ON SOME METRIC PROPERTIES OF POLYNOMIALS WITH REAL ZEROS

Ch. Pommerenke

1. Let $f(z) = \Pi_{\nu=1}^{n}(z - x_\nu)$ be a polynomial of degree n with real zeros x_ν, and let E be the set $|f(z)| \leq 1$. The real axis is denoted by X. A circle (semicircle) that has a segment of X as diameter will be called an orthogonal circle (semicircle).

LEMMA. *Let* L *be an orthogonal semicircle over the real points* a_1 *and* a_2. *If* $z_0 \in E \cap L$ *and* $\Im z_0 > 0$, *then either the arc* $a_1 z_0$ *or the arc* $z_0 a_2$ *of* L *is contained in* E.

Proof. (I owe the idea of this argument to [1, p. 139].) If $x_1 = \cdots = x_n = 0$, the lemma is trivially true. Therefore we can assume that not all of these equations hold. We may take $a_1 = -\rho$, $a_2 = \rho$. Then we have L: $z = \rho e^{i\theta}$ $(0 \leq \theta \leq \pi)$. Consider the function

$$G(\theta) = \log|f(\rho e^{i\theta})| = \frac{1}{2}\sum_{\nu=1}^{n} \log(\rho^2 - 2\rho x_\nu \cos\theta + x_\nu^2)$$

for $0 < \theta < \pi$. Its derivative is

$$G'(\theta) = \rho(\sin\theta)H(\theta), \qquad \text{where } H(\theta) = \sum_{\nu=1}^{n} \frac{x_\nu}{\rho^2 - 2\rho x_\nu \cos\theta + x_\nu^2}.$$

Differentiating $H(\theta)$, we obtain

$$H'(\theta) = \sum_{\nu=1}^{n} \frac{-\rho x_\nu^2 \sin\theta}{(\rho^2 - 2\rho x_\nu \cos\theta + x_\nu^2)^2} < 0$$

for $0 < \theta < \pi$. Hence $G'(\theta) = \rho(\sin\theta)H(\theta)$ has at most one zero σ in $0 < \theta < \pi$. The relation

$$G''(\sigma) = \rho(\cos\sigma)H(\sigma) + \rho(\sin\sigma)H'(\sigma) = \rho(\sin\sigma)H'(\sigma) < 0$$

shows that $G(\theta)$ has a maximum in σ. Therefore the function $G(\theta)$ does not assume a minimum in $0 < \theta < \pi$. Put $z = \rho e^{i\theta_0}$. We have to consider three cases:

1. $G(0) > 0$. Since the function $G(\theta)$ has no minimum in $0 < \theta < \pi$, the inequality $G(\theta_0) \leq 0$ implies that $G(\theta) \leq 0$ for $\theta_0 \leq \theta \leq \pi$. Therefore the arc $a_1 z_0$ of L belongs to E.

2. $G(\pi) > 0$. We can show in a similar way that the arc $z_0 a_2$ of L is contained in E.

3. $G(0) \leq 0$, $G(\pi) \leq 0$. If neither $a_1 z_0$ nor $z_0 a_2$ were contained in E, there would be two values θ_1 and θ_2 such that $0 < \theta_1 < \theta_0 < \theta_2 < \pi$ and $G(\theta_1) > 0$,

Received July 7, 1959.

$G(\theta_2) > 0$. On the other hand we have $G(\theta_0) \leq 0$. Therefore the function $G(\theta)$ would have a minimum in $\theta_1 < \theta < \theta_2$.

THEOREM 1. *Let* $f(z) = \Pi(z - x_\nu)$ *(x_ν real) and* E: $|f(z)| \leq 1$. *Then for every* z_0 *in* E *there exists an orthogonal circle* K *such that* $z_0 \in K$ *and* $K \subset E$. *Hence the set* E *is the union of orthogonal circles.*

Proof. Let $z_0 = x_0 + iy_0$. For $-\infty < a < x_0$, let $L(a)$ denote the orthogonal semicircle through a and through z_0, and let $D_1(a)$ and $D_2(a)$ denote the (left and right) bounded closed regions determined by arcs of $L(a)$ and by segments of the real axis and of the line $x = x_0$. If E contains a point z_1, it contains the entire segment $[z_1, \bar{z}_1]$. Also, $D_1(a)$ and $D_2(a)$ vary monotonically with a, and since E does not contain $D_1(a)$ for large negative values of a, there exists a value a_1 such that $E \supset D_1(a_1)$ and $E \not\supset D_1(a)$ if $a < a_1$. Similarly, there exists a value a_2 such that $E \supset D_2(a_2)$ and $E \not\supset D_2(a)$ if $a > a_2$. Now $a_2 \geq a_1$; for otherwise our lemma would be false for $a = (a_1 + a_2)/2$. Therefore $D_2(a_2) \supset D_2(a_1)$ and $E \supset D_1(a_1) \cup D_2(a_1)$, and the theorem is proved.

2. From Theorem 1 we can obtain information about metric properties of the set $|f(z)| \leq 1$. We shall need the concept of the capacity (also called the transfinite diameter) of a closed bounded set F. The capacity of F is defined as

$$\text{cap } F = \lim_{m \to \infty} \kappa_m ,$$

where

$$\kappa_m = \max_{\zeta_1, \cdots, \zeta_m \in F} \prod_{\mu \neq \nu} |\zeta_\mu - \zeta_\nu|^{1/m(m-1)} . \tag{1}$$

It can be proved that the set $|f(z)| \leq 1$ has capacity 1. We shall consider a class of sets that includes the set $|f(z)| \leq 1$ in the case where all the zeros of $f(z)$ are real.

THEOREM 2. *Let* E *be a closed bounded set that is the union of orthogonal circles and segments of* X. *Let* b *be the (minimal) width of* E, *and* d *the sum of the diameters of the connected components of* E. *Then* cap E = 1 *implies that*

$$b^2 + d^2 \leq 4d \leq 16, \qquad b \leq 2, \qquad b + d \leq 2 + 2\sqrt{2}, \qquad bd \leq 3\sqrt{3}.$$

The sign of equality can occur in each estimate.

Proof. It is easy to see that $d = \text{meas}(X \cap E)$ and that there exists an orthogonal circle K of diameter b that is contained in E. We can take the origin O as the centre of K. Consider first the set

$$F = K \cup (X \cap E).$$

Because $F \subset E$, we have cap F $\leq$ cap E = 1.

Next we obtain a new set F^* in the following way: The set F consists of K and one or more segments of X; we translate these segments towards the point O until they form (without overlapping) one connected set together with K. Then F^* consists of the orthogonal circle K of diameter b and one or two real segments of total length $d - b$. We choose points $z_\nu^* \in F^*$ ($\nu = 1, \cdots, m$) such that

$$\kappa_m^* = \prod_{\mu \neq \nu} |z_\mu^* - z_\nu^*|^{1/m(m-1)}$$

is maximal, and we let z_ν denote the point from which z_ν^* was obtained in the shifting process described above. Then we have $|z_\mu^* - z_\nu^*| \leq |z_\mu - z_\nu|$. Therefore it follows from equation (1) that

$$\kappa_m^* \leq \prod_{\mu \neq \nu} |z_\mu - z_\nu|^{1/m(m-1)} \leq \kappa_m .$$

This implies that

$$\text{cap } F^* \leq \text{cap } F \leq 1 . \tag{2}$$

Let $-c_1$ and c_2 be the endpoints of the segment $X \cap F^*$, so that $c_1 + c_2 = d$. The orthogonal circle K: $|z| \leq b/2$ belongs to F^*. Hence the function

$$w = z + \frac{1}{4} b^2 z^{-1} \tag{3}$$

maps the region exterior to F^* conformally onto the plane slit along the segment

$$S: [-(c_1 + b^2/4c_1), + (c_2 + b^2/4c_2)] .$$

Since the coefficient of z in the function (3) is one, we have cap F^* = cap S. The capacity of a segment is one fourth of its length, that is,

$$\text{cap } F^* = \frac{1}{4}(c_1 + c_2) + \frac{b^2}{16}\left(\frac{1}{c_1} + \frac{1}{c_2}\right) . \tag{4}$$

Using the relations $c_1 + c_2 = d$ and $c_1(d - c_1) \leq \frac{1}{4} d^2$, we see from inequality (2) that

$$1 \geq \frac{1}{4} d + \frac{b^2 d}{16 c_1(d - c_1)} \geq \frac{1}{4} d + \frac{b^2}{4d} = \frac{b^2 + d^2}{4d}$$

holds and therefore $b^2 + d^2 \leq 4d$. This inequality implies that $d^2 \leq 4d$, $4d \leq 16$ and

$$b^2 \leq d(4 - d) \leq 4, \qquad b \leq 2,$$

$$b + d \leq (4d - d^2)^{1/2} + d \leq 2 + 2\sqrt{2},$$

$$bd \leq (4d - d^2)^{1/2} d \leq 3\sqrt{3} .$$

To prove the last sentence of Theorem 2, let $b^2 + d^2 = 4d$, and let E be the set that consists of the orthogonal circle $|z| \leq b/2$ and the two symmetrical segments $[-d/2, -b/2]$ and $[b/2, d/2]$. Then b and d are the width and diameter of E, and since in this case $E = F^*$, equation (4) implies that cap E = 1. Thus the set E satisfies the hypotheses of Theorem 2, and we have equality in the first estimate. To obtain equality in the four other estimates, we take in turn $d = 4, 2, 2 + \sqrt{2}$ and 3.

We note that if the hypothesis on E is replaced by the (essentially weaker) assumption that E is connected, then the last three of our estimates are no longer valid (see [2, p. 225]).

REFERENCES

1. P. Erdös, F. Herzog and G. Piranian, *Metric properties of polynomials*, J. Analyse Math. 6 (1958), p. 125-148.

2. Chr. Pommerenke, *On some problems by Erdös, Herzog and Piranian,* Michigan Math. J. 6 (1959), 221-225.

University of Göttingen