BOUNDED J-FRACTIONS AND UNIVALENCE

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INTRODUCTION

Some attention has recently been given, by Scott, Thale, and Perron, to the problem of finding the domain of univalence of certain well-known classes of continued fractions. In particular, Thale [4] obtained a circular domain of univalence for the class of bounded S-fractions, and one for the class of bounded J-fractions. Perron [3] has established the fact that the result of Thale on bounded S-fractions is sharp. In this paper it is shown that Thale's circular domain of univalence for the class of bounded J-fractions cannot be enlarged. Moreover, some properties related to univalence are obtained for the latter class of continued fractions.

1. THE RADIUS OF UNIVALENCE

Let $M \geq 0$ and $N > 0$ be real numbers. Consider the class $J(M, N)$ of functions of the form

$$\frac{1}{z + b_1} - \frac{a_1^2}{z + b_2} - \cdots - \frac{a_n^2}{z + b_{n+1}} - \cdots, \tag{1.1}$$

where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences of complex numbers such that

$$|a_n| \leq N/3, \qquad |b_n| \leq M/3 \qquad (n = 1, 2, \cdots). \tag{1.2}$$

It is known [5, p. 112] that every function in the class $J(M, N)$ is regular for $|z| > (2N + M)/3$.

Thale [4] has shown that each function of $J(M, N)$ is univalent for

$$|z| > (3\sqrt{2}N + 2M)/6.$$

The function

$$\frac{1}{z +} \frac{N^2/9}{z - M/3 -} \frac{N^2/9}{z - M/3 -} \cdots \frac{N^2/9}{z - M/3 -} \cdots = \frac{6}{9z - M - \sqrt{(3z - M)^2 - 4N^2}},$$

whose derivative is zero at $z = (3\sqrt{2}N + 2M)/6$, shows that *there is no larger circular domain of univalence for the class* $J(M, N)$. By an equivalence transformation, the function $e^{i\theta} f(e^{i\theta} z)$ for fixed θ $(0 \leq \theta < 2\pi)$ is in the class $J(M, N)$ whenever $f(z)$ is in the class. Thus there does not exist a domain of univalence for the class $J(M, N)$ which properly contains the disk $|z| > (3\sqrt{2}N + 2M)/6$.

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2. A COVERING THEOREM

Let $f(z)$ be in the class $J(M, N)$. By an equivalence transformation, the J-fraction representation (1.1) of $f(z)$ can be written as

$$f(z) = \frac{1}{z + b_1}\left[\frac{1}{1} + \frac{-a_1^2/(z + b_1)(z + b_2)}{1} + \cdots + \frac{-a_n^2/(z + b_n)(z + b_{n+1})}{1} + \cdots\right].$$

Since by (1.2) the partial numerators of the continued fraction in brackets have modulus less than 1/4 for $|z| = r > (2N + M)/3$, a result of Paydon and Wall [2] on value regions yields the following covering theorem for the class $J(M, N)$:

THEOREM 2.1. *If* $f(z)$ *is in the class* $J(M, N)$, *then for* $|z| = r > (2N + M)/3$,

$$\frac{3}{3r + M + N\gamma} \le |f(z)| \le \frac{3\gamma}{N}, \tag{2.1}$$

where

$$\gamma = \frac{3r - M - \sqrt{(3r - M)^2 - 4N^2}}{2N}. \tag{2.2}$$

These bounds are sharp.

For fixed θ $(0 \le \theta < 2\pi)$, the lower bound of (2.1) is taken on at the point $z = re^{-i\theta}$ by the modulus of the function

$$\frac{1}{z + Me^{-i\theta}/3} + \frac{e^{-2i\theta}N^2/9}{z - Me^{-i\theta}/3} - \frac{e^{-2i\theta}N^2/9}{z - Me^{-i\theta}/3} - \cdots - \frac{e^{-2i\theta}N^2/9}{z - Me^{-i\theta}/3} - \cdots. \tag{2.3}$$

The function

$$\frac{1}{z + Me^{-i\theta}/3} - \frac{e^{-2i\theta}N^2/9}{z - Me^{-i\theta}/3} - \cdots - \frac{e^{-2i\theta}N^2/9}{z - Me^{-i\theta}/3} - \cdots \tag{2.4}$$

has modulus equal to the upper bound in (2.1) for $z = re^{-i\theta}$.

An analogue for the class $J(M, N)$ of the Koebe-Bieberbach Covering Theorem [1, p. 75] is obtained from (2.1) by setting $z = r = (3\sqrt{2}N + 2M)/6$. Explicitly, we have the following result:

COROLLARY 2.1. *The conformal image, under any mapping by a function in the class* $J(M, N)$, *of the domain of univalence,* $|z| > (3\sqrt{2}N + 2M)/6$, *contains all points of the open disk*

$$|w| < 3/2(\sqrt{2}N + M), \tag{2.5}$$

and it is contained in the open disk

$$|w| < 3\sqrt{2}/2N. \tag{2.6}$$

These results are sharp.

It is easily seen that the function (2.3) omits the value $3e^{i\theta}/2(\sqrt{2}N + M)$ when $|z| > (3\sqrt{2}N + 2M)/6$. Thus the domain (2.5) cannot be enlarged. Moreover, the function (2.4) takes on the value $3\sqrt{2}e^{i\theta}/2N$ at $z = (3\sqrt{2}N + 2M)e^{-i\theta}/6$. This shows that the result (2.6) is the best possible.

3. STARLIKENESS

A lower bound for the radius of starlikeness of the class J(M, N) is given in the following theorem:

THEOREM 3.1. *Each function in the class* J(M, N) *maps the region* $|z| > r$ *onto a region which is starlike with respect to the origin, provided*

$$r \geq r_0 = \frac{2}{9}[2M + \sqrt{M^2 + 12N^2}]. \tag{3.1}$$

Proof. Let f(z) be in J(M, N), and let f(z) have the terminating J-fraction representation

$$\frac{1}{z + b_1} - \frac{a_1^2}{z + b_2} - \cdots - \frac{a_{n-1}^2}{z + b_n}. \tag{3.2}$$

Define the function f_k $(k = 1, 2, \cdots, n)$ by means of the recurrence formulas

$$\begin{aligned} f_1 &= 1/(z + b_n), \\ f_{k+1} &= 1/(z + b_{n-k} - a_{n-k}^2 f_k) \qquad (k = 1, 2, \cdots, n - 1), \end{aligned} \tag{3.3}$$

where $\{a_k\}_{k=1}^{n-1}$ and $\{b_k\}_{k=1}^{n}$ are given in (3.2). For each value of k $(k = 1, 2, \cdots, n)$, the function f_k is in J(M, N). In particular, $f_n \equiv f(z)$.

Formal differentiation of the second formula of (3.3) with respect to z yields

$$f'_{k+1} = -f_{k+1}^2(1 - a_{n-k}^2 f'_k) \qquad (k = 1, 2, \cdots, n - 1). \tag{3.4}$$

By (3.3), this can be rewritten as

$$\frac{zf'_{k+1}}{f_{k+1}} = -1 + b_{n-k}f_{k+1} - a_{n-k}^2 f_k f_{k+1}\left(1 - \frac{zf'_k}{f_k}\right).$$

It follows from this formula and the definition of f_1 that

$$\frac{zf'_n}{f_n} + 1 = \sum_{k=1}^{n} \frac{b_k}{f_n} \prod_{j=0}^{k-1} a_j^2 f_{n-j}^2 - 2\sum_{k=1}^{n-1} \frac{f_n}{f_{n-k}} \prod_{j=1}^{k} a_j^2 f_{n-j}^2, \tag{3.5}$$

where $a_0 = 1$.

By the triangle inequality and the bounds given by (1.2) and (2.1), we obtain from (3.5)

$$\left|\frac{zf'_n}{f_n} + 1\right| \leq \frac{M}{N}\sum_{k=1}^{n} \gamma^{2k-1} + 2\sum_{k=1}^{n} \gamma^{2k} < \frac{\frac{M}{N}\gamma + 2\gamma^2}{1 - \gamma^2}, \tag{3.6}$$

where γ (< 1) is defined by (2.2). The expression on the right does not exceed unity if

$$\gamma \leq \frac{\sqrt{M^2 + 12N^2} - M}{6N}. \tag{3.7}$$

Since by (2.2)

$$r = \frac{1}{3}\left[M + \left(\gamma + \frac{1}{\gamma}\right)N\right], \tag{3.8}$$

we see that (3.7) holds if $|z| = r$ is restricted by (3.1). It follows that the real part of zf_n'/f_n is negative if r is thus restricted, and therefore $f(z)$ is starlike for $|z| > r_0$.

Now if $f(z)$ in $J(M, N)$ has a nonterminating J-fraction representation (1.1), it is the limit of a uniformly convergent sequence of terminating J-fractions in the closed region $|z| \geq r > r_0$. Since each function of the sequence is starlike in this region, the limit function $f(z)$ also has this property. This completes the proof.

From the previous remarks and (3.6), it is evident that, for each $f(z)$ in the class $J(M, N)$,

$$\left|\frac{zf'(z)}{f(z)} + 1\right| \leq \lambda \qquad (|z| > (2N + M)/3), \tag{3.9}$$

where

$$\lambda = \left(\frac{M}{N}\gamma + 2\gamma^2\right)/(1 - \gamma^2). \tag{3.10}$$

4. CONVEXITY

A lower bound for the radius of convexity of the class $J(M, N)$ is furnished by the following result:

THEOREM 4.1. *Let* $f(z)$ *be in the class* $J(M, N)$, *and let* $h = M/N$. *Then* $f(z)$ *maps the circular domain* $|z| > r$ *onto a convex domain if*

$$r \geq r_1 = \frac{1}{3}\left[M + \left(\gamma_1 + \frac{1}{\gamma_1}\right)N\right], \tag{4.1}$$

where γ_1 *is the smallest positive root of the equation*

$$17\gamma^6 + 15h\gamma^5 + (4h^2 - 19)\gamma^4 - 10h\gamma^3 + (11 - 2h^2)\gamma^2 + 3h\gamma - 1 = 0. \tag{4.2}$$

Proof. Suppose $f(z)$ can be represented by the terminating J-fraction (3.2). As before, define the functions f_k $(k = 1, 2, \cdots, n)$ by means of (3.2) and the recurrence formulas (3.3).

If (3.4) is differentiated with respect to z, we see that

$$f''_{k+1} = -f_{k+1}f'_{k+1}(1 - a^2_{n-k}f'_k) + a^2_{n-k}f^2_{k+1}f''_k \ .$$

By (3.4), this can be rewritten in the form

$$\frac{f''_{k+1}}{f'_{k+1}} = 2\frac{f'_{k+1}}{f_{k+1}} + a^2_{n-k}\frac{f^2_{k+1}}{f'_{k+1}}f''_k .$$

It follows that

$$\frac{zf''_{k+1}}{f'_{k+1}} = 2S_{k+1} + a^2_{n-k}f_{k+1}f_k\frac{S_k}{S_{k+1}}\frac{zf''_k}{f'_k} \qquad (k = 1, 2, \cdots, n-1), \tag{4.3}$$

where

$$S_k = \frac{zf'_k}{f_k} \qquad (k = 1, 2, \cdots, n).$$

Successive application of (4.3) yields

$$\frac{zf''_n}{f'_n} + 2 = 2(S_n + 1) + 2\frac{f_n}{S_n}\sum_{k=1}^{n-1}\frac{S^2_{n-k}}{f_{n-k}}\prod_{j=1}^{k}a_j^2 f^2_{n-j} .$$

The modulus of the left-hand side of this equality can now be estimated by applying the triangle inequality and the bounds (1.2), (2.1), and (3.9). Thus

$$\left|\frac{zf''_n}{f_n} + 2\right| \leq 2\lambda + 2\frac{(1+\lambda)^2}{1-\lambda}\sum_{k=1}^{n-1}\gamma^{2k} < 2\lambda + 2\frac{(1+\lambda)^2}{1-\lambda}\frac{\gamma^2}{1-\gamma^2}, \tag{4.4}$$

where λ (< 1) is defined by (3.10) and γ is given by (2.2). The estimate on the right-hand side of (4.4) does not exceed unity if

$$2(1+\lambda)^2\gamma^2 \leq (1-\gamma^2)(1-\lambda)(1-2\lambda).$$

By (3.10) this becomes

$$2\gamma(h+2\gamma)(1-\gamma^2)(1-h\gamma-3\gamma^2) + 2\gamma^2(1+h\gamma+\gamma^2)^2 - (1-\gamma^2)^2(1-h\gamma-3\gamma^2) \leq 0 .$$

The polynomial in γ on the left of this inequality is precisely the polynomial in (4.2). Thus if $\gamma \leq \gamma_1$, where γ_1 is the smallest positive root of (4.2), then the left-hand side of (4.4) does not exceed unity. By (3.8) this implies that

$$\Re\left(\frac{zf''_n}{f'_n} + 1\right) \leq 0 \qquad (|z| > r_1),$$

where r_1 is given in (4.1). Hence the function (3.2) is convex for $|z| > r_1$.

If $f(z)$ in $J(M, N)$ has a nonterminating J-fraction representation, then it is the uniform limit of a sequence of terminating J-fractions. As in the proof of Theorem 3.1, this implies that $f(z)$ is convex for $|z| > r_1$.

In particular, if $f(z)$ is in the class $J(2N, N)$, then it is convex for $|z| > 2.707\,N$. If $f(z)$ is in the class $J(0, N)$, then it is convex for $|z| > 1.155\,N$.

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