clear that R is commutative and has a unit element e (the same unit element as for K, if K is identified with a subset of R).

Then there exists a maximal ideal I among all the ideals of R distinct from R. Since R is complete and is a metric ring, it is a Q-ring [4; p. 155], and therefore the maximal ideal I is closed [4; Theorem 2]. Then R/I is a normed algebra over the real field [3; Chap. IX, Section 3, No. 7, p. 51], and R/I is a field, since I is a maximal ideal. But Mazur's Theorem tells us that a normed division algebra over the real field is isomorphic to the real field, the complex field, or the division ring of real quaternions. Thus, there exists an isomorphism of R/I with the real field or the complex field, hence there exists an isomorphism ϕ of R/I into the complex field $\mathfrak{C}$. If η if the natural homomorphism of R onto R/I, then η followed by ϕ is a homomorphism of R into $\mathfrak{C}$, and it carries e into 1. The restriction to K of this homomorphism is thus a nonzero homomorphism of the field K into $\mathfrak{C}$, and it is therefore an isomorphism of K into $\mathfrak{C}$.

THEOREM 5. *Let* K *be a metric field such that for some integer* $k > 1$, $N(ke) = k$ *and* ke *is* N-*power multiplicative. Then* K *is algebraically isomorphic to a subfield of* $\mathfrak{C}$.

Proof. Let A be the semigroup generated by ke, so that N is multiplicative on A. Theorem 2 shows that there exists a pseudonorm N' subordinate to N, equal to N on A, and homogeneous on A. Thus, $N'(ke) = N(ke) = k$, and

$$N'(kx) = N'((ke)x) = N'(ke)\,N'(x) = kN'(x)$$

for all x. If se set $x = e$ in this last equation, we get $N'(ke) = kN'(e)$, hence $N'(e) = 1$, since $N'(ke) = k$. For x in R and r a natural number, we have

$$rN'(x) + (k^r - r)N'(x) = k^r N'(x) = N'(k^r x) = N'(rx + (k^r - r)x) \leq N'(rx) + (k^r - r)N'(x),$$

whence $rN'(x) \leq N'(rx)$. But $N'(rx) \leq rN'(x)$ always, so that $N'(rx) = rN'(x)$. If in this last statement we set $x = e$, we get $N'(re) = rN'(e) = r$, for every natural number r. It follows that K is of characteristic zero, since re can never be zero.

If m and n are natural numbers and c is the element of the prime field of K identified with the rational number m/n, then $nc = me$, so that

$$nN'(c) = N'(nc) = N'(me) = m,$$

whence $N'(c) = m/n$. Thus, N' reduces to the ordinary absolute value for "positive rational numbers" in K and consequently for all "rational numbers" in K. To complete the proof, we apply Theorem 4.

COROLLARY 1. *Let* K *be a metric field such that* $N(ke) = k$ *for some integer* $k > 1$, *and such that* $N(x^2) = N(x)^2$ *whenever* x *is of the form* ne *with* n *a natural number*, $n \geq k$. *Then* K *is algebraically isomorphic to a subfield of* $\mathfrak{C}$.

Proof. Theorem 1 shows that N is power multiplicative on the semigroup A generated by ke, and the theorem may then be applied.

COROLLARY 2. *Let* K *be a metric field such that* $N(ne) = n$ *for all natural numbers* n *which are sufficiently large. Then* K *is algebraically isomorphic to a subfield of* $\mathfrak{C}$.

Proof. If k is a natural number $(k > 1)$ such that $N(ne) = n$ whenever $n \geq k$, then k clearly satisfies the hypothesis of the theorem.

THEOREM 6. *Let* K *be a metric field such that* $N(ne) = nN(e)$ *for infinitely many natural numbers* n. *Then* K *is algebraically isomorphic to a subfield of* $\mathfrak{C}$.

Proof. Let $N'(x) = \sup \{N(xa)/N(a) \mid N(a) \neq 0\}$ for each x. Then N' is a norm for K such that $N'(e) = 1$. Thus, $N'(ne) \leq nN'(e) = n$ for every natural number n. On the other hand, $N'(ne) \geq N(ne)/N(e) = n$ for infinitely many natural numbers n, so that $N'(ne) = n$ for infinitely many natural numbers n.

Corresponding to any natural number r, let n be a natural number greater than r such that $N'(ne) = n$. Then

$$r + (n - r) = n = N'(ne) = N'(re + (n - r)e) \leq N'(re) + (n - r),$$

so that $r \leq N'(re)$. But $N'(re) \leq rN'(e) = r$, so that $N'(re) = r$. We may now apply Corollary 2 of the preceding theorem to K with the norm N', to obtain the desired result.

THEOREM 7. *Let* K *be a metric field which contains the field* P *of rational numbers, and such that* $N(x) = |x|$ *for all positive rational numbers* x *which are sufficiently small. Then* K *is algebraically isomorphic to a subfield of* $\mathfrak{C}$.

Proof. Let ε be a real number, with $0 < \varepsilon < 1$, such that $N(x) = |x|$ whenever x is a rational number with $0 < x < \varepsilon$. If A is the set of all rational numbers x with $0 < x < \varepsilon$, then A is a semigroup on which N is multiplicative. By Theorem 2, there exists a pseudonorm N' subordinate to N, equal to N on A, and homogeneous on A.

Since N' is nonzero on the field K, $I(N')$ is an ideal in K distinct from K, and it is therefore the zero ideal, so that N' is a norm for K. If n is any natural number greater than ε^{-1}, then $1/n < \varepsilon$, so that $N'(1/n) = N(1/n) = 1/n$. Since N' is homogeneous on A,

$$N'(1) = N'(1/n \cdot n) = N'(1/n) \cdot N'(n) = 1/n \cdot N'(n),$$

and therefore $N'(n) = nN'(1)$ whenever n is a natural number with $n > \varepsilon^{-1}$. Application of Theorem 6 completes the proof, if N' is used as the norm of K.

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New York City