ORIENTATION IN GENERALIZED MANIFOLDS AND APPLICATIONS TO THE THEORY OF TRANSFORMATION GROUPS

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0. INTRODUCTION

This paper is concerned with a type of cohomology manifold which is of importance in the theory of topological transformation groups. These spaces have been studied by C. T. Yang, D. Montgomery, P. A. Smith, E. E. Floyd, A. Borel, and others.

The paper actually consists of two rather disjointed parts. The first consists of Sections 2 through 6 and deals with orientable coverings, and the second consists of Sections 7 through 11 and deals with the theory of transformations of prime period. These parts have a few points of contact in Section 7, and both use the preliminary material of Section 1, which contains some general facts about generalized manifolds.

In Section 2 we define the orientable covering of a cohomology manifold. The definition follows classical lines closely, except that the orientable covering has, in general, more than two sheets. The special case in which the group of coefficients is the integers is studied in Section 4, along with the relationship of this case to the general case. In this case, as in the classical case, the orientable covering has two sheets.

In Section 5 we apply our methods to study the lifting of transformation groups to the orientable covering, and we find that a group may be lifted in a unique manner to be a group of orientation-preserving transformations on the orientable covering. In Section 6 we study conditions under which the lifted group is a topological transformation group of the orientable covering if it is a topological transformation group of the original space.

In Section 7 we study groups of transformations of prime order p on a cohomology manifold over Z_p, that is, P. A. Smith's theory. We obtain, in a modern form, Smith's theorems that the fixed point set is a cohomology manifold over Z_p and is orientable if the space M is orientable; we also obtain a partial converse: If M is paracompact, and if the fixed point set is orientable, then it has an orientable neighborhood in M. We also obtain a new dimensional parity theorem which asserts that if the prime p is odd, then the dimensions of M and of each nonempty component of the fixed point set are of the same parity. Analogous dimensional parity theorems for spaces possessing the homology groups of a sphere have been given by P. A. Smith [10], E. E. Floyd [4], and A. Borel [2]. (See also Section 11 of the present paper.) The analogue of the refinement of the theorem necessary for the case $p = 2$, given by Liao [7], is also proved, and in the course of its proof we find the local groups in the orbit space about a fixed point. The results of this section are mainly obtained by studying the relationships between the orientations "in the small" of the fixed point set and of M.

In Section 9 we give another proof of the dimensional parity theorem of Section 7, under slightly more restrictive assumptions; it is based on entirely different techniques and has the advantage of being somewhat shorter than the preceding proof.

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In Section 10 we apply a result from Section 7 to prove a result for transformation groups of prime order (and for toral groups) which act on orientable generalized manifolds; it says essentially that the set of stationary points, regarded as a subspace of the orbit space, is homologous to zero in the orbit space.

In Section 11 we indicate briefly how the global theorems of Smith on transformations of prime order on cohomology spheres can be obtained in a way analogous to our derivations of the local theorems.

The dimensional parity theorems of Section 7 have subsequently been obtained by A. Borel, by different methods; Borel's proofs will appear in a published version of the notes of the Seminar on Transformation Groups being held at the Institute for Advanced Study.

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1. PRELIMINARIES

L will denote any module over a ring with unit. Z denotes the ring of integers, and Z_p denotes the field of integers modulo the prime number p.

The cohomology of locally compact Hausdorff spaces we use is the Alexander-Spanier cohomology with compact supports; it is well-known to give the same groups as the Čech theory of the one-point compactifications modulo the point at infinity.

Definition 1.1 below is our working definition in this paper. Definition 1.2 is one of several possible definitions of an n-gm over L, and it is easily seen to be equivalent to the definition in the seminar notes mentioned in Section 0. It is given only for a principal ideal ring L. In the case of a principal ideal ring L, we show below that these two definitions are equivalent. This equivalence is not used until Section 7.

DEFINITION 1.1. *A connected, finite-dimensional, locally compact Hausdorff space* M *is said to be an* (L, n)*-manifold if there exists a collection* $\mathfrak{B}$ *of open sets of* M, *called fundamental sets, such that:*

(1) *If* $U \in \mathfrak{B}$, *then* $H^n(U; L) \approx L$.

(2) *If* $U \in \mathfrak{B}$, *and* $V \subset U$ *is open, then the natural map* $H^n(V; L) \to H^n(U; L)$ *is onto. If* $V \in \mathfrak{B}$ *also, then this is an isomorphism.*

(3) *If* V *is open and* $y \in V$, *then there is a set* $W \in \mathfrak{B}$ *with* $y \in W \subset V$, *such that the natural map* $H^k(W; L) \to H^k(V; L)$ *is trivial for all* $k \neq n$.

(4) *If* $\mathfrak{B}'$ *is a collection of open sets satisfying* (1), (2), *and* (3) *with* $\mathfrak{B}'$ *replacing* $\mathfrak{B}$, *then* $\mathfrak{B}' \subset \mathfrak{B}$.

DEFINITION 1.2. *A connected, finite-dimensional, locally compact Hausdorff space* M *is said to be an* n-gm *over the principal ideal ring* L *if for each point* x *there is a neighborhood* N *such that for each open set* $U \subset N$, *each point* $y \in U$, *has a neighborhood* $V \subset U$ *such that for all open* $W \subset V$ *we have*

$$\operatorname{Im}(H^k(W; L) \to H^k(U; L)) = \operatorname{Im}(H^k(V; L) \to H^k(U; L)) \approx \begin{cases} 0 & (k \neq n), \\ L & (k = n). \end{cases}$$

We shall now indicate the proof of the equivalence of these two definitions for a principal ideal ring L. Every (L, n)-manifold is clearly an n-gm over L. Let M be an n-gm over L. Then M is locally connected; for in the definition, we may take N to be compact, and V then is contained in a single component of U. (A more detailed proof of a similar statement can be found in the proof of Theorem 7.4.) Moreover, we see easily that if U is connected, then $\operatorname{Im}(H^n(V; L) \to H^n(U; L))$ is independent of V and y, where V and y are as in the definition.

Lemma 1.3 and Corollary 1.4 below will show that $H^n(U; L) \approx L$ and that $H^n(V; L) \to H^n(U; L)$ is onto, for U connected and as in Definition 1.2, and for all open $V \subset U$. Since L is a principal ideal ring, it now follows that if $U' \subset U$ and if U' and U are both open, connected, and contained in N, then the map

$$H^n(U'; L) \to H^n(U; L)$$

is an isomorphism.

Thus it will follow that the collection $\mathfrak{B}'$ of connected sets U that are as in Definition 1.2 satisfies conditions (1), (2) and (3) of Definition 1.1. Zorn's Lemma now implies easily that there is a unique maximal such collection, and hence M is an (L, n)-manifold.

The collection of fundamental sets will be characterized in Corollary 1.7.

LEMMA 1.3. *Let* U *be a connected, locally compact Hausdorff space such that* $H^{j+1}(Y) = 0$ *for every open set* $Y \subset U$; *suppose also that each point* x *in* U *has a neighborhood* V *such that if* $W \subset V$ *is open, then* $\operatorname{Im}(H^j(W) \to H^j(U)) = G$; *then* $H^j(U) = G$.

Proof. By Zorn's Lemma and the fact that we use compact supports, there exists a maximal open subset $W \subset U$ such that $\operatorname{Im}(H^j(W) \to H^j(U)) = G$. We shall show that $W = U$.

Let $x \in (\overline{W} - W)$, and let N be an open neighborhood of x such that

$$\operatorname{Im}(H^j(N) \to H^j(U)) = G.$$

Consider the diagram

$$\begin{array}{ccccccc} H^j(W \cap N) & \xrightarrow{f_1} & H^j(W) \oplus H^j(N) & \xrightarrow{g_1} & H^j(W \cup N) & \to & 0 \\ \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & \\ H^j(U) & \xrightarrow{f_2} & H^j(U) \oplus H^j(U) & \xrightarrow{g_2} & H^j(U) & \longrightarrow & 0, \end{array}$$

in which each row is part of a Mayer-Vietoris sequence (see [1]).

Now $\operatorname{Im} h_3 = \operatorname{Im} h_3 g_1 = \operatorname{Im} g_2 h_2 = g_2(\operatorname{Im} h_2) = g_2(G \oplus G)$. Since g_2 is the sum of the natural projection maps, it follows that $g_2(G \oplus G) = G$, a contradiction which shows that $W = U$; this completes the proof.

COROLLARY 1.4. *Let* X *be a locally compact, locally connected, finite-dimensional Hausdorff space; suppose moreover that for every connected open set* $U \subset X$, *each point* $x \in U$ *has a neighborhood* V *such that if* $W \subset V$ *is open, then* $\operatorname{Im}(H^i(W) \to H^i(U)) = G_U$, *where* $G_U \approx 0$ *or* $G_U \approx L$ *according as* $i > j$ *or* $i = j$; *then* $H^i(Y)$ *is isomorphic to* 0 *or to* L *according as* $i > j$ *or* $i = j$, *for every connected open subset* $Y \subset X$.

Proof. The property "$H^i(Y) = 0$ for every open subset Y of X" is equivalent to "$H^i(Y) = 0$ for every connected open subset Y of X", since X is locally connected.

Thus, decreasing induction on the hypothesis $H^{j+1}(Y) = 0$ in Lemma 1.3 is possible, and the corollary follows without difficulty.

COROLLARY 1.5. *If* M *is an* (L, n)*-manifold, then* $H^k(U) = 0$ *for all* $k > n$ *and all open subsets* U *of* M.

DEFINITION 1.6. *An open subset* U *of an* (L, n)*-manifold is said to be orientable if for every component* V *of* U *and every fundamental set* $W \subset V$, *the homomorphism* $H^n(W) \to H^n(V)$ *is an isomorphism onto.*

COROLLARY 1.7. *If* U *is a connected open subset of an* (L, n)*-manifold and* $V \subset U$ *is open, then the map* $H^n(V) \to H^n(U)$ *is onto. A subset of* M *is a fundamental set if and only if it is open, connected, and orientable, and every open subset of an orientable set is orientable. Also,* $H^n(C) = 0$ *for every proper closed subset of* U.

Proof. Let $W \subset U$ be a fundamental set, and let $G = \mathrm{Im}(H^n(W) \to H^n(U))$. Then if $W' \subset W$ or $W' \supset W$ and W' is also a fundamental set, then clearly

$$G = \mathrm{Im}(H^n(W') \to H^n(U)).$$

Since U is connected, it is clear that $G = \mathrm{Im}(H^n(V) \to H^n(U))$ for all fundamental sets $V \subset U$, and by Lemma 1.3 and Corollary 1.5, $H^n(U) = G$. The first statement of the corollary follows immediately, and the second follows directly from the first. The last statement follows from the exact cohomology sequence

$$H^n(U\text{-}C) \to H^n(U) \to H^n(C) \to 0.$$

Remark. $\mathrm{Ker}(H^n(W) \to H^n(U))$ will be calculated in Theorem 1.10.

THEOREM 1.8. M *is a* (Z, n)*-manifold* [*respectively, a* (Z_p, n)*-manifold, for a prime* p] *if and only if* M *is a* (K, n) [*respectively,* (Z_p, n)] *(homology) manifold in the sense of* [11], *where* K *is the group of reals modulo one. Moreover, every* (Z, n)*-manifold is also an* (L, n)*-manifold for every abelian group* L.

Proof. The proof of the first statement follows from the proof of Lemma 11 of [11] and, for the case in brackets, from the relation $H^q(U; Z_p) \approx \mathrm{Hom}(H_q(U; Z_p), Z_p)$. Also the last statement can be shown exactly as in the proof of this lemma of Yang, where we must of course use the universal coefficient theorem for cohomology, which says that the following sequence is exact (see [3]):

$$0 \to H^q(U; Z) \otimes L \to H^q(U; L) \to \mathrm{Tor}(H^{q+1}(U; Z), L) \to 0.$$

DEFINITION 1.9. *If* M *is an* (L, n)*-manifold and* $\{f_U; U \in \mathfrak{B}\}$ *is a collection of isomorphisms* $f_U\colon H^n(U; L) \to L$, *then* M *is said to be locally oriented by the collection* $\{f_U\}$.

NOTATION. (a) If U and V $(U \supset V)$ are fundamental sets of the (L, n)-manifold M, then $(U, V)_L$ denotes the natural isomorphism $H^n(V; L) \to H^n(U; L)$. We also put $(V, U)_L = (U, V)_L^{-1}$ and

$$(U_1, \cdots, U_m)_L = (U_1, U_2)_L (U_2, U_3)_L \cdots (U_{m-1}, U_m)_L,$$

when these are defined. Where no confusion can arise, we drop the subscript L.

If $V \subset U$ or $U \subset V$, where U and V are fundamental sets, then the pair U, V is said to be a *step*, and $U_1, U_2, \cdots, U_m$ is called a *path* if each pair U_i, U_{i+1} is a step.

(b) If M is a (Z, n)-manifold locally oriented by the maps f_U, we denote by M_L the corresponding (L, n)-manifold locally oriented by the maps $f_U \otimes L$, where we identify $H^n(U; L)$ with $H^n(U; Z) \otimes L$.

(c) We denote by $\mathfrak{A} = \mathfrak{A}_L$ the automorphism group of L.

(d) If M is an (L, n)-manifold locally oriented by the maps f_U, then we put $\langle U, V\rangle = f_U(U, V) f_V^{-1} \in \mathfrak{A}$, when this is defined. We also use the notation

$$\langle U_1, U_2, \cdots, U_m\rangle = \langle U_1, U_2\rangle \langle U_2, U_3\rangle \cdots \langle U_{m-1}, U_m\rangle .$$

(e) If U is a fundamental set of the (L, n)-manifold M, then we denote by $\mathfrak{A}(U, M)$ the subgroup of the automorphism group of $H^n(U; L)$ consisting of automorphisms of the form $(U, U_1, \cdots, U_m, U)$.

THEOREM 1.10. *If* U *is a fundamental set of the* (L, n)*-manifold* M, *then* $\operatorname{Ker}(H^n(U) \to H^n(M))$ *is the subgroup* $K(U, M)$ *of* $H^n(U)$ *generated by the subset* $(1 - \mathfrak{A}(U, M)) H^n(U)$ *of* $H^n(U)$. *Thus* $H^n(M) \approx H^n(U)/K(U, M)$.

Proof. Denote by $j_{X,Y}$ the natural homomorphism $H^n(Y) \to H^n(X)$ for an open subset Y of X. If P is a path from W to W' in V, then for convenience we shall also denote by P the homomorphism $H^n(W) \to H^n(W')$ induced by P. It is clear that $j_{V,W} = j_{V,W'} P$, and it follows easily that $K(U, V) \subset \operatorname{Ker}(H^n(U) \to H^n(V))$ for each fundamental set U of V. Clearly, by Zorn's lemma and the fact that we are using compact supports, there exists a maximal connected open set V such that $U \subset V \subset M$ and such that

$$\operatorname{Ker}(H^n(U) \to H^n(V)) = K(U, V) .$$

We shall show that $V = M$. If not, then let $p \in \overline{V} - V$, and let W be a fundamental set containing p. Let $g \in \operatorname{Ker} j_{V\cup W,U} - \operatorname{Ker} j_{V,U}$, and put $g' = j_{V,U}(g)$. By the Mayer-Vietoris sequence

$$H^n(V \cap W) \to H^n(V) \oplus H^n(W) \to H^n(V \cup W),$$

there exists an element $g'' \in H^n(V \cap W)$ such that $j_{V,V\cap W}(g'') = g'$, and

$$j_{W,V\cap W}(g'') = 0 .$$

Since $V \cap W$ is a disjoint union of fundamental sets, there exist a finite number of components X_i $(i = 0, 1, \cdots, k)$ of $V \cap W$ and elements $g_i \in H^n(X_i)$ such that $\Sigma\, j_{V\cap W,X_i}(g_i) = g''$.

Let P_i be a path from X_i to U in V, and let P_i' be the path X_i, W. Put $g_i^* = P_i g_i \in H^n(U)$, and consider the element

$$g^* = \sum (1 - P_0 P_0'^{-1} P_i' P_i^{-1})(g_i^*) \in K(U, V \cup W) .$$

We see immediately that

$$g^* = \sum P_i g_i - P_0 P_0'^{-1} \left(\sum P_i' g_i\right) = \sum P_i g_i - P_0 P_0'^{-1} (j_{W,V\cap W}(g'')) = \sum P_i g_i .$$

However, $j_{V,U}(g - \Sigma P_i g_i) = g' - j_{V,U}(\Sigma P_i g_i)$, and from the definition of P_i we see that $j_{V,U}(P_i g_i) = j_{V,X_i}(g_i)$. Thus

$$j_{V,U}(g - g^*) = g' - \sum j_{V,X_i}(g_i) = g' - j_{V,V\cap W}(g'') = g' - g' = 0.$$

Therefore $(g - g^*) \in K(U, V)$, and hence $g \in K(U, V) + K(U, V \cup W) = K(U, V \cup W)$, which implies that $\operatorname{Ker} j_{V\cup W,U} \subset K(U, V \cup W)$. However, we have already remarked that the reverse inclusion holds, and this contradicts the maximality of V and completes the proof.

COROLLARY 1.11. *A* (Z_2, n)-*manifold is always orientable, and a* (Z, n)-*manifold* M *is orientable or not according as* $H^n(M) \approx Z$ *or* $H^n(M) \approx Z_2$.

COROLLARY 1.12. *An* (L, n)-*manifold* M *is orientable if and only if* $\mathfrak{A}(U, M)$ *is trivial.*

Remark. The condition in Corollary 1.12 is Yang's definition of orientability in [11].

DEFINITION 1.13. *If* M *is an* (L, n)-*manifold locally oriented by the maps* f_U, *then* M *is said to be consistently oriented by the* f_U *if* $\langle U, V\rangle$ *is the identity element of* $\mathfrak{A}$ *whenever it is defined.*

Remarks: (a) Every consistently oriented (L, n)-manifold is orientable, and every orientable (L, n)-manifold can be consistently oriented.

(b) If the maps f_U are defined for U in a collection $\mathfrak{B}' \subset \mathfrak{B}$ forming a base for the topology of M, and if $\langle U, V\rangle$ is the identity element of $\mathfrak{A}$ when it is defined, then the collection $\{f_U\}$ can be uniquely extended to be defined for all $U \in \mathfrak{B}$ in such a manner that the new collection gives a consistent orientation of M. Thus we shall use the term "consistently oriented" in this case also.

2. THE FULL ORIENTABLE COVERING AND THE CONNECTED ORIENTABLE COVERING

We assume throughout this section that M is an (L, n)-manifold locally oriented by the maps f_U. Let $\mathfrak{M}$ be the set consisting of elements of the form (p, U, α), where $p \in U$, U is a fundamental set of M, and $\alpha \in \mathfrak{A}$. We say that

$$(p, U, \alpha) \sim (p, V, \beta)$$

if $V \subset U$ and $\beta = \alpha\langle U, V\rangle$. (Notice that $(p, U, \alpha) \sim (p, V, \beta)$ if and only if $(p, U, \gamma\alpha) \sim (p, V, \gamma\beta)$.) Now let E be the equivalence relation generated by the relation $\sim$, and define $M^* = \mathfrak{M}/E$ with the obvious topology.

We now show that M^* is a k-fold orientable covering space of M, where $k = \text{order } \mathfrak{A}$ (possibly infinite). In order to show that M^* is a k-fold covering space of M, it is clearly sufficient to show that if $U_1, U_2, \cdots, U_m, U_1$ is a path in M with $p \in U_i$ for all i, then $\langle U_1, U_2, \cdots, U_m, U_1\rangle$ is e, the identity element of $\mathfrak{A}$.

To prove this, let V be a fundamental set with $p \in V \subset U_i$, for all i. Then $\langle V, U_i\rangle\langle U_i, U_j\rangle = \langle V, U_j\rangle$ for all i, j for which $\langle U_i, U_j\rangle$ is defined. Hence $\langle V, U_1\rangle\langle U_1, U_2, \cdots, U_m, U_1\rangle = \langle V, U_1\rangle$, from which the assertion follows.

If U is a fundamental set of M and $\alpha \in \mathfrak{A}$, let (U, α) denote the fundamental set of M^* which is the image in M^* of the subset $\bigcup_{p\in U} (p, U, \alpha)$ of $\mathfrak{M}$.

Note that the action of $\alpha \in \mathfrak{A}$ on the left of $\mathfrak{M}$ induces an action of α on the left of M^*, and that this action permutes the sheets of M^*. We also denote this action by

α, and we call it the deck transformation induced by α. Also, we denote the covering projection of M^* onto M by π.

We shall consider U_0 to be a fixed fundamental set of M, for the remainder of this section. Let $U_0^* = (U_0, e)$, and let M^0 be the component of M^* containing U_0^*. We shall exhibit a consistent orientation of M^0.

If U^* is a fundamental set of M^0 such that $\pi(U^*) = U$ is a fundamental set of M, then let $f_{U^*} = f_U(\pi^*)^{-1}$, where $\pi^*\colon H^n(U) \to H^n(U^*)$ is the isomorphism induced by π. Note that the deck transformations must preserve this collection of local orientations. Let U^*, V^* be a step in M^0 such that $\pi(U^*) = U$ and $\pi(V^*) = V$ are fundamental sets of M. Then

$$f_{V^*} = f_V \pi^{*-1} = \langle V, U\rangle f_U(U, V)\pi^{*-1} = \langle V, U\rangle f_{U^*}(U^*, V^*). \tag{1}$$

Let $U_0^*, U_1^*, \cdots, U_m^*$ be a path in M^0 such that the $\pi(U_i^*) = U_i$ are fundamental sets of M. Then from (1) we obtain

$$f_{U_m^*} = \langle U_m, \cdots, U_0\rangle f_{U_0^*}(U_0^*, \cdots, U_m^*). \tag{2}$$

We now define

$$g_{U_m^*} = f_{U_0^*}(U_0^*, \cdots, U_m^*) = \langle U_0, \cdots, U_m\rangle f_{U_m^*}, \tag{3}$$

where the latter equality follows from (2). We must show that this definition is consistent.

Now suppose that $V_1^*, \cdots, V_r^*$ is a path in M^0 such that the $\pi(V_i^*) = V_i$ are fundamental sets of M, and suppose further that $\pi(V_1^*) = \pi(V_r^*) = V$.

Since $f_{V_r^*}$ is also the orientation induced from $f_{V_1^*}$ on V_1^* by the deck transformation $\alpha(V_r^*, V_1^*)$ which takes V_1^* onto V_r^*, we see that

$$f_{V_r^*} = f_{V_1^*}\alpha^*(V_1^*, V_r^*), \tag{4}$$

where $\alpha^*(V_1^*, V_r^*)\colon H^n(V_r^*) \to H^n(V_1^*)$ is the isomorphism induced by $\alpha(V_r^*, V_1^*)$.

Now let $U_0^*, U_1^*, \cdots, V_1^*$ be a path from U_0^* to V_1^* such that the $\pi(U_i^*) = U_i$ are fundamental sets of M. We set

$$f_P = (U_0^*, U_1^*, \cdots, V_1^*) \qquad \text{and} \qquad \beta = \langle U_0, U_1, \cdots, V_1\rangle. \tag{5}$$

We then see from (3) that $f_{V_1^*} = \beta^{-1} g_{V_1^*}$. Moreover, by (3) and (4),

$$\begin{aligned} g_{V_r^*} &= \beta\langle V_1, V_2, \cdots, V_r\rangle f_{V_r^*} = \beta\langle V_1, \cdots, V_r\rangle f_{V_1^*}\alpha^*(V_1^*, V_r^*) \\ &= \beta\langle V_1, \cdots, V_r\rangle \beta^{-1} g_{V_1^*}\alpha^*(V_1^*, V_r^*). \end{aligned}$$

Notice that since $U_0^* = (U_0, e)$, we have $V_1^* = (V_1, \beta) = (V, \beta)$ and also

$$V_r^* = (V_r, \beta<V_1, \cdots, V_r>) = (V, \beta<V_1, \cdots, V_r>).$$

However, $(V, \beta<V_1, \cdots, V_r>) = \beta<V_1, \cdots, V_r>\beta^{-1}(V, \beta)$. Therefore the deck transformation taking V_1^* onto V_r^* is

$$\alpha(V_r^*, V_1^*) = \beta<V_1, \cdots, V_r>\beta^{-1}. \tag{6}$$

Thus we have

$$g_{V_r^*} = \alpha(V_r^*, V_1^*)\, g_{V_1^*}\, \alpha^*(V_1^*, V_r^*). \tag{7}$$

Equation (6) shows, in particular, that $<V_1, \cdots, V_r> = e$ if and only if $V_1^* = V_r^*$, which implies that the g_{V^*} are well-defined.

We can now extend the local orientation given by the g_{V^*} to all of M^* by use of (7). Note that with this local orientation of M^*, the orientations on a fundamental set U^* induced by the deck transformations give every possible orientation of U^* exactly once. Clearly, by their definition, the g_{V^*} give a consistent orientation of M^*.

DEFINITION 2.1. M^* *with the orientation defined by the* g_{U^*} *is called the full oriented covering of* M *with respect to* L. *The connected component* M^0 *with the induced orientation is called the connected oriented covering of* M *with respect to* L. *If* M *is a* (Z, n)*-manifold, then we put* $M_L^* = (M_L)^*$ *and* $M_L^0 = (M_L)^0$.

Remark. We leave to the reader the straightforward task of verifying that M^* and the maps g_{U^*} are essentially independent of the choice of the f_U, and also that if $\mathfrak{B}' \subset \mathfrak{B}$ is a base for the topology of M, then M^* can be constructed with reference only to sets in $\mathfrak{B}'$.

DEFINITION 2.2. *We define* $\mathfrak{A}(M)$ *to be the subgroup of* $\mathfrak{A}$ *consisting of the automorphisms* $<U_0, U_1, \cdots, U_j, U_0>$.

Remark. $\mathfrak{A}(M)$ depends on the choice of the basic set U_0. However, if we change basic sets, then $\mathfrak{A}(M)$ changes by an inner automorphism of $\mathfrak{A}$. Note also that the set of deck transformations induced by elements of $\mathfrak{A}(M)$ consists exactly of the deck transformations of M^0, and thus that $\mathfrak{A}(M)$ actually depends only on the choice of the component M^0 of M^*.

3. AN ALTERNATE DEFINITION OF M^0

For an (L, n)-manifold M which admits a simply connected covering space M', we can define M^0 as follows.

First, we orient M' as we oriented M^0 in Section 2. (This is a consistent method, since M' is simply connected.) Next, we let $\mathfrak{D}$ be the set of deck transformations of M' which preserve orientation. Then M^0 is seen to be canonically isomorphic with $M'/\mathfrak{D}$. We leave the details to the reader, since we shall not be concerned with this fact in the rest of this paper.

4. THE ORIENTABLE DOUBLE COVERING

Let M be a (Z, n)-manifold. Then, by Section 1, M is also an (L, n)-manifold denoted by M_L. The full orientable covering M* of M with respect to Z has two sheets, since the automorphism group of Z is of order two.

DEFINITION 4.1. *If* M *is a* (Z, n)-*manifold, then we say that* M* *is the oriented double covering of* M, *and we denote it by* M^d. $M^d_L = (M^d)_L$ *is called the oriented double covering of* M_L.

Example. Let M be the projective plane. Then the full orientable covering of M with respect to Z_p is a collection of (p - 1)/2 2-spheres if p is odd, and it is the projective plane if p = 2. The orientable double covering M^d of M is the 2-sphere.

LEMMA 4.2. *If the* (Z, n)-*manifold* M *is consistently oriented by the maps* g_U, *then* M_L *is consistently oriented by the maps*

$$g_U \otimes L: H^n(U; L) \approx H^n(U; Z) \otimes L \to Z \otimes L \approx L,$$

where, to simplify notation, we consider the isomorphisms on the ends to be identities.

Proof. The condition on the g_U is that the diagram

$$\begin{array}{ccc} H^n(V; Z) & \xrightarrow{(U, V)} & H^n(U; Z) \\ & {\scriptstyle g_V} \searrow \quad \swarrow {\scriptstyle g_U} & \\ & Z & \end{array}$$

is commutative, when it is defined. The lemma follows on tensoring this diagram with L.

THEOREM 4.3. *If* M *is a* (Z, n)-*manifold, then* M^0_L *has either one or two sheets. Furthermore,* M^0_L *is canonically isomorphic to* M_L *or to* M^d_L, *according as* M_L *is orientable* (*with respect to* L) *or not.*

Proof. Let f_U: $H^n(U; Z) \to Z$ be given isomorphisms for fundamental sets U. Then, tensoring with L (see Notation (b) of Section 1), we calculate immediately that $\langle U, V\rangle_L = \langle U, V\rangle_Z \otimes L$. Let $U_0, \cdots, U_m, U_0$ be a path in M. Then we see that

$$\langle U_0, \cdots, U_m, U_0\rangle_L = \langle U_0, \cdots, U_m, U_0\rangle_Z \otimes L.$$

It follows that $\mathfrak{A}_L(M)$ is of order at most two. Moreover, if $\mathfrak{A}_L(M)$ is of order two, then the above canonical correspondence between $\mathfrak{A}_L(M)$ and $\mathfrak{A}_Z(M)$ clearly yields a canonical isomorphism between M^0_L and M^d_L. The theorem follows.

Remark. It is unknown to the author whether or not M^0 is always a single or double covering of M for general (L, n)-manifolds.

5. LIFTING OF HOMEOMORPHISMS AND OF TRANSFORMATION GROUPS

In this section, M is an (L, n)-manifold, and M* is its full oriented covering.

DEFINITION 5.1. *If* X *is an* (L, n)-*manifold, then we denote by* G(X) *the semigroup of all local homeomorphisms of* X *into itself. If* X *is consistently oriented, then we denote by* $G^+(X)$ *the subsemigroup of* G(X) *consisting of orientation-preserving local homeomorphisms.*

THEOREM 5.2. *There is a unique map* $g \to g^*$ *of* G(M) *into* $G^+(M^*)$ *such that* $\pi g^* = g\pi$. *This map is an isomorphism of* G(M) *onto the subsemigroup of* $G^+(M^*)$ *consisting of local homeomorphisms* h *with the property that* $\pi(x) = \pi(y)$ *implies that* $\pi h(x) = \pi h(y)$. *Moreover, if* $g \in G(M)$, *then* $\alpha g^* = g^* \alpha$ *for all deck transformations* α.

Proof. Let g be a member of G(M), and let U* be a fundamental set of M* such that $\pi(U^*) = U$ is a fundamental set of M and is so small that g is a homeomorphism on U. We shall define g* on U*. Note that g(U) is a fundamental set of M. Let $\{V_s^*\}$ be the collection of fundamental sets of M* such that $\pi(V_s^*) = g(U)$. There exists a unique homeomorphism $\pi_s: g(U) \to V_s^*$ such that $\pi\pi_s$ is the identity. Let $g_s = \pi_s g\pi$. These homeomorphisms of U* onto the V_s differ by deck transformations of M*, and hence there is exactly one (call it g_0) which is orientation-preserving. We let g* be g_0 on U*. These definitions are clearly consistent with one another, and they define g* on all of M*.

The fact that $g \to g^*$ is a homomorphism on G(M) follows from the unicity of the transformations g* and the fact that g^*h^* preserves orientation when g* and h* do. That the map is onto the indicated set follows from the fact that any such local homeomorphism h defines by projection a homeomorphism h^π of M and we must have $(h^\pi)^* = h$ by unicity. The last statement in the theorem follows from the fact that $\alpha g^* \alpha^{-1}$ preserves orientation.

DEFINITION 5.3. *If* G *is a group of homeomorphisms of* M *onto itself, then we denote by* G* *the group of homeomorphisms* $\{g^*; g \in G\}$ *of* M*. *The map* $g \to g^*$ *then gives an isomorphism of* G *onto* G*.

THEOREM 5.4. *If* g *is an orientation-preserving local homeomorphism of a locally oriented* (Z, n)-*manifold* M *into itself, then* g *is orientation-preserving on* M_L. *(Note that we do not require* M *to be orientable, but only that maps* $f_U: H^n(U) \to Z$ *be given.)*

Proof. The proof follows on tensoring the following commutative diagram with L:

$$\begin{array}{ccc} H^n(gU; Z) & \xrightarrow{g} & H^n(U; Z) \\ & {}_{f_{gU}}\searrow \quad \swarrow_{f_U} & \\ & Z & \end{array}$$

DEFINITION 5.5. *If* M *is a* (Z, n)-*manifold and if* $g \in G(M)$, *then the induced map* g* *on* M^d *is denoted by* g^d.

COROLLARY 5.6. *If* M *is a* (Z, n)-*manifold and if* $g \in G(M)$, *then* g^d *is orientation-preserving on* M^d_L.

COROLLARY 5.7. *If* M *is a* (Z, n)-*manifold and* M_L *is not orientable, then the connected oriented covering* M_L^0 *of* M_L *is invariant under* g^* *for each* $g \in G(M)$.

Proof. In this case, M_L^0 is canonically isomorphic to M_L^d. Thus, since the action of g^d on M_L^d is orientation-preserving, this action is equivalent to that of g^* on M_L^0.

6. PRESERVATION OF TOPOLOGY

Let G be a topological transformation group on M. Then G^* is a transformation group on M^*, and we should like to know whether or not the topology induced on G^* by that on G will make G^* into a topological transformation group. We shall establish the following partial answer:

THEOREM 6.1. *If* k = order $\mathfrak{A}$ *is finite, then each of the following conditions is sufficient to insure that* G^* *is a topological transformation group with the topology induced from* G.

(1) G *is a Lie group.*

(2) *Each neighborhood* N *of* e *in* G, *contains a neighborhood* N' *of* e *in* G *such that for every element* $g \in N'$, *there exists an element* h *such that* $h^k = g$ *and* $h^i \in N$ *for* $i = 1, 2, \cdots, k$.

(3) M *is locally euclidean.*

Proof. The sufficiency of (1) follows from that of (2). To prove the sufficiency of (2), let $V = \pi(V^*)$ be a fundamental set of M, and let $y \in V^*$. Let $y \in U^* \subset V^*$ and $e \in N \subset G$ be such that $N(U) \subset V$ and $\bigcap_{t \in N} t(U) \neq \emptyset$, where $U = \pi(U^*)$. We shall show that if N' is as in (2), then $(N')^*(U^*) \subset V^*$, which will give our result. Let $g \in N'$ and $h \in N$, as hypothesized in (2). Then, since $h(U) \subset V$, there exists exactly one deck transformation $\alpha \in \mathfrak{A}$ such that $\alpha h^*(U^*) \subset V^*$. Put $h^{*\prime} = \alpha h^*$. Since $U \cap h(U)$ is not empty, we see that $U^* \cap h^{*\prime}(U^*)$ is not empty, and thus also

$$h^{*\prime}(U^*) \cap (h^{*\prime})^2(U^*)$$

is not empty, and therefore $(h^{*\prime})^2(U^*) \subset V^*$. Continuing in this way, we see finally that

$$g^*(U^*) = (h^*)^k (U^*) = \alpha^k (h^*)^k (U^*) = (h^{*\prime})^k (U^*) \subset V^*,$$

which was our assertion.

The sufficiency of (3) follows from the fact that every sufficiently small homeomorphism of a locally euclidean space is locally homotopic to the identity and, therefore, must preserve orientation locally.

Remark. It is not known to the author whether G^* is always a topological transformation group with the topology induced from G; but it seems likely that this is the case.

7. GROUPS OF PRIME ORDER

We concern ourselves, in this section, with groups G of prime order p acting on a (Z_p, n)-manifold M. First of all we give a proof, in modern terminology, of Smith's theorem that every component F of the set F' of fixed points of G is open in F' and is a (Z_p, r)-manifold for some $r < n$. From this proof we also obtain that the orientation maps f_U on M induce, in a natural way, orientation maps $f_{U \cap F}$ for F, at least for a base of fundamental sets $U \cap F$ of F. It then follows that the full orientable covering of F is the subset $\pi^{-1}(F) \subset M^*$, and this result is applied to get a generalization of Smith's theorem that F is orientable whenever M is orientable. We also obtain by our technique the result that if p is odd then n - r is even, which, of course, is analogous but not equivalent to known theorems of Smith [10], Floyd [4], and Borel [2]. We also obtain an analogue of Liao's theorem ([7]) which states that if M is a (Z, n)-manifold on which a transformation T of period two acts, then n - r is even or odd according as T preserves orientation near F or not. We use, throughout this section, the notation G, M, F, and F' as described above.

We shall assume, without giving proofs, the following facts about the Smith theory of transformations of prime period. (Proofs have been obtained by E. E. Floyd, and they will appear in a book by him and P. E. Conner. In the book, the theory will probably be phrased in the language of sheaves; but this is immaterial here. The present author is deeply indebted to Professor Floyd for allowing him to read some preliminary material for this book.) There are the exact sequences of cochain complexes for an open invariant subset V of M:

$$0 \to A_\tau \to A \to A_\sigma \oplus B \to 0,$$

$$0 \to A_\sigma \to A \to A_\tau \oplus B \to 0,$$

which induce the exact cohomology sequences

$$\cdots \to H^k_\rho(V) \xrightarrow{i^*} H^k(V) \xrightarrow{j^*} H^k_{\bar\rho}(V) \oplus H^k(V \cap F') \xrightarrow{\delta^*} H^{k+1}_\rho(V) \to \cdots,$$

where ρ denotes one of σ and τ while $\bar\rho$ denotes the other. The letter τ stands for the cochain map $1 - g^*$, where g is a fixed generator of G, and

$$\sigma = \tau^{p-1} = 1 + g^* + g^{*2} + \cdots + g^{*(p-1)} = \sum_{h \in G} h^*.$$

A_ρ is the group of cochains of the form ρx for $x \in A$, and thus $A_\sigma \subset A_\tau \subset A$. The maps $A_\rho \to A$ in the cochain sequences are inclusions, and the maps $A \to A_\rho \oplus B$ are the direct sums of operation by ρ and restriction to F'. Also,

$$H^k_\sigma(V) \approx H^k((V - F')/G),$$

where $(V - F')/G$ is the orbit space of V - F' under the action of G, so that it is a (Z_p, n)-manifold. Except for the proofs of Lemmas 7.1 and 7.11 and of Theorems 7.7 and 7.8, the only part of the information above which we shall use is the existence of the Smith sequences. We shall use η to denote ρ or $\bar\rho$ when the information which we have is not sufficient to determine which it should be. Usually, η will depend on the parity of n or of n - r.

LEMMA 7.1. *If* V *is a fundamental set, then the homomorphism* $H^n(V) \to H^n_\rho(V)$ *is an isomorphism onto for both* $\rho = \sigma$ *and* $\rho = \tau$. (The author is indebted to E. E. Floyd for pointing out this fact to him.)

Proof. Since $H^k_\sigma(V) = H^k((V - F')/G) = 0$ for $k > n$, it follows from the Smith sequence that $H^k_\tau(V) = 0$ also for $k > n$. Thus $H^n(V) \to H^n_\rho(V) \to 0$ is exact. Thus it suffices to show that $H^n_{\bar\rho}(V) \to H^n(V)$ is trivial, and hence it also suffices to show that the composition $H^n(V) \to H^n_{\bar\rho}(V) \to H^n(V)$ is trivial, since the first map is onto. But this composition is operation by $\bar\rho$ on $H^n(V)$, and this is a power of $\tau = 1 - g^*$. Hence it suffices to show that g induces the identity automorphism of $H^n(V) \approx Z_p$; but there are exactly $p - 1$ automorphisms of Z_p, while g is of order p. Thus g cannot be a nontrivial automorphism of Z_p, and this completes the proof.

LEMMA 7.2. *Let* $U_0 \supset U_1 \supset \cdots \supset U_{n+1}$ *be open invariant subsets of* M *such that the maps* $H^j(U_i) \to H^j(U_{i-1})$ *are trivial for all* i *and for all* $j \neq n$. *Then, for* $0 < k < n$,

$$\operatorname{Im}(H^k_\rho(U_{k+1}) \to H^k_\rho(U_0)) \subset \operatorname{Im}(H^{k-1}_{\bar\rho}(U_k) \oplus H^{k-1}(U_k \cap F') \to$$

$$H^{k-1}_{\bar\rho}(U_0) \oplus H^{k-1}(U_0 \cap F') \to H^k_\rho(U_0)).$$

Proof. This is clear from the following commutative diagram, in which the rows are exact:

$$\begin{array}{ccccc}
 & & H^k_\rho(U_{k+1}) & \to & H^k(U_{k+1}) \\
 & & \downarrow & & \downarrow 0 \\
H^{k-1}_{\bar\rho}(U_k) \oplus H^{k-1}(U_k \cap F') & \to & H^k_\rho(U_k) & \to & H^k(U_k) \\
\downarrow & & \downarrow & & \\
H^{k-1}_{\bar\rho}(U_0) \oplus H^{k-1}(U_0 \cap F') & \to & H^k_\rho(U_0). & &
\end{array}$$

LEMMA 7.3. *Let* $U_0 \supset U_1 \supset \cdots \supset U_n$ *be open invariant subsets of* M *such that the maps* $H^j(U_i) \to H^j(U_{i-1})$ *are trivial for all* i *and for all* $j \neq n$. *Then, for* $0 \leq k < n$,

$$\operatorname{Ker}(H^k(U_n \cap F') \oplus H^k_\rho(U_n) \to H^{k+1}_{\bar\rho}(U_n) \to H^{k+1}_{\bar\rho}(U_{k+1}))$$

$$\subset \operatorname{Ker}(H^k(U_n \cap F') \oplus H^k_\rho(U_n) \to H^k(U_k \cap F') \oplus H^k_\rho(U_k)).$$

Proof. This is clear from the following commutative diagram, in which the rows are exact:

$$\begin{array}{ccccc}
 & & H^k(U_n \cap F') \oplus H^k_\rho(U_n) & \longrightarrow & H^{k+1}_{\bar\rho}(U_n) \\
 & & \downarrow & & \downarrow \\
H^k(U_{k+1}) & \to & H^k(U_{k+1} \cap F') \oplus H^k_\rho(U_{k+1}) & \to & H^{k+1}_{\bar\rho}(U_{k+1}) \\
\downarrow 0 & & \downarrow & & \\
H^k(U_k) & \longrightarrow & H^k(U_k \cap F') \oplus H^k_\rho(U_k). & &
\end{array}$$

THEOREM 7.4. F' *is locally connected, and every component* F *of* F' *is a* (Z_p, r)-*manifold for some* $r < n$.

Proof. It is impossible to choose the sequence $U_0 \supset \cdots \supset U_{n+1}$ of fundamental sets as in Lemma 7.2 such that the maps $H^j(U_i \cap F') \to H^j(U_{i-1} \cap F')$ are trivial for all i and j, since, by Lemma 7.1 and by inductive use of Lemma 7.2, this would imply that

$$0 \neq \operatorname{Im}(H^n_\rho(U_{n+1}) \to H^n_\rho(U_0)) \subset \operatorname{Im}(H^0_\eta(U_1) \to H^0_\eta(U_0) \to H^n_\rho(U_0)),$$

where $H^0_\eta(U_0) \to H^n_\rho(U_0)$ is a composition of the connecting homomorphisms of the Smith sequences, and therefore we would have $H^0_\eta(U_1) \neq 0$. However, U_1 cannot be compact, since otherwise U_0 would not be connected; thus $H^0(U_1) = 0$, and the sequence $0 \to H^0_\eta(U_1) \to H^0(U_1)$ yields a contradiction.

Thus, if $x \in F'$, then there exists a sequence $U_0 \supset \cdots \supset U_n$ of fundamental sets which are neighborhoods of x in M, as in Lemma 7.3, and also such that for some $r < n$ and for all open sets $W \subset U_n$ with $W \cap F' \neq \emptyset$, the map $H^r(W \cap F') \to H^r(U_0 \cap F')$ is nontrivial. By inductive use of Lemma 7.3, we then have

$$\begin{aligned} &\operatorname{Ker}(H^r(U_n \cap F') \oplus H^r_\rho(U_n) \to H^n_\eta(U_n)) \\ &\subset \operatorname{Ker}(H^r(U_n \cap F') \oplus H^r_\rho(U_n) \to H^r(U_r \cap F') \oplus H^r_\rho(U_r)) \\ &< H^r(U_n \cap F') \oplus H^r_\rho(U_n), \end{aligned}$$

where $<$ stands for strict inclusion. However, since $H^n_\eta(U_n) \approx H^n(U_n) \approx Z_p$, it follows that both of the kernels above are of deficiency one, and therefore that

$$\operatorname{Im}(H^r(W \cap F') \to H^r(U_r \cap F')) = \operatorname{Im}(H^r(U_n \cap F') \to H^r(U_r \cap F')) \approx Z_p$$

and that $\operatorname{Im}(H^r_\rho(U_n) \to H^r_\rho(U_r)) = 0$. Moreover, this, together with Lemma 7.3, shows that $\operatorname{Im}(H^k_\rho(U_n) \to H^k_\rho(U_k)) \neq 0$ for $r < k \leq n$, which implies that the r satisfying our demands is unique.

Summing up, we see that for every sufficiently small fundamental set U of M and every point $x \in U \cap F'$, there exists a neighborhood V of x in M and an integer r, such that if W is an open subset of V with $W \cap F' \neq \emptyset$, then

$$\operatorname{Im}(H^k(W \cap F') \to H^k(U \cap F')) \approx \begin{cases} 0 & \text{if } k \neq r, \\ Z_p & \text{if } k = r. \end{cases}$$

Clearly, the integer r is uniquely associated with the point x and does not depend on U and V. It is also clear that the resulting function $r = r(x)$ is constant on $V \cap F'$. Thus, by Corollary 1.4, if we can show that F' is locally connected, it will follow that the component F of F' containing x is a (Z_p, r)-manifold, and the proof of our theorem will be complete.

We may assume that $N = \bar{U}$ is compact and that $\bar{W} \subset U$, where U and W are as above. We claim that $W \cap F'$ is contained in a single component of $N \cap F'$. If not, then, since a component of a compact set is the intersection of its open-closed neighborhoods, there exist two relatively open, disjoint subsets N_1 and N_2 of N, with $(N_1 \cup N_2) \cap F' = N \cap F'$, such that the $N_i \cap W \cap F'$ are both nonempty. Thus, putting $W_i = W \cap N_i$, we see that, since the $N_i \cap U \cap F'$ are disjoint open sets with union $U \cap F'$,

$$\begin{aligned} Z_p &\approx \operatorname{Im}(H^r(W \cap F') \to H^r(U \cap F')) \\ &\approx \operatorname{Im}(H^r(W_1 \cap F') \to H^r(U \cap F')) \oplus \operatorname{Im}(H^r(W_2 \cap F') \to H^r(U \cap F')) \\ &\approx Z_p \oplus Z_p, \end{aligned}$$

which is a contradiction and completes the proof.

THEOREM 7.5. *If we let* $\mathfrak{B}'$ *be the collection of fundamental sets* U *of* M *such that either* $F \cap U = \emptyset$ *or* U *is invariant and* $U \cap F = U \cap F'$ *is connected, then* $\mathfrak{B}'$ *is a base for the topology of* M, *and* $\{U \cap F;\ U \in \mathfrak{B}'\}$ *is a base for the topology of* F. *Moreover, if* $U \in \mathfrak{B}'$ *with* $U \cap F \neq \emptyset$, *then the homomorphism* $\Lambda_U\colon H^r(U \cap F) \to H^n(U)$, *given by* $\Lambda_U = j^{*-1}\Delta_U$, *is an isomorphism; here* j^* *denotes the isomorphism* $H^n(U) \to H^n_\eta(U)$, *and* Δ_U *is the composition*

$$H^r(U \cap F) \to H^{r+1}_\rho(U) \to H^{r+2}_{\bar\rho}(U) \to \cdots \to H^n_\eta(U)$$

of connecting homomorphisms in the Smith sequences.

Proof. The first statement, easily verified, is left to the reader. Let

$$V = U_0 \supset U_1 \supset \cdots \supset U_{n+1}$$

be as in Lemma 7.2, such that $U_i \in \mathfrak{B}'$ and $U_{n+1} \cap F \neq \emptyset$ and such that

$$H^k(U_i \cap F) \to H^k(U_{i-1} \cap F)$$

is trivial for $k \neq r$. If V is sufficiently small, then the $U_i \cap F$ are fundamental sets of F, and the inductive use of Lemma 7.2 gives the relations

$$H^n_\eta(V) = \operatorname{Im}(H^n_\eta(U_{n+1}) \to H^n_\eta(V)) \subset \operatorname{Im}(H^r(V \cap F) \xrightarrow{\Delta_V} H^n_\eta(V)),$$

and thus Δ_V is onto. However, $H^r(V \cap F) \approx Z_p$, and therefore Δ_V is an isomorphism for V sufficiently small. The conclusion follows for arbitrary $U \in \mathfrak{B}'$ by consideration of the diagram

$$\begin{array}{ccc} H^r(V \cap F) & \xrightarrow[\approx]{\Delta_V} & H^n_\eta(V) \\ \downarrow & & \downarrow \approx \\ H^r(U \cap F) & \xrightarrow{\Delta_U} & H^n_\eta(U). \end{array}$$

Remark. There are two homomorphisms Λ_U, obtained by starting with either $\rho = \sigma$ or τ. These will be shown to be identical in Theorem 7.7.

We collect some important properties of the Λ_U:

COROLLARY 7.6. *If* $V \subset U$ *are both in* $\mathfrak{B}'$, *with* $V \cap F \neq \emptyset$, *then*

$$\Lambda_U(U \cap F, V \cap F) = (U, V)\Lambda_V.$$

Thus, if we define $f_U = f_{U \cap F}\Lambda_U^{-1}$ *for* $U \in \mathfrak{B}'$ *and* $U \cap F \neq \emptyset$, *and arbitrarily for* $U \in \mathfrak{B}'$ *and* $U \cap F = \emptyset$, *where the* $f_{U \cap F}$ *are given local orientation maps of* F, *then the induced automorphisms* $\langle U, V\rangle$ *and* $\langle U \cap F, V \cap F\rangle$ *of* Z_p *are identical.*

THEOREM 7.7. *The isomorphism* Λ_U *is independent of the choice of* ρ. *Moreover, if* $p > 2$ *and* $F \neq \emptyset$, *then* $n - r$ *is even.*

Proof. The theorem is trivial for $p = 2$, since $\sigma = \tau$ in this case. Thus we assume that $p > 2$.

Denote by s the inclusion maps $A_\sigma \to A_\tau$, $A \to A$, $B \to B$, and by t the maps $A_\tau \to A_\sigma$, $A \to A$, $B \to B$ induced by operation by τ^{p-2}. We shall also use s and t for the induced maps in cohomology. The following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \to & A_\sigma & \to & A & \to & A_\tau \oplus B & \to & 0 \\
 & & \downarrow s & & \downarrow s & & \downarrow t+s & & \\
0 & \to & A_\tau & \to & A & \to & A_\sigma \oplus B & \to & 0 \\
 & & \downarrow t & & \downarrow t & & \downarrow s+t & & \\
0 & \to & A_\sigma & \to & A & \to & A_\tau \oplus B & \to & 0.
\end{array}$$

Thus we have the commutative cohomology diagram

$$\begin{array}{ccccccccccc}
\cdots & \to & H^i_\sigma(U) & \to & H^i(U) & \to & H^i_\tau(U) \oplus H^i(F\cap U) & \to & H^{i+1}_\sigma(U) & \to & \cdots \\
 & & \downarrow s & & \downarrow s & & \downarrow t+s & & \downarrow s & & \\
\cdots & \to & H^i_\tau(U) & \to & H^i(U) & \to & H^i_\sigma(U) \oplus H^i(F\cap U) & \to & H^{i+1}_\tau(U) & \to & \cdots \\
 & & \downarrow t & & \downarrow t & & \downarrow s+t & & \downarrow t & & \\
\cdots & \to & H^i_\sigma(U) & \to & H^i(U) & \to & H^i_\tau(U) \oplus H^i(F\cap U) & \to & H^{i+1}_\sigma(U) & \to & \cdots.
\end{array}$$

The homomorphism s is the identity on $H^i(U)$ and $H^i(F \cap U)$, and t is trivial on $H^n(U)$ and $H^i(F\cap U)$ (since $t = \tau^{p-2}$, $\tau = 1 - g^*$ and g^*, being an automorphism of Z_p of period p, must be the identity).

Thus we have the commutative diagram

$$\begin{array}{ccccccccccccc}
H^r(F\cap U) & \to & H^{r+1}_\sigma(U) & \to & H^{r+2}_\tau(U) & \to & H^{r+3}_\sigma(U) & \to & \cdots & \to & H^n_\rho(U) & \overset{\approx}{\leftarrow} & H^n(U) \\
\downarrow s & & \downarrow s & & \downarrow t & & \downarrow s & & & & \downarrow \bar{f} & & \downarrow f \\
H^r(F\cap U) & \to & H^{r+1}_\tau(U) & \to & H^{r+2}_\sigma(U) & \to & H^{r+3}_\tau(U) & \to & \cdots & \to & H^n_{\bar\rho}(U) & \overset{\approx}{\leftarrow} & H^n(U),
\end{array}$$

where $f = s$ or $f = t$ according as $n - r$ is even or odd, and where the compositions in the rows are the isomorphisms Λ_U. Thus f cannot be trivial, and it follows that $n - r$ is even, as claimed. The first statement of the theorem also follows directly from this diagram.

The following theorem proves the analogue of Liao's theorem. We also obtain, in the proof, the local groups in the orbit space, and this result is incorporated in the statement of the theorem. The last statement of the theorem is due to Yang [11].

For any invariant subspace S of M, we denote by S' the orbit space S/G of S.

THEOREM 7.8. *Let* M *be a* (Z, n)-*manifold. If* $x \in F$ *and* $N \subset M$ *is a neighborhood of* x, *then there are open neighborhoods* $V \subset U \subset N$ *of* x *in* M *such that if* $W \subset V$ *is an open neighborhood of* x, *then*

$$\text{(i)}\ G^j = \operatorname{Im}(H^j(W'; Z) \to H^j(U'; Z)) \approx \begin{cases} Z_p & \textit{for } j = r + 2k + 1 \leq n,\ k \geq 1, \\ Z & \textit{for } j = n \textit{ and } n - r \textit{ even}, \\ 0 & \textit{otherwise}. \end{cases}$$

(ii) $H^j = \mathrm{Im}(H^j(W'; Z_p) \to H^j(U'; Z_p)) \approx \begin{cases} Z_p & \textit{for } r + 2 \leq j \leq n, \\ 0 & \textit{otherwise}. \end{cases}$

(iii) *The sequence* $\cdots \to G^j \xrightarrow{p} G^i \to H^j \to G^{j+1} \to \cdots$, *induced by the coefficient sequence* $0 \to Z \xrightarrow{p} Z \to Z_p \to 0$, *is exact.*

If $p = 2$, *then* $n - r$ *is even or odd according as the generator* T *of* G *preserves or reverses orientation.*

If $p = 2$ *and* $r = n - 1$, *then* F *is a* $(Z, n - 1)$*-manifold.*

Proof. We first deal with the case $r = n - 1$, and thus also $p = 2$. Let $U \subset V$ be in $\mathfrak{B}'$ (for coefficients in Z_2) and such that the map $H^j(V; L) \to H^j(U; L)$ is trivial for $j \neq n$ and $L = Z$ or $L = Z_2$. The diagram

$$\begin{array}{ccccccccc} H^{n-1}(V; L) & \to & H^{n-1}(F \cap V; L) & \to & H^n(V - F; L) & \to & H^n(V; L) & \to & 0 \\ \downarrow 0 & & \downarrow & & \downarrow & & \downarrow & & \\ H^{n-1}(U; L) & \to & H^{n-1}(F \cap U; L) & \to & H^n(U - F; L) & \to & H^n(U; L) & \to & 0 \end{array}$$

implies that for $L = Z_2$ the sequence $0 \to Z_2 \to H^n(U - F; Z_2) \to Z_2 \to 0$ is exact and hence that $H^n(U - F; Z_2) \approx Z_2 \oplus Z_2$. Thus $U - F$ consists of exactly two components U_1 and U_2. Now T must interchange these two components, since otherwise T could be altered on U to give an involution with fixed point set $\overline{U}_1$, which would clearly be contrary to our general facts about fixed point sets of transformations of prime period.

If we now take coefficients in $L = Z$, the diagram above becomes

$$\begin{array}{ccccccccc} H^{n-1}(V; Z) & \to & H^{n-1}(F \cap V; Z) & \to & Z \oplus Z & \to & Z & \to & 0 \\ \downarrow 0 & & \downarrow & & \downarrow \approx & & \downarrow \approx & & \\ H^{n-1}(U; Z) & \to & H^{n-1}(F \cap V; Z) & \to & Z \oplus Z & \to & Z & \to & 0. \end{array}$$

If we put $G = \mathrm{Im}(H^{n-1}(F \cap V; Z) \to H^{n-1}(F \cap U; Z))$, then this diagram clearly implies that $0 \to G \to Z \oplus Z \to Z \to 0$ is exact, and therefore that $G \approx Z$. It is also easy to see that under our conditions G depends only on U. Thus, by Lemma 1.3,

$$H^{n-1}(F \cap U; Z) = G \approx Z.$$

The isomorphisms $H^n(U_i; Z) \to Z$ may be chosen so that the map $Z \oplus Z \to Z$ in this sequence becomes $(j, k) \to j + k$, and the isomorphism $H^{n-1}(F \cap U; Z) \to Z$ may therefore be chosen so that the map $H^{n-1}(F \cap U; Z) \to Z \oplus Z$ becomes $j \to (j, -j)$. Since T^* is trivial on $H^{n-1}(F \cap U; Z)$, we see that on $H^n(U_1; Z) \oplus H^n(U_2; Z)$ it must take $(j, -j)$ into $(j, -j)$, and therefore it takes (j, k) into $(-k, -j)$; therefore on $H^n(U; Z)$ it becomes $j \to -j$, which shows that T reverses orientation.

Assertion (ii) now follows in this case from the fact that the orbit space U' is represented by $U_i \cup (F \cap U)$, which we shall denote by $\overline{U}_i$, and from the Mayer-Vietoris sequence with coefficients in Z_2:

$$\begin{array}{ccccc} H^j(V) & \to & H^j(\overline{V}_1) \oplus H^j(\overline{V}_2) & \to & H^j(F \cap V) \\ \downarrow & & \downarrow & & \downarrow \\ H^j(U) & \to & H^j(\overline{U}_1) \oplus H^j(\overline{U}_2) & \to & H^j(F \cap U). \end{array}$$

Assertion (i) follows from this, formula (16) (to be proved later), and the cohomology diagram induced by the sequence $0 \to Z \xrightarrow{2} Z \to Z_2 \to 0$. The details are similar to those used later in this proof, and thus we omit them here. Assertion (iii) is trivial, since all the groups are trivial. The last statement in the theorem follows now from the Mayer-Vietoris sequence for coefficients in Z.

We assume now that $r \leq n - 2$, for the remainder of the proof. If $V_0 \in \mathfrak{B}'$ is a neighborhood of $x \in F$, then we can find neighborhoods $V_1, V_2 \in \mathfrak{B}'$ of x such that $V_2 \subset V_1 \subset V_0$ and

(8) $$\operatorname{Im}(H^j(V_i; L) \to H^j(V_{i-1}; L)) = 0 \quad \text{for } j \neq n \text{ and } L = Z \text{ or } L = Z_p,$$

(9) $$\operatorname{Im}(H^j(V_i \cap F; Z_p) \to H^j(V_{i-1} \cap F; Z_p)) = 0 \quad \text{for } j \neq r.$$

We can also assume that

(10) $$\operatorname{Im}(H^j_\rho(V_i; Z_p) \to H^j_\rho(V_{i-1}; Z_p)) \approx \begin{cases} 0 & \text{for } j \leq r \text{ or } j > n, \\ Z_p & \text{for } r < j \leq n, \end{cases}$$

which follows from the proof of Theorem 7.4, in which it was shown that for V_{i-1} sufficiently small this image is nonzero for $r < j \leq n$ and zero otherwise, and from Lemma 7.2, which implies that for $r < j \leq n$ this image is contained in the image of the r-th cohomology of a fundamental set of F under some homomorphism, if V_{i-1} is small.

For the same reason we may also assume that

(11) $$\operatorname{Im}(H^{r+1}_\sigma(V_1) \to H^{r+1}_\sigma(V_0)) \subset \operatorname{Im}(H^r(V_0 \cap F) \to H^{r+1}_\sigma(V_0)).$$

We know that $H^j_\sigma(V_i; Z_p) \approx H^j(V_i' - F; Z_p)$ and thus, using the diagram

$$\begin{array}{ccccccc} \cdots \to & H^j(V_2' - F; Z_p) & \to & H^j(V_2'; Z_p) & \to & H^j(V_2' \cap F; Z_p) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & H^j(V_1' - F; Z_p) & \to & H^j(V_1'; Z_p) & \to & H^j(V_1' \cap F; Z_p) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & H^j(V_0' - F; Z_p) & \to & H^j(V_0'; Z_p) & \to & H^j(V_0' \cap F; Z_p) & \to \cdots, \end{array}$$

we see immediately that

(12) $$\operatorname{Im}(H^j(V_2'; Z_p) \to H^j(V_0'; Z_p)) \approx \begin{cases} 0 & \text{for } j \leq r+1 \text{ or } j > n, \\ Z_p & \text{for } r+1 < j \leq n, \end{cases}$$

where the case $j = r$ follows from the well-known fact that the connecting homomorphisms $H^j(V_i' \cap F) \to H^{j+1}(V_i' - F)$ are the same as the corresponding maps $H^j(V_i \cap F) \to H^{j+1}_\sigma(V_i)$ in the Smith sequence (the proof of this lies outside the scope of this paper, but is not difficult), and where the case $j = r + 1$ follows from (11), which in the other notation reads:

$$\operatorname{Im}(H^{r+1}(V_1' - F) \to H^{r+1}(V_0' - F)) \subset \operatorname{Im}(H^r(V_0' \cap F) \to H^{r+1}(V_0' - F)).$$

It follows from the above that we can find a sequence $U_0 \supset U_1 \supset U_2 \supset \cdots$, in which each U belongs to $\mathfrak{B}'$ and is a neighborhood of x, such that the U_i satisfy the three conditions

(13) $$\operatorname{Im}(H^j(U_i; L) \to H^j(U_{i-1}; L)) = 0 \quad \text{for } j \neq n \text{ and } L = Z \text{ or } L = Z_p,$$

(14) $$\operatorname{Im}(H^j(U_i \cap F; Z_p) \to H^j(U_{i-1} \cap F; Z_p)) = 0 \quad \text{for } j \neq r,$$

(15) $$\operatorname{Im}(H^j(U_i'; Z_p) \to H^j(U_k'; Z_p)) \approx \begin{cases} 0 & \text{for } j \leq r+1 \text{ or } j > n, \\ Z_p & \text{for } r+1 < j \leq n, \end{cases} \quad \text{for all } i > k.$$

We shall use, without proof, the well-known fact that if A_U is the Alexander-Spanier cochain group of an invariant set U and A_U^T is the subgroup of elements invariant under T, then $H^j(A_U^T) = H^j(U')$. Inclusion induces $s: H^j(U'; Z) \to H^j(U; Z)$. (This is the same as the homomorphism induced by the projection of U onto U', but we shall not use this fact.) Operation by $1 + T^* + \cdots + T^{*(p-1)}$ induces

$$\sigma: H^j(U; Z) \to H^j(U'; Z),$$

and we clearly see that $\sigma s(a) = pa$ for $a \in H^j(U'; Z)$. Consider the diagram

$$\begin{array}{ccccc} H^j(U_i'; Z) & \xrightarrow{s} & H^j(U_i; Z) & \xrightarrow{\sigma} & H^j(U_i'; Z) \\ \downarrow & & \downarrow & & \downarrow \\ H^j(U_k'; Z) & \xrightarrow{s} & H^j(U_k; Z) & \xrightarrow{\sigma} & H^j(U_k'; Z). \end{array}$$

The center vertical map is trivial for $j \neq n$ and $i > k$, and thus it follows immediately that

(16) $$\operatorname{Im}(H^j(U_i'; Z) \to H^j(U_k'; Z)) \subset \operatorname{Ker} p \quad \text{for } j \neq n \text{ and } i > k.$$

Now $U_i' - F$ is a (Z, n)-manifold (with Z_p used as coefficients, it follows from the cohomology sequence that $U_i - F$ is connected, since $r \leq n - 2$), and from Theorem 1.10 it is easy to see that

(17) $$H^n(U_i' - F; Z) \approx \begin{cases} Z & \text{if T preserves orientation,} \\ Z_2 & \text{if T reverses orientation (and, necessarily, } p = 2), \end{cases}$$

and, by Corollary 1.7, $H^n(U_i' - F; Z) \to H^n(U_k' - F; Z)$ is always onto, and thus, in this case, is an isomorphism.

By Corollary 1.7, $H^n(U_i - F; Z) \to H^n(U_i; Z)$ is onto, and hence

$$H^{n-1}(F \cap U_i; Z) \to H^n(U_i - F; Z)$$

is trivial. It now follows easily from the conomology sequence and from Lemma 1.3 that $H^{n-1}(F \cap U_i; Z) = 0$. Consequently we also see that $H^{n-1}(F \cap U_i'; Z) = 0$, and thus the map $H^n(U_i' - F; Z) \to H^n(U_i'; Z)$ is an isomorphism. Thus by (17) we have

(18) $$H^n(U_i^!; Z) \approx \begin{cases} Z & \text{if } T \text{ preserves orientation,} \\ Z_2 & \text{if } T \text{ reverses orientation (and } p = 2), \end{cases}$$

and also $H^n(U_i^!; Z) \to H^n(U_k^!; Z)$ is an isomorphism for all $i > k$.

Consider the coefficient sequence $0 \to Z \xrightarrow{p} Z \to Z_p \to 0$ and the induced diagram

$$\begin{array}{ccccccccc} \cdots \to & H^{j-1}(U_i^!; Z_p) & \to & H^j(U_i^!; Z) & \xrightarrow{p} & H^j(U_i^!; Z) & \to & H^j(U_i^!; Z_p) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & H^{j-1}(U_k^!; Z_p) & \to & H^j(U_k^!; Z) & \xrightarrow{p} & H^j(U_k^!; Z) & \to & H^j(U_k^!; Z_p) & \to \cdots . \end{array}$$

By (15) and (16) we have immediately

(19) $$G_{i,k}^j = \operatorname{Im}(H^j(U_i^!; Z) \to H^j(U_k^!; Z)) \approx 0 \text{ or } Z_p, \text{ for } j \neq n \text{ and } i \geq k+2.$$

It is therefore clearly permissible to assume for the $U_i^!$ that

(20) $G_{k+1,k}^j = \operatorname{Im}(H^j(W'; Z) \to H^j(U_k^!; Z))$ for all j, k and all open neighborhoods W of x such that $W \subset U_{k+1}$.

Let $k(j) = \min(k; G_{k+1,k}^j \approx Z_p)$, with the understanding that $k(j) = 0$ for any j such that $\{k; G_{k+1,k}^j \approx Z_p\} = \emptyset$. We define

(21) $$k_0 = \max k(j).$$

We now have, in particular, the relation

(22) $$G_{k_0+1,k_0}^j = G_{k_0+2,k_0}^j \approx G_{k_0+2,k_0+1}^j,$$

in which the isomorphism is induced by the natural map. We put $G^j = G_{k_0+1,k_0}^j$. We also define similarly

(23) $$H_{i,k}^j = \operatorname{Im}(H^j(U_i^!; Z_p) \to H^j(U_k^!; Z_p)).$$

Then by (15) we see that $H_{k+1,k}^j = \operatorname{Im}(H^j(W'; Z_p) \to H^j(U_k^!; Z_p))$ for all open neighborhoods $W \subset U_{k+1}$ of x. We put $H^j = H_{k_0+1,k_0}^j$.

Consider the sequence $\cdots \to H^{j-1} \to G^j \xrightarrow{p} G^j \to H^j \to G^{j+1} \to \cdots$, which is of order two. We claim that this sequence is exact. (This is assertion (iii) of the theorem.) This follows from the following general fact. Say that we have a commutative diagram

$$\begin{array}{ccccc} A_1 & \to & A_2 & \to & A_3 \\ \downarrow & & \downarrow & & \downarrow \\ B_1 & \to & B_2 & \to & B_3 \\ \downarrow & & \downarrow & & \downarrow \\ C_1 & \to & C_2 & \to & C_3 \end{array}$$

in which the rows are exact and in which the vertical maps induce isomorphisms: $\mathrm{Im}(A_i \to B_i) \approx \mathrm{Im}(A_i \to C_i) = \mathrm{Im}(B_i \to C_i)$. Then the subsequence

$$\mathrm{Im}(B_1 \to C_1) \to \mathrm{Im}(B_2 \to C_2) \to \mathrm{Im}(B_3 \to C_3)$$

is exact. To prove this, say that $a \in A_2$ and that a becomes zero in C_3. Then a already becomes zero in B_3, since $\mathrm{Im}(A_3 \to B_3)$ maps isomorphically into C_3. Thus the image of a in B_2 comes from B_1. Hence also the image of a in C_2 comes from $\mathrm{Im}(B_1 \to C_1) = \mathrm{Im}(A_1 \to C_1)$, and this completes the proof.

Now, since $G^j \xrightarrow{p} G^j$ is trivial for $j \neq n$, we have

$$H^{j-1} \approx G^{j-1} \oplus G^j \quad \text{for } j < n. \tag{24}$$

Hence, by (15),

$$G^j = 0 \quad \text{for } j \leq r + 2. \tag{25}$$

From (15), (24) and (25) it follows quite easily that

$$G^j \approx \begin{cases} Z_p & \text{for } j = r + 2q + 1 < n,\ q > 1, \\ 0 & \text{otherwise for } j \neq n. \end{cases} \tag{26}$$

Clearly, we will have finished if we can prove the next-to-the last statement of the theorem. To do this, note that if T preserves orientation, then the exact sequence

$$0 \to G^{n-1} \to H^{n-1} \to G^n \to G^n \to H^n \to 0$$

is

$$0 \to G^{n-1} \to Z_2 \to Z \xrightarrow{2} Z \to Z_2 \to 0,$$

and it follows that $G^{n-1} \approx Z_2$. If T reverses orientation, then this sequence is

$$0 \to G^{n-1} \to Z_2 \to Z_2 \xrightarrow{2} Z_2 \to Z_2 \to 0,$$

and it follows that $G^{n-1} = 0$. Thus, immediately from (26), we see that $n - 1 - r$ is odd or even according as T preserves or reverses orientation; this completes the proof of our theorem.

THEOREM 7.9. *Let* M^* *be the full orientable covering of* M, *and let* G^* *be the transformation group on* M^* *induced by* G. *Then* $\pi^{-1}(F)$ *is left pointwise stationary by* G^*, *and* $\pi^{-1}(F)$ *is the full orientable covering of* F (*with respect to* Z_p).

Proof. That $\pi^{-1}(F)$ is the full orientable covering of F follows directly from Corollary 7.6. Let $g \in G$, and assume that $G^*(x) \neq x$, where $\pi(x) \in F$. Then, since G^* is cyclic of prime order p, $G^*(x)$ contains p distinct points all mapping into $\pi(x)$ by π. This is impossible, since there are only $p - 1$ points of M^* lying over any point of M.

The following theorem generalizes a well-known result. (See [9], [11].)

THEOREM 7.10. *If* F *has an orientable neighborhood in* M, *then* F *is orientable. If* F *is orientable and* M *is paracompact* (*that is,* σ*-compact*), *then* F *has an orientable neighborhood in* M.

Proof. If F is orientable, then $\pi^{-1}(F)$ consists of p - 1 disjoint copies

$$F_1^*, \cdots, F_{p-1}^*$$

of F. If M is paracompact, then so is M*, and hence M* is normal (see [6], p. 159). Thus the F_i^* have neighborhoods N_i^* in M* which are mutually disjoint, and, moreover, the N_i^* may be assumed to be permuted by the deck transformations. Thus $N = \pi(N_i^*)$ is a neighborhood of F for which the full orientable covering consists of p - 1 copies of N. It follows that N is orientable. The converse is proved by going through this procedure backwards.

Remark. "Coefficients in Z_p" is understood in Theorem 7.10, so that this result is vacuous in case p = 2. That it is not true if interpreted in the classical sense (when possible), even for the differentiable case, is shown in the following two examples.

Example A. Let M be the open Moebius strip, and let T be the reflection across the center circle. Then F is the center circle, and it has no neighborhood which is orientable (with respect to Z).

Example B. Let M be the space obtained from $S^2 \times (-\infty, \infty)$ by identifying the points (x, k) and (-x, -k). Then M is orientable, since the identification map $(x, k) \to (-x, -k)$ is orientation-preserving on $S^2 \times (-\infty, \infty)$. Let T be the map taking the point $\{(x, k), (-x, -k)\}$ of M into $\{(x, -k), (-x, k)\}$. Then F is the set of points $\{(x, 0), (-x, 0)\}$, which is the projective plane.

Remark. The examples above can be altered so that T preserves orientation near the fixed point set, as follows. Let $M_1 = M \times (-\infty, \infty)$, and let $T_1 = (T, k \to -k)$. Then the fixed point set of T_1 is $F \times \{0\}$, and T_1 preserves orientation near this set.

LEMMA 7.11. *If* $M' = M - (F' - F)$, *then* M' *is connected.*

Proof. If p > 2, then $\dim_{Z_p} F' \leq n - 2$, and if M is orientable, then also $H^n(M - F') \approx H^n(M) \approx Z_p$, which implies that M - F' is connected. Thus by Theorem 7.9 we have in general that $\pi^{-1}(M - F')$ is connected for p > 2, and hence $M - F = \pi\pi^{-1}(M - F)$ is also connected. This proof is also valid when p = 2 and $\dim_{Z_2} F' \leq n - 2$; therefore we shall assume that p = 2 and $\dim_{Z_2} F' = n - 1$.

Let F'' be the union of some components of F' of dimension n - 1, and put $M'' = M - (F' - F'')$. Then the orbit space $(M - F')/G = (M'' - F'')/G$ is connected, since $H^n((M - F')/G) \approx H^n_\sigma(M) \approx H^n(M) \approx Z_2$, and hence $(M - F'')/G$ must also be connected, since it has a dense connected subspace $(M'' - F'')/G$. If none of the possible F'' separate M, then we have finished. Thus we shall assume that F'' separates M; but if $F^{(3)}$ is a component of F'', then $F'' - F^{(3)}$ does not separate M. Put $M^{(3)} = M - (F'' - F^{(3)})$ and consider

$$H^{n-1}(F^{(3)}) \to H^n(M^{(3)} - F^{(3)}) \to H^n(M^{(3)}) \to 0,$$

which is

$$Z_2 \to H^n(M^{(3)} - F^{(3)}) \to Z_2 \to 0.$$

It follows that $M^{(3)} - F^{(3)} = M - F''$ has exactly two components, and the nonzero element of G must interchange these components, since $(M - F'')/G$ is connected. It follows that F'' = F', and thus the component F is a candidate for $F^{(3)}$, and the lemma follows.

In case M is orientable, it is desirable to have a natural global isomorphism $H^r(F) \to H^n(M)$. In fact, we have:

THEOREM 7.12. *If* M *is orientable and* $M' = M - (F' - F)$, *then the composition*

$$\Lambda: H^r(F) \to H_\rho^{r+1}(M') \to H_{\bar\rho}^{r+2}(M') \to \cdots \to H_\eta^n(M') \xrightarrow{j^{*-1}} H^n(M') \to H^n(M)$$

is an isomorphism.

Proof. By Lemma 7.11, orientability, and Corollary 1.7, we see that $M' \in \mathfrak{B}'$, and thus by Theorem 7.5 the homomorphism $\Lambda_{M'}: H^r(F) \to H^n(M')$ is an isomorphism. That the last map is also an isomorphism follows from the fact that M' is connected (Lemma 7.11).

Remark. The proof of Theorem 7.12 is independent of Theorem 7.10, and thus it gives another proof of the first part of that theorem.

An interesting fact noticed by G. D. Mostow and communicated to the author by E. E. Floyd is that P. A. Smith's proof of the orientability of F when M is orientable actually also shows that if M is compact and orientable, then F' can not consist of exactly one point. It seems desirable to include a proof of this fact here.

COROLLARY 7.13. *If* M *is orientable and compact, then the set of fixed points of* G *can not consist of exactly one point.*

Proof. Suppose that the corollary is false. Then $H^0(M) \to H^0(F)$ is onto, and by the exact sequence

$$0 \to H_\rho^0(M) \to H^0(M) \to H_{\bar\rho}^0(M) \oplus H^0(F)$$

it follows that $H_\rho^0(M) = 0$. Thus the sequence

$$H^0(M) \to H^0(F) \to H_\rho^1(M)$$

is exact, and thus the second map is trivial, contrary to Theorem 7.12.

Remark. An obvious generalization of Corollary 7.13 is that

$$\mathrm{Im}(H^r(M') \to H_\rho^r(M') \oplus H^r(F)) \cap H^r(F) = 0$$

whenever M is orientable. Another generalization will be found in Section 10.

8. A REMARK

All the results of Section 7, with the exception of Theorem 7.8 and Corollary 7.13, can be generalized to arbitrary transformation groups of order p^a, where p is a prime. The method of proving this statement is due to E. E. Floyd and, in a less general form, to P. A. Smith. In fact, since such a group G is solvable, there exists a sequence $G = G_0 \supset G_1 \supset \cdots \supset G_{a-1} \supset G_a = 0$ such that G_{i+1} is normal in G_i and G_i/G_{i+1} is of order p. Let $F = F_0$ be, as usual, a component of the set of stationary points of $G = G_0$, and let F_i be the component of the set of stationary points of G_i which contains F. Then G_{a-1} is a group of order p on M, so that the theorems are true for F_{a-1} and G_{a-1} in place of F and G. Now, G_{a-2} leaves invariant the set of points stationary under G_{a-1}, and since F_{a-1} contains a point which is stationary under all of G, G_{a-2} leaves F_{a-1} invariant. Thus the theorems are true with

F_{a-2}, F_{a-1}, and G_{a-2} in place of F, M, and G, since G_{a-2} is effectively of order p on F_{a-1}. Continuing in this way we get, for example, isomorphisms

$$\Lambda_U^i: H^{r_i}(F_i \cap U) \to H^{r_{i+1}}(F_{i+1} \cap U).$$

(Of course, U must be chosen so that it works at all stages, but this is not hard to do.) Thus, we find that $\Lambda_U = \Lambda_U^{a-1} \cdots \Lambda_U^1 \Lambda_U^0$ is an isomorphism of $H^r(F \cap U)$ onto $H^n(U)$. The rest of the results are generalized in the same way.

Corollary 7.13 does not remain true in this more general context, since Z_4 acts on P^2 with only one fixed point and, of course, P^2 is orientable mod 2. The author does not know whether it would remain true for odd p or for p = 2 and M an orientable (Z, n)-manifold. In this connection, see also Section 10, in which this corollary is generalized in a different way.

9. ANOTHER PROOF OF DIMENSIONAL PARITY

In this section we give an independent proof of the dimensional parity part of our Theorem 7.7. The proof is more restrictive than the previous "local" proof, since we shall have to make the global assumptions of paracompactness, orientability, and of finitely generated cohomology.

We say that a space X is *of finite type over* Z_p if $\Sigma \dim(H^i(X; Z_p)) < \infty$.

THEOREM 9.1. *Let* G *be a group of prime order* $p > 2$ *acting on a paracompact, orientable* (Z_p, n)*-manifold* M *of finite type. Then, if* F *is a component of the set* F' *of stationary points and is of dimension* r *over* Z_p, F *is of finite type and* n - r *is even.*

Proof. Let $H_i(X)$ denote the direct limit of the Čech homology groups of compact subsets of X with respect to inclusion. (Note that this is the group used in [4].) We shall use the notation

$$\chi^*(X) = \sum (-1)^i \dim H^i(X), \qquad \chi(X) = \sum (-1)^i \dim H_i(X),$$

when these are defined. Note that if X is a (Z_p, m)-manifold and is orientable and paracompact, then we have, by Theorem 7.2 of [1] and Section 7.4 of [1], the Poincaré duality

$$H^i(X) \approx H_{m-i}(X),$$

and hence also

$$\chi^*(X) = (-1)^m \chi(X).$$

It follows from Theorem 4.4 of [4] that F' is of finite type. Hence also F and F' - F are of finite type. From the exact sequence

$$\cdots \to H^{i-1}(F' - F) \to H^i(M - (F' - F)) \to H^i(M) \to \cdots$$

it follows that M - (F' - F) is of finite type. Thus it suffices to prove the theorem for the case in which F = F' is connected.

Consider the exact sequence $\cdots \to H^i(M - F) \to H^i(M) \to H^i(F) \to \cdots$. It follows easily that $\chi^*(M) = \chi^*(F) + \chi^*(M - F)$. This shows, through the Poincaré duality, that

$\chi(M) = (-1)^{n-r}\chi(F) + \chi(M - F)$. However, it follows from Theorem 4.2 of [4] that $\chi(M - F) \equiv 0 \pmod p$ and that $\chi(M) \equiv \chi(F) \pmod p$. Thus we see that

$$\chi(F) \equiv (-1)^{n-r}\chi(F) \pmod p.$$

If $\chi(F) \not\equiv 0 \pmod p$, then $n - r$ is even and we have finished. If $\chi(F) \equiv 0 \pmod p$, let $x \in F$. (If $F = \emptyset$, the theorem must be regarded as vacuous.) G acts on the space $M - \{x\}$ with fixed point set $F - \{x\}$. Moreover, the sequence

$$0 \to H^0(F - \{x\}) \to H^0(F) \to H^0(\{x\}) \to H^1(F - \{x\}) \to \cdots$$

shows that $\chi^*(F - \{x\}) = \chi^*(F) - 1$, and hence that

$$\chi(F - \{x\}) = \chi(F) - (-1)^r \equiv (-1)^{r+1} \pmod p.$$

Thus the result above applied to $M - \{x\}$ shows that $n - r$ is even in this case, also; this concludes the proof.

Remark. Using the orientable covering, one can extend the proof above to the nonorientable case by applying the results of Section 5 and the fact that points of M^* lying above a point of F are left fixed by G^* (see Theorem 7.9). However, the assumption of finite type must be made for M^*.

10. THE POSITION OF THE FIXED POINT SET

In this section we again consider the situation of Section 7, and use Theorem 7.12 to derive a result which, in the language of homology, says that in some sense the fixed point set is homologous to zero in the orbit space M/G of M, if M is orientable; we also derive some consequences of this. The result is, of course, a generalization of Corollary 7.13. Some examples are given which show that it does not hold in general, if we replace M/G by M in the statement. We also use this theorem to prove the analogous result for a toral group acting on a compact orientable manifold.

As usual, we shall state and prove our results by means of the language of cohomology. We shall, however, state the analogous results for homology, since it is not immediately clear how these results should read, and we shall only indicate the proofs. The reader should have no trouble filling in the proofs for homology; but it requires the establishment of the analogues of some of the results in Section 7. The homology theory which we use in this section is the Čech theory of the one-point compactifications modulo the point at infinity.

THEOREM 10.1. *If* G *is a group of prime order* p *acting on the orientable* $(Z_p\text{-}n)$-*manifold* M *with orbit space* M/G, *and if* $F_1, \cdots, F_k$ *are the components of dimension* r *of the fixed point set* F, *where* r *is some definite integer, then there exist elements* $0 \neq y_i \in H^r(F_i)$ *such that if* $a_i \in Z_p$ *and* $\Sigma\, a_i y_i \in \mathrm{Im}(H^r(M/G) \to H^r(F))$, *then* $\Sigma\, a_i = 0$.

Proof. Let $0 \neq x \in H^n(M)$, and put $y_i = \Lambda_i^{-1} x$, where Λ_i is the isomorphism $H^r(F_i) \to H^n(M)$ given by Theorem 7.12. Then the homomorphism $\Lambda\colon H^r(F) \to H^n(M)$ given by $\Lambda = j^{*-1}\triangle$, where j^* is the isomorphism $H^n(M) \to H^n_\rho(M)$, and where $\triangle$ is the composition

$$H^r(F) \to H^{r+1}_\sigma(M) \to \cdots \to H^n_\rho(M)$$

of connecting homomorphisms in the Smith sequences, carries the element $\Sigma\, a_i y_i$

into $\Sigma a_i x$. As remarked before, the homomorphism $H^r(F) \to H_\sigma^{r+1}(M)$ is the same as the connecting homomorphism $H^r(F) \to H^{r+1}(M/G - F)$ in the sequence

$$\cdots \to H^r(M/G - F) \to H^r(M/G) \to H^r(F) \to H^{r+1}(M/G - F) \to \cdots,$$

and therefore it follows that if $\Sigma a_i y_i \in \mathrm{Im}(H^r(M/G) \to H^r(F))$, then

$$\sum a_i x = \Lambda \sum a_i y_i = 0$$

and consequently $\Sigma a_i = 0$, as claimed.

We now formulate the corresponding result in homology.

THEOREM 10.2. *With the notation of Theorem* 10.1, *there exist elements* $0 \neq z_i \in H_r(F_i)$ *and an element* $z \in H_r(F - \bigcup F_i)$ *(perhaps zero) such that*

$$z + \sum z_i \in \mathrm{Ker}(H_r(F) \to H_r(M/G)).$$

Proof. In analogy to the results in Section 7, there exists an isomorphism $\Lambda_i: H_n(M) \to H_r(F_i)$ given by $\Lambda_i = \Delta_i j_*^{-1}$, where j_* is the isomorphism

$$H_n^\rho(M) \to H_n(M),$$

and where Δ_i is the composition

$$H_n^\rho(M) \to H_{n-1}^{\bar\rho}(M) \to \cdots \to H_{r+1}^\sigma(M) \to H_r(F_i)$$

of connecting homomorphisms in the Smith homology sequences. There exists a corresponding homomorphism $\Lambda': H_n(M) \to H_r(F - \bigcup F_i)$ which may be trivial.

We let $0 \neq x \in H_n(M)$, and we define $z = \Lambda' x$ and $z_i = \Lambda_i x$. Thus, if Λ is the combined homomorphism $H_n(M) \to H_r(F)$, then $z + \Sigma z_i = \Lambda x$. In particular, $z + \Sigma z_i \in \mathrm{Im}(H_{r+1}^\sigma(M) \to H_r(F))$. However, $H_{r+1}^\sigma(M) = H_{r+1}(M/G - F)$, and the homomorphism $H_{r+1}^\sigma(M) \to H_r(F)$ is the same as the connecting homomorphism $H_{r+1}(M/G - F) \to H_r(F)$ in the exact sequence

$$\cdots \to H_{r+1}(M/G - F) \to H_r(F) \to H_r(M/G) \to H_r(M/G - F) \to \cdots,$$

and the conclusion follows directly.

Remark. The author does not know whether these theorems are true for transformations of prime power period, or indeed whether Corollary 7.13, which is an immediate consequence of these theorems, holds in the general case.

Before going on to the corollaries of these theorems, we shall give two examples showing that the corresponding statements with M replacing M/G are not valid. The first example is due to T. T. Frankel.

Example A. Let M be the complex projective plane; that is, if C denotes the complex number field, then M is the space obtained from $C \times C \times C$ through the equivalence relation $(z_1, z_2, z_3) \sim (zz_1, zz_2, zz_3)$ for all $0 \neq z \in C$. We regard Z_p as a subgroup of the circle group T, and T as the group of complex numbers of absolute value one. We let T act on M by $t(z_1, z_2, z_3) = (z_1, z_2, tz_3)$. Clearly, the fixed point set of T and of all Z_p is the point $P = \{(0, 0, z)\}$, together with the complex projective line $L = \{(z_1, z_2, 0)\}$, which is a two-sphere. It is well known that $H_2(L) \to H_2(M)$ is an isomorphism, and thus L is "not homologous to zero" in M.

The corresponding statement in cohomology is that $H^2(M) \to H^2(L)$ is onto. Clearly, the orbit space M/T is a closed 3-cell with boundary corresponding to L. Thus L is homologous to zero in M/T, and the analogue of our theorem for toral groups is valid in this example. Later, this will be shown to be true in general.

We shall now alter Example A so that F has two components of dimension two.

Example B. Let M be as in Example A, let E be a small invariant cell around the isolated fixed point P, and denote by S the 3-sphere boundary of E. Let M' be the space obtained by pasting two copies M_1 and M_2 of M - Int(E) along the sets corresponding to S. We see that if F_i is the set corresponding to the 2-sphere L in M_i, then $H_2(F_i) \to H_2(M_i)$ is an isomorphism and $H^2(M_i) \to H^2(F_i)$ is onto. However, by the Mayer-Vietoris sequences

$$H_2(S) \to H_2(M_1) \oplus H_2(M_2) \to H_2(M') \to H_1(S),$$

$$H^1(S) \to H^2(M') \to H^2(M_1) \oplus H^2(M_2) \to H^2(S),$$

it follows that $H_2(M_1) \oplus H_2(M_2) \to H_2(M')$ is an isomorphism into and

$$H^2(M') \to H^2(M_1) \oplus H^2(M_2)$$

is onto, since S is a 3-sphere. Thus $H_2(F) \to H_2(M')$ is an isomorphism into and $H^2(M') \to H^2(F)$ is onto, and thus the analogues of our theorems with M' replacing M'/G do not hold in this example.

We now prove the corresponding results for toral groups.

COROLLARY 10.3. *If* T *is a toral group acting on the orientable* (Z, n)*-manifold* M, *and if* T *has only a finite number of isotropy subgroups (for example, if* M *is compact), then Theorems* 10.1 *and* 10.2 *hold with* T *in place of* G *and coefficients in* Z_p *for all sufficiently large primes* p.

Proof. Since the subgroups of T of prime order are dense in T, the condition on isotropy subgroups implies that there is a subgroup G of T of order p for all sufficiently large primes p, such that the fixed point set F of G is the same as that of T. Thus, since we have the factorizations

$$H^r(M/T) \to H^r(M/G) \to H^r(F) \quad \text{and} \quad H_r(F) \to H_r(M/G) \to H_r(M/T),$$

the results follow immediately.

Remark. It is unknown to the author whether Corollary 10.3 is true for all primes p.

For coefficients in Z, we can prove the following partial analogue of Theorem 10.1 for toral groups.

COROLLARY 10.4. *If, in the situation of Corollary* 10.3, F' *is a component of dimension* r *of the set* F *of stationary points of* T, *then*

$$\operatorname{Im}(H^r(M/T; Z) \to H^r(F; Z)) \cap H^r(F'; Z) = 0.$$

Thus, if F = F', *then* $H^r(M/T; Z) \to H^r(F; Z)$ *is trivial.*

Proof. We shall use the fact that F' is an orientable (Z, n)-manifold, which follows from some results of Floyd and Conner. First note that for sufficiently large p, the conclusion of the corollary is true for coefficients in Z_p instead of Z. Let y be a generator of $H^r(F'; Z)$, and let $x = my$, $m \in Z$. Let $p > |m|$ be large, and consider the element $z \in H^r(F'; Z_p)$ which is the image of x under the homomorphism $H^r(F'; Z) \to H^r(F'; Z_p)$. We see that $z \neq 0$, since $p > |m|$. Thus it follows immediately from Corollary 10.3 that $z \notin \operatorname{Im}(H^r(M/T; Z_p) \to H^r(F; Z_p))$. Hence we see immediately from the diagram

$$\begin{array}{ccc} H^r(M/T; Z) & \to & H^r(M/T; Z_p) \\ \downarrow & & \downarrow \\ H^r(F; Z) & \longrightarrow & H^r(F; Z_p) \end{array}$$

that $x \notin \operatorname{Im}(H^r(M/T; Z) \to H^r(F; Z))$, and this completes the proof.

COROLLARY 10.5. *Let* G *be a group of prime order* p [*respectively a toral group*] *acting on a compact orientable* (Z_p, n)-*manifold* [*respectively on a compact orientable* (Z, n)-*manifold*] *with fixed point set* F, *and let* F' *be the union of the components of* F *of highest dimension. Then* F' *is not a retract of* M/G.

Proof. The only nontrivial case is that for which F' is connected. Thus $H^r(M/G) \to H^r(F')$ is trivial, where coefficients are in Z_p [respectively, in Z]. However, if F' is a retract of M/G, then there is a homomorphism

$$H^r(F') \to H^r(M/G)$$

such that the composition $H^r(F') \to H^r(M/G) \to H^r(F')$ is the identity, from which it follows that $H^r(F') = 0$, contrary to the fact that F' is orientable modulo Z_p [respectively, modulo Z].

COROLLARY 10.6. *If* M *is a compact* (Z_2, n)-*manifold with an involution* $g: M \to M$ $(g^2 = \text{identity})$ *such that there exist only a finite number* k *of fixed points of* g, *then* k *is even. Also, the Euler characteristic of* M *is even, when it is defined.*

Proof. Let G be the group consisting of g and the identity, and let $\{P_i\}$ $(i = 1, \cdots, k)$ be the fixed points of g. If y_i is the nonzero element of $H^0(P_i; Z_2)$ and x is the nonzero element of $H^0(M/G; Z_2)$, then the map $H^0(M/G; Z_2) \to H^0(F; Z_2)$ carries x onto Σy_i. Thus, by Theorem 10.1, $k = \Sigma_{i=1}^k 1 = 0$ in Z_2, which implies that k is even. The last statement follows from the formula $\chi(M) \equiv \chi(F) \pmod 2$, proved in [4]. The analogous proof for homology is obvious.

11. THE GLOBAL SMITH THEOREMS

In this section we shall give a method for deriving the global Smith theorems. The procedure is well known, and our only justification for including it here is to point out the fact that if the procedure is based on spaces with the cohomology groups of euclidean spaces rather than of spheres, the proofs become analogous to (though considerably simpler than) our derivation of the local theorems. We regard p to be a fixed prime, in this section, and we always regard coefficients to be in Z_p.

For convenience, we shall say that a locally compact, finite-dimensional space X is a CM^n if $H^k(X) \approx H^k(M^n)$ for all k, where M^n is some definite n-manifold. (M^n will always be either S^n or E^n, here.) If X is compact, then we denote by cX the open cone over X. Clearly, X is a CS^n if and only if cX is a CE^{n+1}.

If X is a CE^n on which a group G of order p operates with fixed point set F, then it is clear from the Smith sequences and from finite-dimensionality that $H_\rho^k(X)$

and $H^k(F)$ are trivial for $k > n$. Thus $H^n(X) \to H^n_\rho(X) \oplus H^n(F)$ is onto. If $H^n(F) \neq 0$, then $H^n_\rho(X) = 0$ and this map is an isomorphism. If $H^n(F) = 0$, then the proof of Lemma 7.1 applies to show that this map is again an isomorphism. Thus, in place of Lemma 7.1, we have

LEMMA 11.1. *With the above notation,* $H^n(X) \to H^n_\rho(X) \oplus H^n(F)$ *is an isomorphism.*

By the method of Section 7 (with considerable simplification, since we do not have to resort to subspaces), it is now easy to verify that the following theorem holds.

THEOREM 11.2. *If* X *is a* CE^n *on which a group* G *of order* p *acts with fixed point set* F, *then* F *is a* CE^r *for some* $0 \leq r \leq n$, *and furthermore the map* $\Lambda: H^r(F) \to H^n(X)$ *given by* $\Lambda = j^{*-1}\Delta$ *is an isomorphism, where* j^* *is the isomorphism* $H^n(X) \to H^n_\eta(X) \oplus H^n(F)$ *and* Δ *is the composition*

$$H^r(F) \to H^{r+1}_\rho(X) \to H^{r+2}_{\bar\rho}(X) \to \cdots \to H^n_\eta(X)$$

of connecting homomorphisms of the Smith sequences.

From this result we see that the proof of Theorem 7.7 goes over word for word and gives

THEOREM 11.3. *The isomorphism* Λ *of Theorem* 11.2 *is independent of the choice of* ρ. *Moreover, if* $p > 2$, *then* $n - r$ *is even.*

In order to apply these results to a space which is a CS^n, we merely have to pass to the cone over the space. Thus if G, of order p, acts on X which is a CS^n with fixed point set F, then G acts naturally on cX which is a CE^{n+1} with fixed point set cF. Thus cF is a CE^{r+1}, for some $-1 \leq r \leq n$, and it follows that F is a CS^r. The dimensional parity also holds. Note that if $F = \emptyset$, then cF reduces to one point and is thus a CE^0, and if p is odd, then n must also be odd. Hence, with the convention that the empty set is a CS^{-1}, the dimensional parity holds in this case also. Summing up, we have the well-known

THEOREM 11.4. *If* X *is a* CS^n *on which a group* G *of order* p *acts with fixed point set* F, *then* F *is a* CS^r *for some* $-1 \leq r \leq n$. *If* $p > 2$, *then* $n - r$ *is even.*

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