ON A THEOREM OF A. JAKIMOVSKI ON LINEAR TRANSFORMATIONS

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Dr. Jakimovski defined, in [1], the $[F, d_n]$ transformations in the following way. Let $\{d_n\}$ $(n \geq 1)$ be a fixed sequence of complex numbers. Let the matrix (P_{mn}) be defined by $P_{00} = 1$, $P_{mn} = 0$ when $m > n$, and

$$\prod_{k=1}^{n} (x + d_k) = \sum_{m=0}^{n} P_{nm} x^m \qquad (0 \leq m \leq n = 1, 2, 3, \cdots).$$

Then the $[F, d_n]$ transform $\{t_n\}$ of $\{s_n\}$ is defined by $t_0 = s_0$ and

$$t_n = \left(\prod_{k=1}^{n} (1 + d_k)^{-1} \right) \sum_{m=0}^{n} P_{nm} s_m \qquad (n \geq 1).$$

The following result was proved in [1].

THEOREM. *If* $\{d_n\}$ $(d_n \neq 0,\ d_n \neq -1$ *for* $n \geq 1)$ *satisfies*

(A)
$$\lim_{n \to \infty} \prod_{k=1}^{n} \left| 1 + \frac{1}{d_k} \right| = +\infty$$

and

(B)
$$\prod_{k=1}^{n} \frac{1 + |d_k|}{|1 + d_k|} \leq H < +\infty,$$

then the $[F, d_n]$ *transformation is regular. Condition* (A) *is also necessary.*

The question of the necessity of condition (B) was left open by Dr. Jakimovski. I give here an example which proves that (B) is *not* necessary. Let

$$d_{2k-1} = \frac{1}{k+1}; \qquad d_{2k} = -\frac{1}{k+1} \qquad (k = 1, 2, \cdots).$$

Then, as $N \to \infty$,

$$\prod_{k=1}^{2N} \frac{1 + |d_k|}{|1 + d_k|} = \prod_{k=1}^{2N+1} \frac{1 + |d_k|}{|1 + d_k|} = \prod_{k=1}^{N} \frac{1 + \frac{1}{k+1}}{1 - \frac{1}{k+1}} = \prod_{k=1}^{N} \frac{k+2}{k} = (N+1)(N+2)/2 \to +\infty,$$

and therefore condition (B) is not fulfilled. Let $c_{00} = 1$ and

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$$c_{nm} = \left(\prod_{k=1}^{n} (1 + d_k)^{-1} \right) P_{nm} \quad \text{for } n = 1, 2, \cdots; \; m = 0, 1, \cdots.$$

The conditions for regularity of the transformation are

$$(\alpha) \qquad \lim_{n \to \infty} \sum_{m=0}^{n} c_{nm} = 1$$

$$(\beta) \qquad \lim_{n \to \infty} c_{nm} = 0 \qquad (m = 0, 1, \cdots),$$

$$(\gamma) \qquad \sum_{m=0}^{n} |c_{nm}| < H < +\infty \qquad (n = 0, 1, \cdots).$$

The definition implies that (α) holds. It is easy to show that $P_{nn} = 1$ and that, for $n = 1, 2, \cdots$ and $m = 0, 1, 2, \cdots$,

$$(1) \qquad \begin{cases} P_{n+1,m} = P_{n,m-1} + d_{n+1} P_{nm}, \\ P_{nm} = \sum_{1 \le j_1 < \cdots < j_{n-m} \le n} d_{j_1} \cdots d_{j_{n-m}}. \end{cases}$$

Since by definition $d_{2k} = -d_{2k-1}$, we see easily that for $v = 0, 1, \cdots$ and $u = 0, 1, \cdots$, $P_{2v,2u+1} = 0$ and therefore

$$(2) \qquad c_{2v,2u+1} = 0.$$

From (1) and (2) we obtain

$$c_{2v+1,2u+1} = \frac{1}{1 + d_{2v+1}} c_{2v,2u},$$

$$c_{2v+1,2u} = \frac{d_{2v+1}}{1 + d_{2v+1}} c_{2v,2u}.$$

Since $d_{2v+1} > 0$, it follows that

$$(3) \qquad |c_{2v+1,2u+1}| + |c_{2v+1,2u}| = |c_{2v,2u}|.$$

But

$$|c_{2v,2u}| = \frac{|P_{2v,2u}|}{\prod_{k=1}^{v} (1-(k+1)^{-2})} = \frac{\sum_{2 \le j_1 < j_2 < \cdots < j_{v-u} \le v+1} j_1^{-2} j_2^{-2} \cdots j_{v-u}^{-2}}{\prod_{k=1}^{v} (1 - (k + 1)^{-2})}$$

$$< \binom{v}{u} 2^{u-v} \prod_{k=2}^{v+1} (1 - k^{-2})^{-1}.$$

This implies that

$$\lim_{v \to \infty} |c_{2v,2u}| = 0 \qquad (u = 0, 1, \cdots).$$

By (2) and (3) we see that (β) holds. Now, from (2) and (3), it follows that

$$\text{(4)} \qquad \sum_{m=0}^{2v+1} |c_{2v+1,m}| = \sum_{m=0}^{2v} |c_{2v,m}| = \sum_{u=0}^{v} |c_{2v,2u}|$$

and

$$\sum_{u=0}^{v} |c_{2v,2u}| = \left(\prod_{k=2}^{v+1} (1 - k^{-2})^{-1} \right) \cdot \sum_{u=0}^{v} \sum_{2 \leq j_1 < \cdots < j_{v-u} \leq v+1} j_1^{-2} j_2^{-2} \cdots j_{v-u}^{-2}$$

$$\leq \left(\prod_{k=2}^{v+1} (1 - k^{-2})^{-1} \right) \cdot \left(\prod_{k=2}^{v+1} (1 + k^{-2}) \right) < H < +\infty ,$$

where H does not depend on v. By (4) we see that (γ) is true.

REFERENCE

1. A. Jakimovski, *A generalization of the Lototsky method of summability*, Michigan Math. J. 6 (1959), 277-290.

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